Research Article

A Stochastic Switched Epidemic Model with Two Epidemic Diseases

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In this paper, we study a stochastic epidemic model with double epidemics which includes white noise and telegraph noise modeled by Markovian switching. Sufficient conditions for the extinction and persistence of the diseases are established. In the end, some numerical simulations are presented to demonstrate our analytical results.

1. Introduction

Compartmental models represent an important mathematical tool to describe the spread of infectious diseases [1–6]. A mathematical model that describes the spread of infectious diseases currently known as the SIR model divides the population into three classes, the sensitive S , infectious I , and recovered R , and was firstly developed by Kermack and McKendrick [7] in 1927. To take into account the case of recovered individuals who lose their immunity and return to the susceptible compartment, we use SIRS epidemic models investigated by many scientists (see, for example, [8–10]). In classic SIRS epidemic models, the disease caused by one virus but in the real world, there are certain diseases to which the population is infected with two or more viruses (see [11–14]). Zhao et al. [15] established the following SIRS model describing the dynamics of two viruses (see Figure 1):

$$\begin{cases} S'(t) = \mu - \beta_1 S(t)I_1(t) - \beta_2 S(t)I_2(t) - \mu S(t) + \delta R(t), \\ I_1'(t) = \beta_1 S(t)I_1(t) - (\mu + \gamma_1)I_1(t), \\ I_2'(t) = \beta_2 S(t)I_2(t) - (\mu + \gamma_2)I_2(t), \\ R'(t) = \gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu + \delta)R(t), \end{cases} \quad (1)$$

where $S(t)$ denotes the number of susceptible individuals at time t , $I_1(t)$ and $I_2(t)$ are, respectively, the numbers of

infected individuals with virus A and virus B , $R(t)$ denotes the number of removed individuals at time, the parameters β_1 and β_2 are the disease transmission coefficient with virus A and virus B , respectively, μ represents the birth rate of the population and the natural death rate of the population, γ_1 and γ_2 are the recovery rates of infectious individuals with virus A and with virus B , and δ is the lost immunity rate.

However, the biological systems are necessarily touched via the environmental noise, which can present an additional degree of realism compared with their corresponding deterministic models. Thus, it is an important component in an ecosystem. Consequently, many works existing in the literature studied the effects of white noise on epidemic models (see, for example, [16–18]). El Fatini et al. [19] studied a stochastic epidemic model. They assumed that environmental perturbations are of the white noise type. They showed the existence and the uniqueness of the solution of their proposed model. They established sufficient conditions for extinction by constructing some suitable Lyapunov function. And they showed sufficient conditions for the persistence of the disease under some conditions on parameters of the model. Liu et al. [20] investigated the dynamical behavior of a higher-order stochastically perturbed SIRS epidemic model with relapse and media coverage. They showed sufficient measures for the existence and uniqueness of an ergodic stationary distribution of their stochastic model by constructing a suitable Lyapunov function. And

FIGURE 1: The compartmental diagram for the SIRS model with two epidemic diseases.

they obtained sufficient conditions for the elimination of the infectious disease. Tang et al. [21] investigated the threshold of a class of the stochastic SIRS epidemic model with nonlinear incidence, which determines the extinction and persistence of the disease. In [22], Qi et al.

investigated the dynamics of a nonautonomous stochastic SIS epidemic model with nonlinear incidence rate and double epidemic. Rajasekar and Pitchaimani [23] have formulated the following stochastic SIRS model with two viruses:

$$\begin{cases} dS(t) = [\mu - \beta_1 S(t)I_1(t) - \beta_2 S(t)I_2(t) - \mu S(t) + \delta R(t)]dt + \sigma_1 S(t)dB_1(t), \\ dI_1(t) = [\beta_1 S(t)I_1(t) - (\mu + \gamma_1)I_1(t)]dt + \sigma_2 I_1(t)dB_2(t), \\ dI_2(t) = [\beta_2 S(t)I_2(t) - (\mu + \gamma_2)I_2(t)]dt + \sigma_3 I_2(t)dB_3(t), \\ dR(t) = [\gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu + \delta)R(t)]dt + \sigma_4 R(t)dB_4(t), \end{cases} \quad (2)$$

where $B_i(t)$ for $i = 1, 2, 3, 4$ are independent standard Brownian motions which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions and σ_i^2 for $i = 1, 2, 3, 4$ denote the intensity of the white noise.

On the other hand, the white noise cannot express the phenomena that the population may submit sudden catastrophic collisions in nature. The telegraph noise can be illustrated as a switching between two or more regimes of environment, which differ by factors such as nutrition or socio-cultural factors [24–26]. In fact, switching between environmental regimes is frequently memoryless and the waiting time for the next switching follows the exponential distribution [27]. So, the regime switching can be modeled by a continuous-time Markov chain $\{r(t)\}_{t \geq 0}$ taking values in a finite-state space $\mathbb{S} = \{1, \dots, m\}$. Many works have introduced Markov process into their models. For example, in [28], Zhang and Peng addressed a stochastic cholera epidemic model with vaccination. They presented the threshold of a stochastic cholera epidemic model which determines the extinction of the disease and the existence of a unique ergodic stationary distribution. Wang et al. [29]

discussed the asymptotic properties of the solution of a stochastic SIRS epidemic model with Beddington–DeAngelis incidence rate and Markovian switching. They proved that the proposed model is stochastically asymptotically stable in the large. In addition, they presented threshold values and conditions which determine disease extinction and persistence. In [30], Li et al. proposed a stochastic SIRS model with vaccination and regime switching. They proved sufficient conditions for the extinction and the existence of the stationary distribution of the population. Motivated by the above facts, in this work, we will initially include telegraph noise into stochastic model (1) to obtain a more realistic model. Then, the following stochastic white and telegraphic noise perturbations on the diseases transmission coefficient are assumed:

$$\begin{aligned} \beta_1 dt &\longrightarrow \beta_1(r(t))dt + \sigma_1(r(t))dB_1(t), \\ \beta_2 dt &\longrightarrow \beta_2(r(t))dt + \sigma_2(r(t))dB_2(t). \end{aligned} \quad (3)$$

Then, the stochastic SIRS epidemic model with two viruses (1) can be expressed as the following stochastic system under regime switching:

$$\begin{cases} dS(t) = [\mu(r(t)) - \beta_1(r(t))S(t)I_1(t) - \beta_2(r(t))S(t)I_2(t) - \mu(r(t))S(t) \\ \quad + \delta(r(t))R(t)]dt - \sigma_1(r(t))S(t)I_1(t)dB_1(t) - \sigma_2(r(t))S(t)I_2(t)dB_2(t), \\ dI_1(t) = [\beta_1(r(t))S(t)I_1(t) - (\mu(r(t)) + \gamma_1(r(t)))I_1(t)]dt \\ \quad + \sigma_1(r(t))S(t)I_1(t)dB_1(t), \\ dI_2(t) = [\beta_2(r(t))S(t)I_2(t) - (\mu(r(t)) + \gamma_2(r(t)))I_2(t)]dt \\ \quad + \sigma_2(r(t))S(t)I_2(t)dB_2(t), \\ dR(t) = \gamma_1(r(t))I_1(t) + \gamma_2(r(t))I_2(t) - (\mu(r(t)) + \delta(r(t)))R(t). \end{cases} \quad (4)$$

This article proposes a stochastic SIRS epidemic model with regime switching and double epidemic hypothesis. Our main purpose is to investigate the effect of the white and telegraph noises on the spread dynamics of the disease in the population. The rest of the paper is as follows. In Section 3, we show the existence and uniqueness of a global positive solution to system (4). In Sections 4 and 5, we study the existence of a stochastic threshold for the extinction and the persistence in mean of the disease. In last section, we present some numerical simulations to demonstrate our main theoretical results.

2. Existence and Uniqueness of the Nonnegative Solution

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Let $\{r(t)\}_{t \geq 0}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ taking values in a finite-state space $\mathbb{S} = \{1, 2, \dots, m\}$ with the generator $\Phi = (\phi_{uv})_{1 \leq u, v \leq N}$ given, for $\delta > 0$, by

$$P(r(t + \delta) = v | r(t) = u) = \begin{cases} \phi_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \phi_{uu}\delta + o(\delta), & \text{if } u = v. \end{cases} \quad (5)$$

Here, ϕ_{uv} is the transition rate from u to v and $\phi_{uv} \geq 0$ if $u \neq v$, while

$$\phi_{uu} = - \sum_{u \neq v} \phi_{uv}. \quad (6)$$

Suppose that the Markov chain $r(t)$ is independent of the Brownian motion $B(\cdot)$ and it is irreducible. Under this condition, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \dots, \pi_N)$, which can be determined by solving the linear equation $\pi\Phi = 0$, subject to $\sum_{i=1}^N \pi_i = 1$, and $\pi_i > 0, \forall i \in \mathbb{S}$. Thereafter, for any vector $h = (h(1), \dots, h(N))^T$, let $\hat{h} = \min_{i \in \mathbb{M}} \{h(i)\}$ and $\bar{h} = \max_{i \in \mathbb{M}} \{h(i)\}$.

We consider the following stochastic system:

$$d\vartheta(t) = f(t, \vartheta(t), r(t))dt + g(t, \vartheta(t), r(t))dB(t), \quad (*) \quad (7)$$

where $B(t)$ is a d -dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Denoted by $C^{1,2}(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$, the family of all nonnegative functions $\mathcal{Q}$ is defined on $\mathbb{R}^d \times \mathbb{S}$ such that they are continuously twice differentiable in ϑ . The operator $\mathcal{L}$ associated with $(*)$ is defined as follows:

$$\begin{aligned} \mathcal{L}\mathcal{Q}(\vartheta, i) &= \mathcal{Q}_t(t, \vartheta, i) + \mathcal{Q}_\vartheta(t, \vartheta, i)f(t, \vartheta, i) \\ &\quad + \frac{1}{2}g^T(t, \vartheta, i)\mathcal{Q}_{\vartheta, \vartheta}(t, \vartheta, i)g(t, \vartheta, i) \\ &\quad + \sum_{j \in \mathbb{S}} \phi_{ij}\mathcal{Q}(t, \vartheta, j), \end{aligned} \quad (8)$$

where $\mathcal{Q}_\vartheta$ and $\mathcal{Q}_{\vartheta, \vartheta}$ represent the gradient and Hessian of $\mathcal{Q}$ and T is the transpose of a matrix.

By Itô's formula, if $\mathcal{Q}$ defined on $\mathbb{R}^d \times \mathbb{S}$, we have

$$d\mathcal{Q}(\vartheta(t), i) = \mathcal{L}\mathcal{Q}(\vartheta(t), i)dt + \mathcal{Q}_\vartheta(\vartheta(t), i)g(\vartheta(t), i)dB(t). \quad (9)$$

The total population in system (4) verifies the following equation:

$$\begin{aligned} d(S(t) + I_1(t) + I_2(t) + R(t)) \\ = [\mu - \mu(S(t) + I_1(t) + I_2(t) + R(t))]dt. \end{aligned} \quad (10)$$

Thus,

$$\begin{aligned} S(t) + I_1(t) + I_2(t) + R(t) \\ = 1 + (S(0) + I_1(0) + I_2(0) + R(0) - 1)e^{-\mu(r(t))t}, \quad \text{for all } t \geq 0. \end{aligned} \quad (11)$$

Since, for $X(0) \in \mathbb{D}$, we have

$$S(t) + I_1(t) + I_2(t) + R(t) = 1, \quad \text{for all } t \geq 0 \text{ a.s.} \quad (12)$$

Let

$$\mathbb{D} = \{(S + I_1 + I_2 + R) \in \mathbb{R}_+^4 : S + I_1 + I_2 + R = 1\}. \quad (13)$$

By using the same method as in [31, 32], we can easily prove the following theorem.

Theorem 1. *Let $X(0) = (S(0), I_1(0), I_2(0), R(0)) \in \mathbb{D}$, then there is a unique solution $(S(t), I_1(t), I_2(t), R(t))$ to system (4) on $t \geq 0$, and the solution will remain in $\mathbb{D}$ with probability 1.*

Firstly, we define the following expressions which we use to determine the extinction of epidemics:

$$\begin{aligned}\sum_{i=1}^m \pi_i D_i &= \sum_{i=1}^m \pi_i \left(-\sigma_1^2(i) + \beta_1(i) - (\mu(i) + \gamma_1(i)) \right), \\ \sum_{i=1}^m \pi_i A_i &= \sum_{i=1}^m \pi_i \left(-\sigma_2^2(i) + \beta_2(i) - (\mu(i) + \gamma_2(i)) \right).\end{aligned}\quad (14)$$

3. Extinction

This section is consecrated to give sufficient conditions so that the epidemics die out. According to [33], we have the following lemma.

Lemma 1. *Suppose that there exists a function $U(u, i) \in C^{1,2}(\mathbb{R}^3 \times \mathbb{S}; \mathbb{R}_+)$ and $\rho(u) \in (\mathbb{R}^3 \times; \mathbb{R}_+)$ disappears only at E_0 such that*

$$\begin{aligned}\mathcal{L}U(u, i) &\leq -\rho(u), \quad \text{for all } (u, i) \in \mathbb{R}^3 \times \mathbb{S}, \\ \lim_{|x| \rightarrow \infty} \inf_{j \in \mathbb{S}} U(u, i) &= \infty.\end{aligned}\quad (15)$$

Then, the equilibrium E_0 of system (4) is globally asymptotically stable in probability.

By Lemma 1, we identify the conditions for stochastic asymptotic stability of the disease-free equilibrium which are presented by the following theorem.

Theorem 2. *If $\beta_2(i) \geq \sigma_2^2(i)$ and $\beta_1(i) \geq \sigma_1^2(i)$ for all $i \in \mathbb{S}$, and*

$$\begin{aligned}\sum_{i=1}^m \pi_i D_i &< 0, \\ \sum_{i=1}^m \pi_i A_i &< 0,\end{aligned}\quad (16)$$

then the disease-free equilibrium E_0 is globally stochastically asymptotically stable on $\mathbb{D}$.

Proof. We consider the following Lyapunov function:

$$\begin{aligned}\mathcal{H}(S, I, R, i) &= m_1(1-S)^2 + (\eta + \omega_i)I_1^{(1/\eta)} \\ &\quad + (\theta + \bar{\omega}_i)I_2^{(1/\theta)} + m_2R^2, \quad \text{for all } i \in \mathbb{S},\end{aligned}\quad (17)$$

where m_1 , m_2 , η , and θ are real positive constants to be determined later. By Itô's formula, we calculate $\mathcal{L}\mathcal{H}$ as follows:

$$\begin{aligned}\mathcal{L}\mathcal{H} &= -2m_1(1-S)[\mu - \mu S - \beta_1 S I_1 - \beta_2 S I_2 + \delta R] + m_1 \sigma_1^2 S^2 I_1^2 \\ &\quad + m_2 \sigma_2^2 S^2 I_2^2 + \frac{1}{\eta}(\eta + \omega_i)I_1^{(1/\eta)}[\beta_1 S - (\mu + \gamma_1)] \\ &\quad + \frac{1}{2\eta}\left(\frac{1}{\eta} - 1\right)(\eta + \omega_i)\sigma_1^2 S^2 I_1^{(1/\eta)} \\ &\quad + \frac{1}{\theta}(\theta + \bar{\omega}_i)I_2^{(1/\theta)}[\beta_2 S - (\mu + \gamma_2)] + \frac{1}{2\theta}\left(\frac{1}{\theta} - 1\right)(\theta + \bar{\omega}_i)\sigma_2^2 S^2 I_2^{(1/\theta)} \\ &\quad + 2m_2 R[\gamma_1 I_1 + \gamma_2 I_2 - (\mu + \delta)R] + \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i + \bar{\omega}_k - \bar{\omega}_i) \\ &= -2m_1(1-S)^2 + 2m_1\beta_1 S I_1(1-S) + 2m_1\beta_2 S I_2(1-S) \\ &\quad - 2m_1\delta(1-S)R + m_1\sigma_1^2 S^2 I_1^2 + m_1\sigma_2^2 S^2 I_2^2 + \frac{1}{\eta}(\eta + \omega_i)\beta_1 I_1^{(1/\eta)}S \\ &\quad - \frac{1}{\eta}(\eta + \omega_i)(\mu + \gamma_1)I_1^{(1/\eta)} + \frac{1}{2\eta}\left(\frac{1}{\eta} - 1\right)(\eta + \omega_i)\sigma_1^2 S^2 I_1^{(1/\eta)} \\ &\quad + \frac{1}{\theta}(\theta + \bar{\omega}_i)\beta_2 I_2^{(1/\theta)}S - \frac{1}{\theta}(\theta + \bar{\omega}_i)(\mu + \gamma_2)I_2^{(1/\theta)} \\ &\quad + \frac{1}{2\theta}\left(\frac{1}{\theta} - 1\right)(\theta + \bar{\omega}_i)\sigma_2^2 S^2 I_2^{(1/\theta)} + m_2\gamma_1 R I_1 + m_2\gamma_2 R I_2 - m_2(\mu + \delta)R^2 \\ &\quad + I_1^{(1/\eta)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) + I_2^{(1/\theta)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\bar{\omega}_k - \bar{\omega}_i).\end{aligned}\quad (18)$$

Using the fact that $I \leq 1 - S$ and $S, I \in \mathbb{D}$, we obtain

$$\begin{aligned}
\mathcal{LH} \leq & -2m_1(1-S)^2 + 2m_1\beta_1 I_1^{(1/\eta)} + 2m_1\beta_2 I_2^{(1/\theta)} + m_1\sigma_1^2 I_1^{(1/\eta)} + m_1\sigma_2^2 I_2^{(1/\theta)} \\
& + \frac{1}{\eta}(\eta + \omega_i)\beta_1 I_1^{(1/\eta)} S - \frac{1}{\eta}(\eta + \omega_i)(\mu + \gamma_1) I_1^{(1/\eta)} \\
& + \frac{1}{2\eta}\left(\frac{1}{\eta} - 1\right)(\eta + \omega_i)\sigma_1^2 S^2 I_1^{(1/\eta)} + \frac{1}{\theta}(\theta + \omega_i)\beta_2 I_2^{(1/\theta)} S \\
& - \frac{1}{\theta}(\theta + \omega_i)(\mu + \gamma_2) I_2^{(1/\theta)} + \frac{1}{2\theta}\left(\frac{1}{\theta} - 1\right)(\theta + \omega_i)\sigma_2^2 S^2 I_2^{(1/\theta)} \\
& + (m_2\gamma_1 - m_1\delta)RI_1 + (m_2\gamma_2 - m_1\delta)RI_2 - m_2(\mu + \delta)R^2 \\
& + I_1^{(1/\eta)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) + I_2^{(1/\theta)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i),
\end{aligned} \tag{19}$$

for all η and $\theta \geq 1$. Since we can choose $m_2 < \min_{i \in \mathbb{S}} \{(m_1\delta(i)/\gamma_1(i)), (m_1\delta(i)/\gamma_2(i))\}$, we obtain

$$\begin{aligned}
\mathcal{LH} \leq & -2m_1(1-S)^2 - m_2(\mu + \delta)R^2 + \frac{(\eta + \omega_i)}{\eta} I_1^{(1/\eta)} \left[\frac{\eta m_1}{(\eta + \omega_i)} (\sigma_1^2 + 2\beta_1) \right. \\
& \left. - (\mu + \gamma_1) + \frac{1}{2\eta} \sigma_1^2 + \beta_1 S - \sigma_1^2 S^2 + \frac{\eta}{(\eta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right] \\
& + \frac{(\theta + \omega_i)}{\theta} I_2^{(1/\theta)} \left[\frac{\theta m_1}{(\theta + \omega_i)} (\sigma_2^2 + 2\beta_2) - (\mu + \gamma_2) + \frac{1}{2\theta} \sigma_2^2 + \beta_2 S \right. \\
& \left. - \sigma_2^2 S^2 + \frac{\theta}{(\theta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right] \\
= & -2m_1(1-S)^2 - m_2(\mu + \delta)R^2 + \frac{(\eta + \omega_i)}{\eta} I_1^{(1/\eta)} \left[\frac{\eta m_1}{(\eta + \omega_i)} (\sigma_1^2 + 2\beta_1) \right. \\
& \left. + \frac{1}{2\eta} \sigma_1^2 + f_i(S) + \frac{\eta}{(\eta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right] \\
& + \frac{(\theta + \omega_i)}{\theta} I_2^{(1/\theta)} \left[\frac{\theta m_1}{(\theta + \omega_i)} (\sigma_2^2 + 2\beta_2) + \frac{1}{2\theta} \sigma_2^2 + g_i(S) \right. \\
& \left. + \frac{\theta}{(\theta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right],
\end{aligned} \tag{20}$$

where $f_i(x) = -\sigma_1^2(i)x^2 + \beta_1(i)x - (\mu(i) + \gamma_1(i))$ and $g_i(x) = -\sigma_2^2(i)x^2 + \beta_2(i)x - (\mu(i) + \gamma_2(i))$. Note that the functions $f_i(x)$ and $g_i(x)$ are all increasing on $(0, 1)$, which means

$$\begin{aligned}
f_i(x) & \leq f_i(1) =: D_i, \\
g_i(x) & \leq g_i(1) =: A_i,
\end{aligned} \tag{21}$$

then

$$\begin{aligned}
\mathcal{LH} \leq & -2m_1(1-S)^2 - m_2(\mu + \delta)R^2 + \frac{(\eta + \omega_i)}{\eta} I_1^{(1/\eta)} \left[\frac{\eta m_1}{(\eta + \omega_i)} (\sigma_1^2 + 2\beta_1) \right. \\
& + \left. \frac{1}{2\eta} \sigma_1^2 + D_i + \frac{\eta}{(\eta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right] \\
& + \frac{(\theta + \omega_i)}{\theta} I_2^{(1/\theta)} \left[\frac{\theta m_1}{(\theta + \omega_i)} (\sigma_2^2 + 2\beta_2) + \frac{1}{2\theta} \sigma_2^2 + A_i \right. \\
& + \left. \frac{\theta}{(\theta + \omega_i)} \sum_{k \neq i, k \in \mathbb{S}} \phi_{ik}(\omega_k - \omega_i) \right].
\end{aligned} \tag{22}$$

Since the generator matrix Φ is irreducible, then for $D = (D_1, \dots, D_m)^T$, there exists $\Xi = (\omega_1, \dots, \omega_m)^T$ solution of the Poisson system:

$$\Phi \Xi = -D + \left(\sum_{i=1}^m \pi_i D_i \right) \vec{E}. \tag{23}$$

And for $A = (A_1, \dots, A_m)^T$, there exists $\Theta = (\omega_1, \dots, \omega_m)^T$ solution of the Poisson system:

$$\Phi \Theta = -A + \left(\sum_{i=1}^m \pi_i A_i \right) \vec{E}, \tag{24}$$

where $\vec{E}$ denotes the column vector with all its entries equal to 1.

Substituting (23) and (24) in (22), we get

$$\begin{aligned}
\mathcal{LH} \leq & -2m_1(1-S)^2 - m_2(\mu + \delta)R^2 + \frac{(\eta + \omega_i)}{\eta} I_1^{(1/\eta)} \left[\frac{\eta m_1}{(\eta + \omega_i)} (\sigma_1^2 + 2\beta_1) \right. \\
& + \left. \frac{1}{2\eta} \sigma_1^2 + \frac{\eta}{(\eta + \omega_i)} \left(D_i - \sum_{i=1}^m \pi_i D_i \right) + \sum_{i=1}^m \pi_i D_i \right] \\
& + \frac{(\theta + \omega_i)}{\theta} I_2^{(1/\theta)} \left[\frac{\theta m_1}{(\theta + \omega_i)} (\sigma_2^2 + 2\beta_2) + \frac{1}{2\theta} \sigma_2^2 \right. \\
& + \left. \frac{\theta}{(\theta + \omega_i)} \left(A_i - \sum_{i=1}^m \pi_i A_i \right) + \sum_{i=1}^m \pi_i A_i \right].
\end{aligned} \tag{25}$$

Choosing η_0 and θ_0 , such that

$$\begin{aligned}
C_1 &:= \max_{i \in \mathbb{S}} \left\{ \frac{\sigma_1^2}{2\eta_0} + \frac{\eta_0}{(\eta_0 + \omega_i)} \left(D_i - \sum_{i=1}^m \pi_i D_i \right) + \sum_{i=1}^m \pi_i D_i \right\} < 0, \\
C_2 &:= \max_{i \in \mathbb{S}} \left\{ \frac{\sigma_2^2}{2\theta_0} + \frac{\theta_0}{(\theta_0 + \omega_i)} \left(A_i - \sum_{i=1}^m \pi_i A_i \right) + \sum_{i=1}^m \pi_i A_i \right\} < 0.
\end{aligned} \tag{26}$$

Since we choose $\eta > \max\{-\omega_i, \eta_0, 1\}$ and $\theta > \max\{-\omega_i, \theta_0, 1\}$ such that

$$\begin{aligned}
m_1 &< \min \left\{ -\frac{\eta + \min \omega_i}{\eta(\max \sigma_1^2 + 2 \max \beta_1)} \times C_1, \right. \\
&\quad \left. -\frac{\theta + \min \omega_i}{\theta(\max \sigma_2^2 + 2 \max \beta_2)} \times C_2 \right\}.
\end{aligned} \tag{27}$$

According to Lemma 1, the proof is completed.

Theorem 3. Let $(S(t), I_1(t), I_2(t), R(t))$ be the solution of system (4), then for any given initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{D}$, model (4) has the following property:

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sup \frac{1}{t} \ln(1 - S + I_1 + I_2 + R) \\
& \leq \sum_{i=1}^m \pi_i \left(\frac{\beta_1^2(i) - \mu(i)\sigma_1^2(i)}{2\sigma_1^2(i)} + \frac{\beta_2^2(i) - \mu(i)\sigma_2^2(i)}{2\sigma_2^2(i)} \right), \quad \text{a.s.}
\end{aligned} \tag{28}$$

If

$$\sum_{i=1}^m \pi_i \left(\frac{\beta_1^2(i) - \mu(i)\sigma_1^2(i)}{2\sigma_1^2(i)} + \frac{\beta_2^2(i) - \mu(i)\sigma_2^2(i)}{2\sigma_2^2(i)} \right) < 0 \tag{29}$$

holds, then the disease-free E_0 is almost surely exponentially stable in $\mathbb{D}$. Namely, the diseases die out with probability one.

$$\mathcal{G}(S, I_1, I_2, R, i) = \ln(1 - S + I_1 + I_2 + R). \quad (30)$$

By Itô's formula, we obtain

Proof. Consider the following function:

$$\begin{aligned} d\mathcal{G} = & \frac{1}{1 - S + I_1 + I_2 + R} [-\mu(r)(1 - S) - \mu(r)I_1 - \mu(r)I_2 \\ & - (\mu(r) + 2\delta(r))(r)R + 2\beta_1(r)SI_1 + 2\beta_2(r)SI_2] dt \\ & - 2\sigma_1^2(r) \left(\frac{SI_1}{1 - S + I_1 + I_2 + R} \right)^2 dt - 2\sigma_2^2(r) \left(\frac{SI_2}{1 - S + I_1 + I_2 + R} \right)^2 dt \\ & + \frac{2\sigma_1(r)SI_1}{1 - S + I_1 + I_2 + R} dB_1 + \frac{2\sigma_2(r)SI_2}{1 - S + I_1 + I_2 + R} dB_2. \end{aligned} \quad (31)$$

Note that

$$\frac{1}{(1 - S + I_1 + I_2 + R)} [-\mu(r)(1 - S) - \mu(r)I_1 - \mu(r)I_2 - (\mu(r) + 2\delta(r))R] \leq -\mu(r). \quad (32)$$

Thus, we have

$$\begin{aligned} d\mathcal{G} \leq & \left(-2\sigma_1(r) \left(\frac{SI}{1 - S + I_1 + I_2 + R} - \frac{\beta_1(r)}{2\sigma_1^2(r)} \right) + \frac{\beta_1^2(r)}{2\sigma_1^2(r)} - \frac{\mu(r)}{2} \right) dt \\ & + \left(-2\sigma_2(r) \left(\frac{SI}{1 - S + I_1 + I_2 + R} - \frac{\beta_2(r)}{2\sigma_2^2(r)} \right) + \frac{\beta_2^2(r)}{2\sigma_2^2(r)} - \frac{\mu(r)}{2} \right) dt \\ & + \frac{2\sigma_1(r)SI_1}{1 - S + I_1 + I_2 + R} dB_1 + \frac{2\sigma_2(r)SI_2}{1 - S + I_1 + I_2 + R} dB_2. \\ \leq & \left[\frac{\beta_1^2(r) - \mu(r)\sigma_1^2(r)}{2\sigma_1^2(r)} + \frac{\beta_2^2(r) - \mu(r)\sigma_2^2(r)}{2\sigma_2^2(r)} \right] dt + \frac{2\sigma_1(r)SI_1}{1 - S + I_1 + I_2 + R} dB_1 \\ & + \frac{2\sigma_2(r)SI_2}{1 - S + I_1 + I_2 + R} dB_2. \end{aligned} \quad (33)$$

Integrating both sides of (33) from 0 to t and dividing by t , we get

$$\begin{aligned} \frac{\mathcal{G}(t)}{t} &\leq \frac{\mathcal{G}(0)}{t} \\ &+ \frac{1}{t} \int_0^t \left(\frac{\beta_1^2(r(u)) - \mu(r(u))\sigma_1^2(r(u))}{2\sigma_1^2(r(u))} + \frac{\beta_2^2(r(u)) - \mu(r(u))\sigma_2^2(r(u))}{2\sigma_2^2(r(u))} \right) du \\ &+ \frac{1}{t} \int_0^t \frac{2\sigma_1(r(u))S(u)I_1(u)}{1 - S(u) + I_1(u) + I_2(u) + R(u)} dB_1(u) \\ &+ \frac{1}{t} \int_0^t \frac{2\sigma_2(r(u))S(u)I_2(u)}{1 - S(u) + I_1(u) + I_2(u) + R(u)} dB_2(u). \end{aligned} \quad (34)$$

Let $F_i(t) = \int_0^t ((2\sigma_i(r)S(u)I_i(u))/(1 - S(u) + I_1(u) + I_2(u) + R(u)))dB_i(u)$ ($i = 1, 2$), then

$$\langle F_i(t), F_i(t) \rangle$$

$$= \int_0^t \left(\frac{2\sigma_i(r(u))S(u)I_i(u)}{1 - S(u) + I_1(u) + I_2(u) + R(u)} \right)^2 du \leq 4\tilde{\sigma}_i, \quad t < \infty. \quad (35)$$

Then, by strong law of large numbers, we obtain

$$\lim_{t \rightarrow \infty} \frac{F_i(t)}{t} = 0, \quad \text{a.s.} \quad (36)$$

By the ergodic theory of the Markov chain, we can obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\frac{\beta_1^2(r(u)) - \mu(r(u))\sigma_1^2(r(u))}{2\sigma_1^2(r(u))} + \frac{\beta_2^2(r(u)) - \mu(r(u))\sigma_2^2(r(u))}{2\sigma_2^2(r(u))} \right) du \\ \leq \sum_{i=1}^m \pi_i \left(\frac{\beta_1^2(i) - \mu(i)\sigma_1^2(i)}{2\sigma_1^2(i)} + \frac{\beta_2^2(i) - \mu(i)\sigma_2^2(i)}{2\sigma_2^2(i)} \right), \quad \text{a.s.} \end{aligned} \quad (37)$$

Hence, if $\sum_{i=1}^m \pi_i ((\beta_1^2(i) - \mu(i)\sigma_1^2(i))/2\sigma_1^2(i)) + ((\beta_2^2(i) - \mu(i)\sigma_2^2(i))/2\sigma_2^2(i)) < 0$, then the disease-free E_0 is almost surely exponentially stable in $\mathbb{D}$.

4. Persistence

In this section, we explain sufficient conditions for the persistence of two epidemics.

We define $\langle h \rangle_t = (1/t) \int_0^t h(u)du$ if h is an integrable function on $[0, +\infty)$.

Definition 1. System (4) is said to be persistent in the mean, if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(u)du > 0, \quad \text{a.s.} \quad (38)$$

Let

$$R_0^s = \sum_{i=0}^m \pi_i \left(2\mu(i) + \gamma_1(i) + \gamma_2(i) + \frac{\sigma_1(i)}{2} + \frac{\sigma_2(i)}{2} - k_1(i)\mu(i) \right). \quad (39)$$

Theorem 4. Let $(S(t), I_1(t), I_2(t), R(t))$ be the solution of model (4). If $R_0^s > 0$, then

$$\liminf_{t \rightarrow \infty} \langle I_1 + I_2 \rangle_t \geq I^*,$$

$$\liminf_{t \rightarrow \infty} \langle R \rangle_t \geq \frac{\min\{\hat{\gamma}_1, \hat{\gamma}_2\}}{\check{\mu} + \check{\delta}} I^*, \quad (40)$$

$$\liminf_{t \rightarrow \infty} \langle 1 - S \rangle_t \geq \left(\frac{\check{\mu} + \check{\delta} + \min\{\hat{\gamma}_1, \hat{\gamma}_2\}}{\check{\mu} + \check{\delta}} \right) I^*,$$

where $I^* = (R_0^s/\bar{D})$ with $\bar{D}$ is positive constant.

Proof. By system (4), one can obtain that

$$\mathcal{L}(-\ln I_1) = -\beta_1(i)S + \mu(i) + \gamma_1(i) + \frac{\sigma_1^2(i)}{2}S^2, \quad (41)$$

$$\mathcal{L}(-\ln I_2) = -\beta_2(i)S + \mu(i) + \gamma_2(i) + \frac{\sigma_2^2(i)}{2}S^2.$$

Moreover,

$$\begin{aligned}
\mathcal{L}(S + I_1 + I_2) &= \mu(i) - \mu(i)S - (\mu(i) + \gamma_1(i)) \\
&\quad I_1 - (\mu(i) + \gamma_2(i))I_2 + \delta(i)R, \\
\mathcal{L}R &= \gamma_1(i)I_1 + \gamma_2(i)I_2 - (\mu(i) + \delta(i))R.
\end{aligned} \tag{42}$$

$$\begin{aligned}
\beta_1(i) + \beta_2(i) - k_1(i)\mu(i) + \sum_{l=1}^m \phi_{il}k_1(l) &= 0, \\
k_1(i)\delta(i) - k_2(i)(\mu(i) + \delta(i)) + \sum_{l=1}^m \phi_{il}k_2(l) &= 0.
\end{aligned} \tag{44}$$

So, we consider the Lyapunov function $\mathcal{T}$ defined on $\mathbb{D}$ by

$$\begin{aligned}
\mathcal{T}(S, I_1, I_2, i) &= -\ln I_1 - \ln I_2 \\
&\quad - k_1(i)(S + I_1 + I_2) - k_2(i)R - v(i),
\end{aligned} \tag{43}$$

Using Itô's formula, one can get that

$$\begin{aligned}
d\mathcal{T} &= \mathcal{L}\mathcal{T}dt - [\sigma_1(i)S + k_1(i)SI_1]dB_1(t) \\
&\quad - [\sigma_2(i)S + k_2(i)SI_2]dB_2(t),
\end{aligned} \tag{45}$$

where

where $(k_1(i), k_2(i))^T$ is the unique positive solution of the following system:

$$\begin{aligned}
\mathcal{L}\mathcal{T} &= -\beta_1S + \mu(i) + \gamma_1(i) + \frac{\sigma_1^2(i)}{2}S^2 - \beta_2(i)S + \mu(i) + \gamma_2(i) + \frac{\sigma_2^2(i)}{2}S^2 \\
&\quad - k_1(i)[\mu(i) - \mu(i)S - (\mu(i) + \gamma_1(i))I_1 - (\mu(i) + \gamma_2(i))I_2 + \delta(i)R] \\
&\quad - k_2(i)[\gamma_1(i)I_1 + \gamma_2(i)I_2 - (\mu(i) + \delta(i))R] - \sum_{l=1}^m \phi_{il}k_1(l)(S + I_1 + I_2) \\
&\quad - \sum_{l=1}^m \phi_{il}k_2(l)R - \sum_{l=1}^m \phi_{il}v(l).
\end{aligned} \tag{46}$$

Using the fact that $S \in (0, 1)$, we deduce that

$$\begin{aligned}
\mathcal{L}\mathcal{T} &\leq 2\mu(i) + \gamma_1(i) + \gamma_2(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} - k_1(i)\mu(i) - \sum_{l=1}^m \phi_{il}v(l) \\
&\quad - \left[\beta_1(i) + \beta_2(i) - k_1(i)\mu(i) + \sum_{l=1}^m \phi_{il}k_1(l) \right] S \\
&\quad - \left[k_1(i)\delta(i) - k_2(i)(\mu(i) + \delta(i)) + \sum_{l=1}^m \phi_{il}k_2(l) \right] R \\
&\quad + \left[k_1(i)(\mu(i) + \gamma_1(i)) - k_2(i)\gamma_1(i) - \sum_{l=1}^m \phi_{il}k_1(l) \right] I_1 \\
&\quad + \left[k_1(i)(\mu(i) + \gamma_2(i)) - k_2(i)\gamma_2(i) + \sum_{l=1}^m \phi_{il}k_1(l) \right] I_2.
\end{aligned} \tag{47}$$

From (44), we get that

$$\begin{aligned} \mathcal{L}\mathcal{T} \leq & 2\mu(i) + \gamma_1(i) + \gamma_2(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} - k_1(i)\mu(i) - \sum_{l=1}^m \phi_{il}v(l) \\ & + [\beta_1(i) + \beta_2(i) + k_1(i)\gamma_1(i)]I_1 + [\beta_1(i) + \beta_2(i) + k_1(i)\gamma_2(i)]I_2. \end{aligned} \quad (48)$$

In addition, we also have

$$\begin{aligned} \mathcal{L}\mathcal{T} \leq & 2\mu(i) + \gamma_1(i) + \gamma_2(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} - k_1(i)\mu(i) - \sum_{l=1}^m \phi_{il}v(l) \\ & + [\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{k}_1\tilde{\gamma}_1]I_1 + [\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{k}_1\tilde{\gamma}_2]I_2 \\ \leq & 2\mu(i) + \gamma_1(i) + \gamma_2(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} - k_1(i)\mu(i) - \sum_{l=1}^m \phi_{il}v(l) \phi_{il}v(l) + \overline{D}(I_1 + I_2), \\ & + \overline{D}(I_1 + I_2) \\ := & -R_0(i) - \sum_{l=1}^m \end{aligned} \quad (49)$$

where

$$\overline{D} = \max\{\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{k}_1\tilde{\gamma}_1, \tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{k}_1\tilde{\gamma}_2\}. \quad (50)$$

Since the generator matrix Φ is irreducible, then for $R_0 = (R_0(1), \dots, R_0(m))^T$, there exists $Y = (v(1), \dots, v(m))^T$ solution of the Poisson system:

$$\Phi Y = -R_0 + \left(\sum_{i=1}^m \pi_i R_0(i) \right) \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}, \quad (51)$$

which implies that

$$\mathcal{L}\mathcal{T} \leq -R_0^s + \overline{D}(I_1 + I_2). \quad (52)$$

Integrating (45) from 0 to t and dividing by t on both sides, we obtain

$$\frac{\mathcal{T}(t)}{t} \leq \frac{\mathcal{T}(0)}{t} - R_0^s + \overline{D}\langle I_1 + I_2 \rangle_t - \frac{M_1(t)}{t} - \frac{M_2(t)}{t}, \quad (53)$$

where $M_j(t) = \sigma_j(i)SdB_j(t)$ (for $j = 1, 2$) is a continuous martingale with $M(0) = 0$ and its quadratic variation is as follows:

$$\langle M_j(t), M_j(t) \rangle = \int_0^t (\sigma_j(i)S(u))^2 du \leq (\tilde{\sigma}_j)^2 t. \quad (54)$$

Thus, the strong law of large number for local martingales [27] implies that

$$\lim_{t \rightarrow \infty} \frac{M_j(t)}{t} = 0, \quad \text{a.s.} \quad (55)$$

Since $(S, I_1, I_2, R) \in \mathbb{D}$, we can derive

$$\begin{aligned} \mathcal{T}(S, I_1, I_2, i) = & -\ln I_1 - \ln I_2 - k_1(i)(S + I_1 + I_2) \\ & - k_2(i)R - v(i) \geq -\tilde{k}_1 - \tilde{k}_2 - \tilde{v} := C. \end{aligned} \quad (56)$$

Taking the superior limit on both sides of (45) and combining with (52), (55), and (56), we deduce that

$$\liminf_{t \rightarrow \infty} \langle I_1 + I_2 \rangle_t \geq \frac{R_0^s}{\overline{D}} := I^*, \quad \text{a.s.} \quad (57)$$

Integrating the third equation of (4) and dividing both sides by t , we have

$$\langle R \rangle_t \geq \frac{\min\{\hat{\gamma}_1, \hat{\gamma}_2\}}{(\tilde{\mu} + \tilde{\delta})} \langle I_1 + I_2 \rangle_t - \frac{R(t) - R(0)}{(\tilde{\mu} + \tilde{\delta})t}. \quad (58)$$

Since $\lim_{t \rightarrow \infty} ((R(t) - R(0))/t) = 0$ and using (57), we get

$$\liminf_{t \rightarrow \infty} \langle R \rangle_t \geq \frac{\min\{\hat{\gamma}_1, \hat{\gamma}_2\}}{\tilde{\mu} + \tilde{\delta}} I^*. \quad (59)$$

Furthermore, it follows by $1 - S = I + R$ that

TABLE 1: Parameter values used in numerical simulations.

States	Coefficients							
	μ	β_1	β_2	δ	γ_1	γ_2	σ_1	σ_2
1	0.25	0.5	0.6	0.5	0.4	0.3	0.2	0.1
2	0.3	0.6	0.4	0.4	0.2	0.4	0.12	0.2

FIGURE 2: Sample paths of $S(t)$ in blue, $I_1(t)$ in red, $I_2(t)$ in black, $R(t)$ in green, and $r(t)$ in brown in Example 1.

TABLE 2: Parameter values used in numerical simulations.

States	Coefficients							
	μ	β_1	β_2	δ	γ_1	γ_2	σ_1	σ_2
1	0.01	0.9	0.8	0.4	0.2	0.3	0.6	0.8
2	0.01	0.8	0.8	0.4	0.3	0.2	0.7	0.7

FIGURE 3: Sample paths of $S(t)$ in blue, $I_1(t)$ in red, $I_2(t)$ in black, $R(t)$ in green, and $r(t)$ in brown in Example 1.

$$\liminf_{t \rightarrow \infty} \langle 1 - S \rangle_t \geq \left(\frac{\tilde{\mu} + \tilde{\delta} + \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}}{\tilde{\mu} + \tilde{\delta}} \right) I^*. \quad (60)$$

This completes the proof.

5. Simulation

In the following, we give numerical simulations to illustrate the main theoretical results.

Example 1. We consider the Markov chain $(r(t))_{t \geq 0}$ taking values in $\mathbb{S} = \{1, 2\}$, with generator defined by the following:

$$\Phi = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \quad (61)$$

The parameter values of (4) are presented in Table 1.

Exact computations supply us $\sum_{i=1}^2 \pi_i D_i = -0.0352$ and $\sum_{i=1}^2 \pi_i A_i = -0.0467$. Consequently, by Theorem 2, the disease-free equilibrium E_0 is globally stochastically asymptotically stable on $\mathbb{D}$. Figure 2 confirms this result.

Example 2. In this example, we use the parameter values given in Table 2 and the Markov chain used in Example 1. By simply computing, we obtain $R_0^* > 0$. Theorem 4 shows that the two epidemics persist in the population. Figure 3 confirms this result.

6. Conclusion

This paper investigates a stochastic switched epidemic model with double epidemic diseases and cure rate affected with a composite of environmental perturbations modeled by white and color noises. We established conditions for extinction and persistence of diseases.

The theoretical result given in this paper shows that the stationary distribution of Markov chain $r(t)$ controls the propagation of the epidemic in the population. To know, if the Markov chain $r(t)$ takes the values which lead to find that $\sum_{i=1}^m \pi_i D_i < 0$ and $\sum_{i=1}^m \pi_i A_i < 0$, then the epidemic diseases die out. We have given numerical simulations in order to support and illustrate the main results of this paper.

In this perspective, we prepare to include Lévy noises into epidemic models (4) to describe the effect of sudden and dangerous phenomena such as pandemic, tsunami, and volcano.

Data Availability

The data were curated by the authors and are available upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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