

Research Article

Global Attractor for Second-Order Nonlinear Evolution Differential Inclusions

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In this paper, we address the model of global attractor formulated in the form of evolution differential inclusions with second order in Banach spaces. Firstly, based on the fixed point theorem, the existence result of mild solutions is deduced. Then, by implementing the measure of noncompactness, the existence of global attractor associated with m -semiflow is validated. Finally, a concrete application of the main result is demonstrated to enhance the practical signification.

1. Introduction

Consider a Banach space $(E, \|\cdot\|)$. We attempt to handle the following problem:

$$\begin{cases} u''(t) - Au(t) \in F(t, u(t)), & t > 0, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (1)$$

where A is a closed, linear, and densely defined operator from E to E that generates a strongly continuous cosine family $\{C(t)\}_{t \geq 0}$; $F: [0, +\infty) \times E \rightarrow \mathcal{P}(E)$ is measurable and upper semicontinuous in terms of both first-order and second-order variables, and it also satisfies the natural growth condition.

The nonlinear evolution differential inclusions in Banach spaces have drawn more and more attention of a large number of experts and scholars during the past decades, see [1]. At present, there are substantial outcomes focusing on the existence of solutions for differential equations with second or higher order [2, 3]. For instance, semilinear evolution differential inclusions were explored by Cardinali and Rubbioni in [4]. Liu, Li, and Migórski et al. discussed the problems of one and second evolution differential

variational inequalities, and they attained several existence results about the mild solution, see [5, 6] and references therein.

It is known that global attractor has played an important role in studying the asymptotic feature of solutions for diverse types of differential systems. Melnik, Valero, and Tran et al. were devoted to researching on global attractor of multivalued semiflows and differential inclusions, see [7–13] and references therein. In [10], the authors studied the global attractor of the following functional differential inclusions:

$$\begin{cases} u'(t) - Au(t) \in F(u(t), u_t), & t > 0, \\ u(s) = \varphi(s), & s \in [-h, 0], \end{cases} \quad (2)$$

where u represents the state function and u_t describes the history of the state function, that is, $u_t(s) = u(t+s)$ for $s \in [-h, 0]$. They proved the existences of solutions of system (2) and global attractor by applying noncompactness measure. Global attractor is a gradual state of solution set, and it is also an important research content of a dynamic system. Compared with the first-order differential inclusions, the cosine family generated by operator A is more complex and more meaningful. There are relatively few

studies on this type of second-order differential inclusions, so it is more important to study the attractor of the second-order differential inclusions.

The first aim of the work is to explore the existence of solutions in terms of system (1) in infinite dimensional spaces. Secondly, we are devoted to show the existence of a global attractor for system (1). To our knowledge, there is currently a research gap in the aforementioned problem. However, it is not easy to establish a connection between mild solution and m -semiflows for second-order differential inclusions.

The work is structured as follows. Section 2 presents several relevant definitions involving noncompactness measures and m -semiflows with its global attractor. In Section 3, several sufficient conditions with respect to the mild solutions are invoked. In Section 4, the presence of a global attractor of system (1) is examined in Banach spaces. Finally, a concrete application is presented to emphasize the magnitude of the main results.

2. Preliminaries

This section recalls some fundamental definitions and properties of nonlinear analysis, for more details, see [14]. The symbol \mathbb{R} (resp., \mathbb{R}_+) represents the set of real (resp., positive real) numbers. The strong and weak convergences of $\{u_k\}$ to u are denoted by $u_k \rightarrow u$ and $u_k \rightharpoonup u$, respectively.

Consider a Banach space E whose dual space is denoted by E^* . Let $I = [0, T]$ with $0 < T < \infty$ and let $L^1(I, E)$ be the Banach space of all (Bochner) integrable functions on I endowed with the norm $\|f\| = \int_I \|f(s)\|_E ds$. The notations mentioned as follows will be employed:

$$\begin{aligned} \mathcal{P}_{f(c)}(E) &:= \{Q \subset E: Q \text{ is nonempty, closed (convex)}\}, \\ \mathcal{P}_{wk(c)}(E) &:= \{Q \subset E: Q \text{ is nonempty, weakly compact (convex)}\}. \end{aligned} \quad (3)$$

Definition 1. Suppose that E_1 and E_2 are Banach spaces. A multivalued mapping $F: E_1 \rightarrow \mathcal{P}(E_2)$ is referred to as follows:

- (1) Upper semicontinuous (u.s.c), provided that, for an arbitrary closed set Q in E_2 ,

$$F^{-1}(Q) = \{e \in E_1: F(e) \cap Q \neq \emptyset\}, \quad (4)$$

is closed in E_1 .

- (2) Weakly upper semicontinuous (weakly u.s.c), if for arbitrary weakly closed set Q in E_2 ,

$$F^{-1}(Q) = \{e \in E_1: F(e) \cap Q \neq \emptyset\}, \quad (5)$$

is closed in E_1 .

- (3) Closed, if its graph $\Gamma_F = \{(e_1, e_2): e_2 \in F(e_1)\}$ is closed in $E_1 \times E_2$.
- (4) Compact, if $F(D)$ is a relatively compact subset of E_2 for every bounded set D .

A weakly upper semicontinuous mapping with a weakly compact convex value has good properties. The following conclusion can be found in [15].

Lemma 1. Suppose that E_1 and E_2 are Banach spaces and $F: E_1 \rightarrow \mathcal{P}_{wk(c)}(E_2)$ is a mapping. Then, F is weakly u.s.c if and only if for any sequence $\{u_n\}$ of E_1 satisfying $u_n \rightarrow u_0 \in E_1$ and $v_n \in F(u_n)$, it implies $v_n \rightarrow v_0 \in F(u_0)$ up to a subsequence.

In the subsequent arguments, we consider the presence of mild solutions about system (1) from the viewpoint of evolution differential inclusions with second order. We only need to deal with the following differential equation:

$$\begin{cases} u''(t) = Au(t) + f(t), & t > 0, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases} \quad (6)$$

where $f(t) \in F(t, u(t))$.

Definition 2. Assume that $\mathcal{L}(E)$ is the space of all bounded linear operators defined from E to E^* . A mapping $C: \mathbb{R} \rightarrow \mathcal{L}(E)$ is called a strongly continuous cosine family, provided that $C(0) = I_0$ (I_0 is an identity operator) and $C(r_1 + r_2) + C(r_1 - r_2) = 2C(r_1)C(r_2)$ for every $r_1, r_2 \in \mathbb{R}$.

Given a strongly continuous cosine family $C(t)$ with $t \in \mathbb{R}$. Then, its generator is an operator $A: E \rightarrow E$ formulated as

$$Au = \frac{d^2}{dt^2} C(t)u|_{t=0}, \quad \forall u \in D(A), \quad (7)$$

where the domain $D(A) = \{e \in E: C(t)e \in C^2(\mathbb{R}, E)\}$. By the definition of generator A , it is known that A is closed, linear, and densely defined over E , see [2]. The sine operator $S: \mathbb{R} \rightarrow \mathcal{L}(E)$, which is closely relevant to the cosine operator C , is expressed by

$$S(t)u = \int_0^t C(s)uds, \quad \forall t \in \mathbb{R}, u \in E. \quad (8)$$

Definition 3. A mapping $u \in C(I, E)$ is referred to as a mild solution of system (6), provided that there is an $f \in L^1(I, E)$ with $f(t) \in F(t, u(t))$ and

$$u(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)f(s)ds, \quad \forall t \in I. \quad (9)$$

Given an arbitrary bounded set B in E , the Hausdorff measure of noncompactness (MNC, for short) is given by

$$\chi(B) = \inf\{\varepsilon > 0: B \text{ has a finite } \varepsilon\text{-net}\}. \quad (10)$$

Let $\Omega \subset L^1(I, E)$. We say that Ω is integrably bounded, if any element f of Ω satisfies that $\|f(t)\| \leq g(t)$ for a.e. $t \in I$ with some $g \in L^1(I, \mathbb{R}_+)$. An integrably bounded Ω is said to be semicompact, provided that $\{\Omega(t)\}$ is relatively compact for a.e. $t \in I$.

The norm of product spaces $E \times E$ is represented by

$$\|(u, v)\| = \max\{\|u\|_E, \|v\|_E\}, \quad \forall (u, v) \in E \times E. \quad (11)$$

It is straightforward to see that, for any bounded set $B_1 \times B_2 \subset E \times E$,

$$\tilde{\chi}(B_1 \times B_2) = \max\{\chi(B_1), \chi(B_2)\}, \quad (12)$$

where $\tilde{\chi}$ is measure of noncompactness on $E \times E$.

Definition 4. Presume that χ is a Hausdorff measure of noncompactness and $0 \leq \kappa < 1$. A multivalued mapping $F: X \rightarrow \mathcal{P}_\kappa(E)$ is referred to as κ -condensing, if

$$\chi(F(B)) \leq \kappa \cdot \chi(B), \quad \forall B \subset E. \quad (13)$$

The open (resp., closed) ball with centrum 0 and radius $r > 0$ is denoted by $B_r(0)$ (resp., $\overline{B_r(0)}$). The fixed point theorem used in our main results is introduced in [16] as follows.

Lemma 2. Define two operators $\mathcal{A}: \overline{B_r(0)} \rightarrow X$ and $\mathcal{B}: \overline{B_r(0)} \rightarrow \mathcal{P}_{\kappa c}(E)$ by coupling with the following properties:

- (1) \mathcal{A} is a contraction mapping with a single value and a coefficient $\lambda < 1/2$
- (2) \mathcal{B} is compact and u.s.c

Then, each of the following conclusions holds:

- (i) There exists an element $\omega \in \overline{B_r(0)} \setminus B_r(0)$ such that $\rho\omega \in \mathcal{A}\omega + \mathcal{B}\omega$ for some $\rho > 1$
- (ii) The operator inclusion $u \in \mathcal{A}u + \mathcal{B}u$ admits a solution in $\overline{B_r(0)}$

Next, we introduce related theories involving global attractors and m -semiflow, see [8, 10, 11]. Suppose that Γ is a nontrivial subgroup of the additive group of \mathbb{R} and $\Gamma_+ = \Gamma \cap \mathbb{R}_+$.

Definition 5. A mapping $\Phi: \Gamma \times E \rightarrow \mathcal{P}(E)$ is said to be a multivalued flow (m -flow, in short) if the following conditions are guaranteed:

- (1) $\Phi(0, u) = u$ for every $u \in E$
- (2) $\Phi(t + s, u) \subset \Phi(t, \Phi(s, u))$ for any pair $t, s \in \Gamma$ and $u \in E$ with $\Phi(\tau, B) = \bigcup_{u \in B} \Phi(\tau, u)$

Remark 1. A mapping $\Phi: \Gamma_+ \times E \rightarrow \mathcal{P}(E)$ is referred to as an m -semiflow, if conditions (1) and (2) are satisfied for any $t_1, t_2 \in \Gamma_+$.

A mapping Φ is said to be a strict m -semiflow if for every $t, s \in \Gamma_+$, $u \in E$, we have $\Phi(t + s, u) = \Phi(t, \Phi(s, u))$. A map $u(\cdot): \Gamma_+ \rightarrow E$ is called a trajectory of m -semiflow with respect to the initial condition $u(0) = u_0$, if $u(t + s) \in \Phi(t, u(s))$ for all $t, s \in \Gamma_+$.

Definition 6. Let Φ be an m -semiflow. A bounded set $B_0 \subset E$ is referred to as an absorbing set for Φ , provided that, for every bounded $B_1 \subset E$, $\gamma_{T(B_1)}^{(+)} \subset B_0$ holds for some $T = T(B_1) \geq 0$, where $\gamma_{T(B_1)}^{(+)} = \bigcup_{t \geq T(B_1)} \Phi(t, B_1)$.

For a given $e \in E$ and $B_1, B_2 \subset E$, we define $d(e, B_i) = \inf_{x \in B_i} \{d(e, x)\}$. The metric excess of B_1 over B_2 is given as $\eta(B_1, B_2) = \sup_{e \in B_1} \{d(e, B_2)\}$. It is well known that the Hausdorff distance between B_1 and B_2 can be represented in the following way:

$$d_H(B_1, B_2) = \max\{\eta(B_1, B_2), \eta(B_2, B_1)\}. \quad (14)$$

Definition 7. Let Φ be an m -semiflow. A set \mathfrak{R} is said to be a global attractor of Φ , provided that the following conditions are satisfied:

- (1) \mathfrak{R} attracts any bounded set B in E , namely, $\eta(\Phi(t, B), \mathfrak{R}) \rightarrow 0$ as $t \rightarrow +\infty$
- (2) \mathfrak{R} is negatively semi-invariant, namely, $\mathfrak{R} \subset \Phi(t, \mathfrak{R})$, for all $t \in \Gamma_+$

We now give an existence result about a global attractor for a given m -semiflow Φ , see [11].

Lemma 3. Assume that a mapping $\Phi: \Gamma_+ \times E \rightarrow \mathcal{P}(E)$ satisfies the following conditions:

- (i) For every $t \in \Gamma_+$, $u \rightarrow \Phi(t, u)$ is u.s.c. with closed values
- (ii) Φ has an absorbing set
- (iii) Φ is asymptotically upper semicompact, i.e., for every bounded $B \subset E$ with bounded $\gamma_{T(B)}^{(+)}$ for some positive value $T(B)$, any sequence $\{\xi_n \in E: \xi_n \in \Phi(t_n, B)\}$ is relatively compact as $t_n \rightarrow +\infty$

Then, Φ admits a compact global attractor \mathfrak{R} in E . In particular, \mathfrak{R} is invariant if it is a strict m -semiflow, i.e., $\mathfrak{R} = \Phi(t, \mathfrak{R}), \forall t \in \Gamma_+$.

3. The Existence Result of Mild Solutions

This section is based on the idea of [3] for nonlinear evolution hemivariational inequalities with second order, which is adapted for evolution differential inclusions in our context. We will handle problem (6) under the following basic assumptions:

- (A) The operator $A: D(A) \subset E \rightarrow E$ is a generator of a cosine operator $\{C(t)\}_{t \in \mathbb{R}}$, and $\|C(t)\|, \|S(t)\| \leq M_A e^{w|t|}$ for some $M_A \geq 1$ and $w \geq 0$. Moreover, $S(t)$ is compact for arbitrary $t \in I$.
- (F) The multivalued mapping $F: I \times E \rightarrow \mathcal{P}_{\kappa c}(E)$ is u.s.c., and there are $a \in L^2(I, \mathbb{R}_+)$ and $b > 0$ with

$$\|F(t, u)\| = \sup\{\|\xi\|: \xi \in F(t, u)\} \leq a(t) + b\|u\| \quad \text{for every } (t, u) \in I \times E. \quad (15)$$

By combining the uniform boundedness principle with the assumption (A), it is inferred that $C(t)$ (resp., $S(t)$) is uniformly bounded on I subject to some upper-bound M_C (resp., M_S). We set $M = \max\{M_C, M_S\}$.

In the sequel, let a multivalued operator $\mathcal{P}_F: C(I, E) \rightarrow \mathcal{P}(L^2(I, E))$ be

$$\mathcal{P}_F(u) = \left\{ f \in L^2(I, E) : f(t) \in F(t, u(t)) \text{ for a.e. } t \in I \right\}. \quad (16)$$

It is well known that $F(t, u)$ has a measurable selection for every $u \in C(I, E)$ and a.e. $t \in I$ under the assumption F , which means that \mathcal{P}_F is nonempty and well-defined.

We also construct a multivalued operator $\mathcal{B} : C(I, E) \rightarrow C(I, E)$ by

$$\mathcal{B}(u) = \left\{ \phi \in C(I, E) : \phi(t) = \int_0^t S(t-s)f(s)ds, \quad f \in \mathcal{P}_F(u) \right\}. \quad (17)$$

Set $\mathcal{W} : L^2(I, E) \rightarrow C(I, E)$ by

$$\mathcal{W}(f)(t) = \int_0^t S(t-s)f(s)ds. \quad (18)$$

Lemma 4. *Assume that (A) and (F) hold. Then, for an arbitrary sequence $\{f_n\}_{n=1}^\infty \subset L^2(I, E)$ with semicompactness, the sequence $\{\mathcal{W}f_n\}_{n=1}^\infty$ is relatively compact. In particular, if $f_n \rightharpoonup f_0$, then $\mathcal{W}f_n \rightarrow \mathcal{W}f_0$.*

Proof. It is shown by analogizing the proof of Theorem 2 of [4].

Aiming to validate the presence of solutions for system (1), we now clarify a property of the multivalued operator \mathcal{B} , which is inspired by Lemma 5 of [3]. \square

Lemma 5. *Assume that (A) and (F) hold. Then, the multivalued mapping $\mathcal{B} : C(I, E) \rightarrow C(I, E)$ is compact and u.s.c. with convex and compact values.*

Proof. Note that the operator \mathcal{B} has convex values for any $u \in C(I, E)$ due to the convexity of \mathcal{P}_F . The verification process will be divided into four steps as follows:

Step 1: \mathcal{B} is a bounded operator on $C(I, E)$. Given $u \in B_r(0)$ and $\phi \in \mathcal{B}(u)$, from (F) and Hölder inequality, it follows that

$$\begin{aligned} \|\phi(t)\|_E &\leq \int_0^t \|S(t-s)f(s)\|_E ds \leq M \int_0^t a(s) + b\|u(s)\|_E ds \\ &\leq M \left(\|a\|_{L^2(I, \mathbb{R}_+)} \sqrt{T} + brT \right). \end{aligned} \quad (19)$$

Hence, $\mathcal{B}(B_r(0))$ is a bounded subset of $C(I, E)$, i.e., the operator \mathcal{B} is a bounded operator.

Step 2: $\{\mathcal{B}(u) : u \in B_r(0)\}$ is equicontinuous. Presume that $0 < t_1 < t_2 \leq T$, $\delta > 0$ is sufficiently small, then

$$\begin{aligned} \|\phi(t_2) - \phi(t_1)\|_E &= \left\| \int_0^{t_2} S(t_2-s)f(s)ds - \int_0^{t_1} S(t_1-s)f(s)ds \right\|_E \\ &\leq \int_0^{t_1} \| [S(t_2-s) - S(t_1-s)]f(s) \|_E ds + \int_{t_1}^{t_2} \| S(t_2-s)f(s) \|_E ds \\ &\leq \int_0^{t_1} \| [S(t_2-s) - S(t_1-s)] \|_E \| [a(s) + br] \|_E ds + M \int_{t_1}^{t_2} [a(s) + br] ds \\ &\leq \sup_{s \in [0, t_1 - \delta]} \| [S(t_2-s) - S(t_1-s)] \|_E \left(\|a\|_{L^2(I, \mathbb{R}_+)} \sqrt{T} + brT \right) \\ &\quad + M \left[\|a\|_{L^2(I, \mathbb{R}_+)} (2\sqrt{\delta} + \sqrt{t_2 - t_1}) + br(2\delta + t_2 - t_1) \right]. \end{aligned} \quad (20)$$

From the continuity of the sine operator $S(t)$, we get easily that the above inequality is independent of u and it converges to 0 as $t_2 \rightarrow t_1$. This implies the equicontinuity of $\{\mathcal{B}(u) : u \in B_r(0)\}$ in $C(I, E)$.

Step 3: $\Pi(t) = \{\phi(t) : \phi \in \mathcal{B}(B_r(0))\}$ is relatively compact in E for a.e. $t \in I$. Firstly, the compactness of $\Pi(0) = \{0\}$ is trivially checked. Then, fix $t \in (0, T]$. For any $u \in B_r(0)$ and $\phi \in \mathcal{B}(B_r(0))$, there exists $f \in \mathcal{P}_F(u)$ with

$$\phi(t) = \int_0^t S(t-s)f(s)ds. \quad (21)$$

For every $\varepsilon \in (0, t)$, we define

$$\phi^\varepsilon(t) = \int_0^{t-\varepsilon} S(t-s)f(s)ds = S(\varepsilon) \int_0^{t-\varepsilon} S(t-s-\varepsilon)f(s)ds. \quad (22)$$

It can be seen that the compactness of $S(\cdot)$ with the boundedness of $\int_0^{t-\varepsilon} S(t-s-\varepsilon)f(s)ds$ together assures that $\Pi_\varepsilon(t) = \{\phi^\varepsilon(t) : \phi \in \mathcal{B}(B_r(0))\}$ is relatively compact in E . Furthermore, we have

$$\begin{aligned} \|\phi(t) - \phi^\varepsilon(t)\|_E &\leq M \int_{t-\varepsilon}^t a(s) + b\|u(s)\|_E ds \\ &\leq M \left(\|a\|_{L^2(I, \mathbb{R}_+)} \sqrt{\varepsilon} + br\varepsilon \right). \end{aligned} \quad (23)$$

Thereby, $\Pi(t) (t > 0)$ is totally bounded, and thus it is relatively compact in E .

To this end, by applying Ascoli-Arzelà theorem and taking account of Steps 1-2, it can be verified that the mapping \mathcal{B} is compact.

Step 4: \mathcal{B} has a closed graph. In fact, let $\{u_n\}, \{\phi_n\} \subset C(I, E)$, $u_n \rightarrow \bar{u}$, $\phi_n \in \mathcal{W} \circ \mathcal{P}_F(u_n)$, and $\phi_n \rightarrow \bar{\phi}$. Choose any sequence $f_n \subset L^2(I, E)$ such that $f_n \in \mathcal{P}_F(u_n)$, $\phi_n = \mathcal{W}(f_n)$. From (F), we conclude that the sequence f_n is integrable bounded. Hence, f_n is weakly compact in $L^2(I, E)$. It is rational to presume that $f_n \rightharpoonup \bar{f}$. From Lemma 4, we have

$$\phi_n = \mathcal{W}(f_n) \longrightarrow \mathcal{W}(\bar{f}) = \bar{\phi}. \quad (24)$$

Besides, $\bar{f} \in \mathcal{P}_F(\bar{u})$ due to Lemma 1. Therefore, \mathcal{B} has a closed graph. \square

$$\mathcal{F}(u) = \left\{ \varphi \in C(I, E) : \varphi(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)f(s)ds, \quad f \in \mathcal{P}_F(u) \right\}. \quad (25)$$

For convenience, we write $\mathcal{F} = \mathcal{A} + \mathcal{B}$, where \mathcal{B} is given by (17) and \mathcal{A} is defined as

$$\mathcal{A}(u) = C(t)u_0 + S(t)v_0, \quad t \in I. \quad (26)$$

Obviously, u is a fixed point of \mathcal{F} if only and if it is a mild solution of system (6). According to Lemma 2, we need to prove that (i) of Lemma 2 is not satisfied.

In fact, suppose that $\rho u \in \mathcal{A}u + \mathcal{B}u$ with $\rho > 1$ and there exists $\eta \in \mathcal{P}_F(u)$ with

$$\rho u(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)\eta(s)ds. \quad (27)$$

This yields that

$$\begin{aligned} \|u(t)\|_E &\leq \|C(t)u_0\|_E + \|S(t)v_0\|_E + \left\| \int_0^t S(t-s)\eta(s)ds \right\|_E \\ &\leq M\|u_0\|_E + M\|v_0\|_E + M \int_0^t a(s) + b\|u(s)\|_E ds \\ &\leq d + Mb \int_0^t \|u(s)\|_E ds, \end{aligned} \quad (28)$$

where $d = M\|u_0\|_E + M\|v_0\|_E + M\|a\|_{L^2(I, \mathbb{R}_+)}\sqrt{T}$. Employing Gronwall inequality, one gets

$$\Sigma(\xi) = \{u \in C([0, +\infty), E) : u \text{ is a mild solution of (1.1) on } [0, T], \xi = (u_0, v_0)\}, \quad (32)$$

where $u(0) = u_0, u'(0) = v_0$. Obviously, $\pi_T \circ \Sigma(\xi) = \text{Fix}(\mathcal{F})$ and $\pi_T \circ \Sigma(\xi) \subset S(\cdot)\xi + \mathcal{W} \circ \mathcal{P}_F(\pi_T \circ \Sigma(\xi))$. Here, $\text{Fix}(\mathcal{F})$ stands for the solution set of the fixed points of system (6). \square

Lemma 6. *Let $U = E \times E$ and all the assumptions of Theorem 1 be fulfilled. Then, $\pi_T \circ \Sigma(\xi_n)$ is relatively compact in $C([0, T], E)$ for arbitrary sequence $\xi_n \longrightarrow \xi$ in U . Especially, for every $\xi \in U$, $\pi_T \circ \Sigma(\xi)$ is compact in $C([0, T], E)$.*

Proof. It is verified by analogizing the proof of Lemma 3.7 of [8].

The m -semiflow associated with system (1) is given as follows:

Theorem 1. *Presume that (A) and (F) hold. Then, there is at least one mild solution related to system (6) for each pair of initial values $u_0, v_0 \in E$.*

Proof. Construct a multivalued mapping $\mathcal{F} : C(I, E) \longrightarrow C(I, E)$ by

$$\|u(t)\|_E \leq de^{Mbt}. \quad (29)$$

This implies

$$\|u\|_{C(I, E)} \leq de^{MbT} =: r. \quad (30)$$

Furthermore, we write

$$Q_r = \{u \in C(I, E) : \|u\|_{C(I, E)} < r + 1\}. \quad (31)$$

Obviously, Q_r is open in $C(I, E)$. Utilizing Lemma 5, we know that $\mathcal{B} : \overline{Q}_r \longrightarrow \mathcal{P}_{kc}(E)$ is compact and u.s.c. We also see easily that $\mathcal{A} : \overline{Q}_r \longrightarrow E$ is a contraction mapping with single value and coefficient $\lambda < 1/2$. Therefore, according to the choice of \overline{Q}_r , none of $u \in E$ with $\|u\| = r$ ensures that $\lambda u \in \mathcal{A}u + \mathcal{B}u$ with some $\rho > 1$.

Applying Lemma 3, one knows that \mathcal{F} has a fixed point due to all the conditions being fulfilled. Hence, there is a mild solution for system (6).

Next, let us discuss some features of the solution set. For every $T > 0$, we say that π_T is a truncation operator from $C([0, +\infty), E)$ to $C([0, T], E)$ if for any $z \in C([0, +\infty), E)$, we have $\pi_T(z) \in C([0, T], E)$. Put

$$\Phi : \mathbb{R}_+ \times U \longrightarrow \mathcal{P}(U)$$

$$\Phi(t, \xi) = \{u(t), u(t) : x \text{ is a mild solution of (1.1), } \xi = (u_0, v_0)\}. \quad (33)$$

By applying a similar discussion in [11], we can obtain that Φ is a strict m -semiflow, i.e., for arbitrary $t, s \in \mathbb{R}_+$ and $\xi \in U$, it holds that

$$\Phi(t + s, \xi) = \Phi(t, \Phi(s, \xi)). \quad (34)$$

\square

Lemma 7. *Assume that all the hypotheses of Theorem 1 are fulfilled. For all $t > 0$, $\Phi(t, \cdot)$ is u.s.c. with compact values.*

Proof. It is verified by analogizing the proof of Lemma 3.8 of [8]. \square

4. The Existence Result of Global Attractors

In the sequel, we will present a severe validation for the existence of a global attractor of m -semiflow Φ induced by system (1). The conditions (A) and (F) are replaced by (A') and (F') as follows, respectively:

(A') The cosine and sine functions are exponentially stable with exponent $\alpha > 0$, that is, there exist $M_0 \geq 1$ and $\alpha \geq 0$ with

$$\|C(t)\|_{\mathcal{L}(E)} \leq M_0 e^{-\alpha t}, \|S(t)\|_{\mathcal{L}(E)} \leq M_0 e^{-\alpha t}, \quad \forall t > 0. \quad (35)$$

(F') The multivalued mapping $F: \mathbb{R}_+ \times E \rightarrow \mathcal{P}_{kc}(E)$ is u.s.c. and satisfies the following conditions:

- (1) There exists $a_1 \in L^2(\mathbb{R}_+, \mathbb{R}_+)$ and $b_1 > 0$, ensuring that

$$\|F(t, x)\| = \sup\{\|\xi\|: \xi \in F(t, u)\} \leq a_1(t) + b_1\|u\|, \quad \forall (t, u) \in \mathbb{R} \times E. \quad (36)$$

- (2) There exists $q \in L^2(\mathbb{R}_+, \mathbb{R}_+)$ ensuring that, for every bounded Ω in E ,

$$\chi(F(t, \Omega)) \leq q(t)\chi(\Omega) \quad \text{for almost all } t \in \mathbb{R}_+. \quad (37)$$

Given $T > 0$, for convenience, the translation operator $\Phi(T, \cdot)$ is defined by Φ_T . The following condensing property of Φ_T is an essential result to reveal that Φ is asymptotically upper semicompact.

Lemma 8. *Suppose that (A') and (F') are satisfied. Then, there are $\zeta \in [0, 1)$ and $T_0 > 0$, guaranteeing that, for all $T \geq T_0$ and all bounded $B \subset U$, it is inferred that*

$$\tilde{\chi}(\Phi_T(B)) \leq \zeta \cdot \tilde{\chi}(B), \quad (38)$$

whenever

$$\alpha - 4M_0\|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} > 0. \quad (39)$$

Proof. Let $U = E \times E$, $B = B_1 \times B_2$ be bounded in U . Denote $D = \Sigma(B)$, we obtain

$$D(t) \subset C(t)B_1 + S(t)B_2 + \int_0^t S(t-s)\mathcal{P}_F(D)(s)ds, \quad t > 0. \quad (40)$$

Therefore, we get

$$\begin{aligned} \tilde{\chi}(\Phi_t(B)) &= \chi(D(t)) \leq M_0 e^{-\alpha t} \chi(B_1) + M_0 e^{-\alpha t} \chi(B_2) \\ &\quad + \chi\left(\int_0^t S(t-s)\mathcal{P}_F(D)(s)ds\right) \\ &\leq 2M_0 e^{-\alpha t} \tilde{\chi}(B) + 4M_0 \int_0^t e^{-\alpha(t-s)} \chi(\mathcal{P}_F(D)(s))ds \\ &\leq 2M_0 e^{-\alpha t} \tilde{\chi}(B) + 4M_0 e^{-\alpha t} \int_0^t e^{\alpha s} q(s) \chi(D(s))ds \\ &\leq 2M_0 e^{-\alpha t} \tilde{\chi}(B) + 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} e^{-\alpha t} \\ &\quad \int_0^t e^{\alpha s} \chi(D(s))ds. \end{aligned} \quad (41)$$

Hence,

$$e^{\alpha t} \chi(D(t)) \leq 2M_0 \tilde{\chi}(B) + 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} \int_0^t e^{\alpha s} \chi(D(s))ds. \quad (42)$$

Applying the Gronwall inequality, we have

$$e^{\alpha t} \chi(D(t)) \leq 2M_0 e^{4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} t} \tilde{\chi}(B). \quad (43)$$

Equivalently,

$$\chi(D(t)) \leq 2M_0 e^{-(\alpha - 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)})t} \tilde{\chi}(B). \quad (44)$$

Note that $\tilde{\chi}(\Phi_t(B)) = \chi(D(t))$, we get

$$\tilde{\chi}(\Phi_T(B)) \leq \zeta_t \cdot \tilde{\chi}(B), \quad (45)$$

where $\zeta_t = 2M_0 e^{-(\alpha - 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)})t}$.

Taking $T_0 > \ln 2M_0 / \alpha - 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)}$ and $\zeta = \zeta_{T_0}$, we conclude the proof. \square

Lemma 9. *Presume that (A') and (F') are fulfilled. Then, Φ is asymptotically upper semicompact.*

Proof. Let $U = E \times E$ and $B = B_1 \times B_2$ be a bounded subset of U and $\Delta(B)$ be a collection consisted of sequences ξ_n with $\xi_n \in \Phi(t_n, B)$, $t_n \rightarrow \infty$. Put

$$\eta = \sup\{\tilde{\chi}(Q): Q \in \Delta(B)\}. \quad (46)$$

We claim that $\eta = 0$ by using reduction. Assume that there exists $Q_\theta = \{\xi_n \in \Delta(B)\}$ such that

$$\chi(Q_\theta) > \eta - \theta, \quad \forall \theta \in (0, (1 - \zeta)\eta), \quad (47)$$

where ζ is the same as in Lemma 8. Choose $T > 0$ as described in Lemma 8; then, there exists $m_n \in \mathbb{N}$ such that $t_n = m_n T + r_n$, $r_n \in [0, T)$ for any $t_n \in (T, \infty)$. Let $\tau_n = (m_n - 1)T + r_n$. Then,

$$\xi_n \in \Phi(t_n, B) = \Phi(T + \tau_n, B) = \Phi_T(\Phi(\tau_n, B)). \quad (48)$$

By taking $\sigma_n \in \Phi(\tau_n, B)$ such that $\xi_n \in \Phi(T, \sigma_n)$, we obtain

$$\tilde{\chi}(Q_\theta) = \tilde{\chi}(\xi_n) \leq \tilde{\chi}(\Phi_t(\sigma_n)) \leq \zeta \tilde{\chi}(\sigma_n) \leq \zeta \eta < \eta - \theta. \quad (49)$$

This is a contradiction. \square

Lemma 10. *Presume that (A') and (F') are fulfilled. If $\alpha > bM_0$, then an absorbing set exists for m -semiflow Φ .*

Proof. Let $t > 0$ and $B = B_1 \times B_2$ be bounded in $U = E \times E$. Then, for each $\xi = (u_0, v_0) \in U$ and $u \in \Sigma(\xi)$, there is $M_B > 0$ with

$$\begin{cases} u(t) = C(t)u_0 + S(t)v_0 + \int_0^t S(t-s)f(s)ds, \\ \|u_0\| \leq M_B, \|v_0\| \leq M_B, \end{cases} \quad (50)$$

for some $f \in \mathcal{P}_F(u)$. The combination of Hölder inequality, (A') and the part (1) of (F') gives

$$\begin{aligned} \|u(t)\|_X &\leq 2M_B M_0 e^{-\alpha t} + M_0 \int_0^t e^{-\alpha(t-s)} [a_1(s) + b_1 \|u(s)\|_E] ds \\ &\leq 2M_B M_0 e^{-\alpha t} + M_0 \int_0^t e^{-\alpha(t-s)} a_1(s) ds + b_1 M_0 \int_0^t e^{-\alpha(t-s)} \|u(s)\|_E ds. \\ &\leq 2M_B M_0 e^{-\alpha t} + M_0 e^{-\alpha t} \left[\|a_1\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} \frac{1}{\sqrt{2\alpha}} e^{\alpha t} \right] + b_1 M_0 \int_0^t e^{-\alpha(t-s)} \|u(s)\|_E ds \\ &\leq 2M_B M_0 e^{-\alpha t} + M_0 \left[\|a_1\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} \frac{1}{\sqrt{2\alpha}} \right] + b_1 M_0 \int_0^t e^{-\alpha(t-s)} \|u(s)\|_E ds. \end{aligned} \quad (51)$$

Putting $c_1 = M_0 [\|a_1\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)} 1/\sqrt{2\alpha}]$ and $c_2 = b_1 M_0$, we obtain

$$e^{\alpha t} \|u(t)\|_E \leq 2M_B M_0 + c_1 e^{\alpha t} + c_2 \int_0^t e^{\alpha s} \|u(s)\|_X ds. \quad (52)$$

The employment of Gronwall inequality yields

$$\begin{aligned} e^{\alpha t} \|u(t)\|_E &\leq 2M_B M_0 + c_1 e^{\alpha t} + c_2 \int_0^t [2M_B M_0 + c_1 e^{\alpha s}] e^{c_2(t-s)} ds \\ &\leq 2M_B M_0 + c_1 e^{\alpha t} + c_2 \left[\int_0^t 2M_B M_0 e^{c_2(t-s)} ds + \int_0^t c_1 e^{\alpha s + c_2 t - c_2 s} ds \right] \\ &\leq 2M_B M_0 + c_1 e^{\alpha t} + 2M_B M_0 (e^{c_2} - 1) + \frac{c_1 c_2}{\alpha - c_2} [e^{\alpha t} - e^{c_2 t}]. \end{aligned} \quad (53)$$

This implies

$$\begin{aligned} \|u(t)\|_E &\leq 2M_B M_0 e^{-\alpha t} + c_1 + 2M_B M_0 e^{(c_2 - \alpha)t} \\ &\quad + \frac{c_1 c_2}{\alpha - c_2} \left[1 - e^{(c_2 - \alpha)t} \right]. \end{aligned} \quad (54)$$

The above inequality holds subject to the ball with the radius R as

$$R = c_1 + \frac{c_1 c_2}{\alpha - c_2} + 1. \quad (55)$$

Moreover, it is actually an absorbing set in terms of Φ . \square

Theorem 2. *Assume that (A') and (F') are fulfilled. Then, the m -semiflow Φ induced by system (1) has a compact global attractor, if the following holds:*

$$\min \left\{ \alpha - 4M_0 \|q\|_{L^2(\mathbb{R}_+, \mathbb{R}_+)}, \alpha - b_1 M_0 \right\} > 0. \quad (56)$$

Proof. By Lemmas 7, 9, and 10, we obtain that all the conditions of Lemma 3 are satisfied. Therefore, the m -semiflow Φ admits a compact global attractor. \square

5. An Illustrative Example

In this section, a concrete example is illustrated for the application of evolution partial differential equations.

$$\begin{cases} Z_{tt}(t, u) - \Delta Z(t, u) = f(t, u), & t > 0, u \in \Omega, \\ f(t, u) = \lambda f_1(t, Z(t, u)) + (1 - \lambda) f_2(t, Z(t, u)), & \lambda \in [0, 1], t > 0, u \in \Omega, \\ Z(t, u) = 0, & t \geq 0, u \in \partial\Omega, \\ Z(0, u) = Z_0(u), & Z_t(0, u) = Z_1(u), \end{cases} \quad (57)$$

where $f_1, f_2: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. This system describes a classical wave equation with a free bounded issue [1].

Put $E = L^2(\Omega)$. The multivalued mapping $F: \mathbb{R}_+ \times E \rightarrow \mathcal{P}(E)$ is described as

$$F(\bar{Z})(t) = \{\lambda f_1(t, \bar{Z}(t)) + (1 - \lambda) f_2(t, \bar{Z}(t)): \lambda \in [0, 1]\}. \quad (58)$$

Then, (57) can be reformulated as

$$\begin{cases} Z''(t) - AZ(t) \in F(t, Z(t)), & t \geq 0, \\ Z(0) = Z_0, & Z_t(0) = Z_1, \end{cases} \quad (59)$$

in which $A = \Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and $Z(t) \in E$.

Suppose that there exists $a_1, a_2 \in L^2(\Omega)$ and positive constants b_1 and b_2 such that

$$\begin{aligned} |f_1(t, p)| &\leq a_1(t) + b_1|p|, \\ |f_2(t, p)| &\leq a_2(t) + b_2|p|, \quad \forall x \in \Omega, p \in \mathbb{R}. \end{aligned} \quad (60)$$

Note that the mapping F has convex, closed, and compact values. In addition,

$$\|F(t, \bar{Z})\| \leq \max\{a_1(t), a_2(t)\} + \max\{b_1, b_2\} \|\bar{Z}\|_X. \quad (61)$$

By the continuity of f_1 and f_2 , we can obtain that there is a closed graph for F . Furthermore, by exploiting Lebesgue dominated convergence theorem, one can verify the quasi-compactness of F . Therefore, F is u.s.c., and $(F')(1)$ is testified. If $S(\cdot)$ is exponentially stable and compact, then (A') and $(F')(2)$ is also satisfied. This indicates the presence of a global attractor, which belongs to the m -semiflow induced by (57).

Data Availability

The data used to support the findings of this study are available within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Take an open set Ω which is bounded in \mathbb{R}^n and denote its boundary by $\partial\Omega$. The following system is worthy of survey:

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