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Research Article

Limit Cycles of a Class of Perturbed Differential Systems via the First-Order Averaging Method

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By means of the averaging method of the first order, we introduce the maximum number of limit cycles which can be bifurcated from the periodic orbits of a Hamiltonian system. Besides, the perturbation has been used for a particular class of the polynomial differential systems.

1. Introduction

As we know that the second part of the 16 Hilbert problem ([1, 2]) wants to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree, we refer the readers to see [3, 4]. The limit cycles problem and the center problem are fastened on specified classes of systems. For instance, we refer to Kukles systems (see, for example, [5–9]) and Liénard systems given by

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -u - f(u), \end{cases} \tag{1}$$

where $f(u)$ is a polynomial in the variable u of degree m. For these systems, in 1977, Lins et al. [10] presented the conjecture that if $f(u)$ has degree $m \geq 1$, then system (1) has at most $[m/2]$ limit cycles where $[\cdot]$ denotes the integer part function. They prove this conjecture for $m = 1, 2$. The conjecture for $m = 3$ has been proved recently by Chengzi and Llibre in [11].

Suppose that polynomials $f(u)$ and $g(u)$ are in the variable u of degrees n and m, respectively; then, Llibre et al. [12] established the following generalized Liénard polynomial differential system:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -g(u) - f(u)v, \end{cases} \tag{2}$$

where $[n + m - 1/2]$ is limit cycles.

Llibre and Makhlouf [13] studied the number of limit cycles of the following generalized Liénard polynomial differential system:

$$\begin{cases} \dot{u} = -v^{2p-1}, \\ \dot{v} = u^{2q-1} - \epsilon f(u) v^{2n-1}, \end{cases} \tag{3}$$

where $f(u)$ is a polynomial of degree m, p, q, and n are positive integers, and ϵ is a small parameter.

They introduced the following theorem.

Theorem 1 (see [13]). *Let m be the degree of the polynomial $f(u)$, and $\epsilon \neq 0$; then, the polynomial differential system (3) can have at least $[m/2]$ limit cycles.*

Also, Jianyuan and Shuliang [14] investigated the maximum number of limit cycles of the following polynomial differential system:

$$\begin{cases} \dot{u} = -v^{2p-1}, \\ \dot{v} = u^{2mp-1} + \epsilon\left(pu^{2mp} + mpv^{2p}\right)(g(u,v) - A), \end{cases} \quad (4)$$

where ϵ is a small parameter, $A > 0$, $p, m \in \mathbb{N}$, and $g(u,v)$ is a polynomial of degree n with $g(0,0) = 0$.

In this manuscript, we discuss the maximum number of limit cycles of the following polynomial differential system:

$$\begin{cases} \dot{u} = -v^{2p-1} - \epsilon puf(u,v), \\ \dot{v} = u^{2q-1} - \epsilon qvf(u,v), \end{cases} \quad (5)$$

where $f(u,v)$ is a polynomial of degree m, ϵ is a small parameter, and $p, q \in \mathbb{N}$. Clearly, system (5) with $\epsilon = 0$ is a Hamiltonian system with Hamiltonian

$$H(u,v) = \frac{1}{2q}u^{2q} + \frac{1}{2p}v^{2p}. \quad (6)$$

More precisely, our main results are the following.

Theorem 2. *For the sufficiently small $|\varepsilon|$, system (5) has at most*

$$\lambda = \left[\frac{m}{2}\right]\max\{p,q\} \quad (7)$$

limit cycles bifurcating from the periodic orbits of the center $\dot{u} = -v^{2p-1}$, $\dot{v} = u^{2q-1}$, by using the averaging theory of first order.

The proof of Theorem 2 is given in Section 3.

Theorem 3. *Consider system (5) with $q = np$, where n is a positive integer; then, for $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial differential system (5) bifurcating from the periodic orbits of the center $\dot{u} = -v^{2p-1}$, $\dot{v} = u^{2np-1}$ using the averaging theory of first order is*

$$\begin{aligned} &(\mathbf{a}).\mu_1 = \frac{1}{2}\left(\left[\frac{m}{2}\right]\left(\left[\frac{m}{2}\right] + 3\right)\right), \quad \text{if} \left[\frac{m}{2}\right] \leq n-1, \\ &(\mathbf{b}).\mu_2 = n\left[\frac{m}{2}\right] - \frac{n(n-3)+2}{2}, \quad \text{if} \left[\frac{m}{2}\right] \geq n. \end{aligned} \quad (8)$$

The proof of Theorem 3 is given in Section 4. Also, an example is given with its limit cycles (see Figure 1).

2. First-Order Averaging Method

Here, we state the basic outcomes from the averaging theory of first order, which will be used to prove the main outcomes.

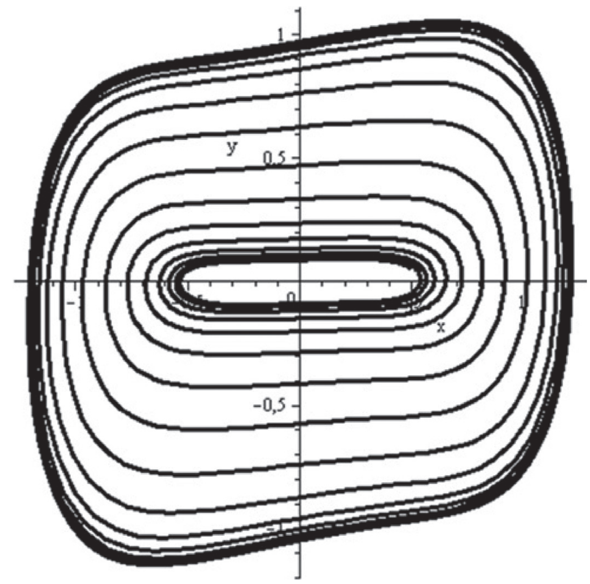

Figure 1: Limit cycles for the system in Example 1 with $\epsilon = 0.01$.

Theorem 4. *Consider the following two initial-value problems:*

$$\begin{aligned} \dot{u} &= \epsilon R(t,u) + \epsilon^2 G(t,u,\epsilon), \\ u(0) &= u_0, \end{aligned} \quad (9)$$

$$\begin{aligned} \dot{v} &= \epsilon f^0(v), \\ v(0) &= u_0, \end{aligned} \quad (10)$$

where u, v and $u_0 \in D$ which is an open domain of $\mathbb{R}$, $t \in [0,\infty)$, $\epsilon \in (0,\epsilon_0]$, R and G are periodic functions with their period T with its variable t, and $f^0(v)$ is the average function of $R(t,v)$ with respect to t, that is,

$$f^0(v) = \frac{1}{T}\int_0^T R(t,v)\mathrm{d}t. \quad (11)$$

Assume that

(i) R, $\partial R/\partial u$, $\partial^2 R/\partial u^2$, G, and $\partial G/\partial u$ are well defined, continuous, and bounded by a constant independent by $\epsilon \in (0,\epsilon_0]$ in $[0,\infty) \times D$.

(ii) T is a constant independent of ϵ.

(iii) $v(t)$ belongs to D on the time scale $1/\epsilon$. Then, the following statements hold:

(a) On the time scale $1/\epsilon$, we have

$$u(t) - v(t) = O(\epsilon), \text{ as } \epsilon \longrightarrow 0 \quad (12)$$

(b) If p is an equilibrium point of the averaged system (10) such that

$$\frac{\partial f^0}{\partial v}\Big|_{v=p} \neq 0, \quad (13)$$

then system (9) has a T-periodic solution $\phi(t,\epsilon) \longrightarrow p$ as $\epsilon \longrightarrow 0$

(c) If (13) is negative, the corresponding periodic solution $\phi(t,\epsilon)$ of equation (9) according to (t,u) is asymptotically stable for all ϵ sufficiently small; if (13) is positive, then it is unstable

For more information about the averaging theory, see, e. g., [15–17].

3. Proof of Theorem 2

The $(p.q)$-trigonometrical functions were defined by Liapunov [18]. Let $u(\theta)=Cs\theta$ and $v(\theta)=Sn\theta$ be the solution of the following initial value problem:

$$\begin{aligned}\dot{u}&=-v^{2p-1},\\ \dot{v}&=u^{2q-1},\\ u(0)&=\sqrt[2]{q}\frac{1}{p}\ \text{and}\ v(0)=0.\end{aligned}\tag{14}$$

Furthermore, the following properties are satisfied:

(a) The functions $Cs\theta$ and $Sn\theta$ are T–periodic with

$$T=2p^{(-1/2q)}q^{(-1/2p)}\frac{\Gamma(1/2p)\Gamma(1/2q)}{\Gamma((1/2p)+(1/2q))},\tag{15}$$

where Γ is the gamma function.

(b) For $p=q=1$, we have $Cs\theta=\cos\theta$ and $Sn\theta=\sin\theta$

(c) $pCs^{2p}\theta+qSn^{2q}\theta=1$

(d) Let $Cs\theta$ and $Sn\theta$ be the $(1.q)$-trigonometrical functions, when i and j are both even (see [19])

$$\int_0^T Cs^i\theta Sn^j\theta d\theta=2q^{-(j+1/2)}\frac{\Gamma(i+1/2q)\Gamma(j+1/2)}{\Gamma((i+1/2q)+(j+1/2))}.\tag{16}$$

We shall need the first-order averaging theory to prove Theorem 2. We write system (5) in (p,q)-polar coordinates (r,θ), where $u=r^pCs\theta$ and $v=r^qSn\theta$. In this way, system (5) will become written in the standard form for applying the averaging theory. If we write $f(u,v)=\sum_{i+j=0}^m a_{i,j}u^iv^j$, then system (5) becomes

$$\left\{\dot{r}=-\epsilon r\sum_{i+j=0}^m\left(a_{i,j}r^{pi+qj}Cs^i\theta Sn^j\theta\right)\right),\dot{\theta}=r^{2pq-p-q}.\tag{17}$$

Treating θ as the independent variable, we get from system (17) the following:

$$\frac{dr}{d\theta}=\epsilon R(r,\theta),\tag{18}$$

where

$$R(r,\theta)=-r^{-2pq+p+q+1}\sum_{i+j=0}^m\left(a_{i,j}r^{pi+qj}Cs^i\theta Sn^j\theta\right).\tag{19}$$

By using the notation which is introduced in Section 2, we get

$$f^0(r)=-\frac{r^{-2pq+p+q+1}}{T}\sum_{i+j=0}^m\left(a_{i,j}r^{pi+qj}\int_0^T Cs^i\theta Sn^j\theta d\theta\right),\tag{20}$$

and we write

$$f^0(r)=-\frac{r^{-2pq+p+q+1}}{T}\sum_{i+j=0}^m a_{i,j}\alpha_{i,j}r^{pi+qj},\tag{21}$$

where

$$\alpha_{i,j}=\int_0^T Cs^i\theta Sn^j\theta d\theta.\tag{22}$$

It is known that

$$\begin{aligned}\alpha_{i,j}&=0,\ \text{if } i \text{ or } j \text{ is odd},\\ \alpha_{i,j}&>0,\ \text{if } i \text{ and } j \text{ are even}.\end{aligned}\tag{23}$$

Hence,

$$f^0(r)=-\frac{r^{-2pq+p+q+1}}{T}\sum_{s+k=0}^{[m/2]}a_{2s,2k}\alpha_{2s,2k}r^{2(ps+qk)}.\tag{24}$$

For the simplicity of calculation, let $A_{s,k}=a_{2s,2k}\alpha_{2s,2k}$; therefore, (24) can be reduced to

$$f^0(r)=-\frac{r^{-2pq+p+q+1}}{T}\sum_{s+k=0}^{[m/2]}A_{s,k}r^{2(ps+qk)}.\tag{25}$$

The positive zeros number of $f^0(r)$, as we know, is equal to the following:

$$K(r)=\sum_{s+k=0}^{[m/2]}A_{s,k}r^{2(ps+qk)},\tag{26}$$

and then, to find the real positive roots of $K(r)$, we must find the zeros of a polynomial in the variable $t=r^2$:

$$H(t)=\sum_{s+k=0}^{[m/2]}A_{s,k}t^{ps+qk}.\tag{27}$$

Now, we stretch the polynomial (27) as follows:

$$\begin{aligned}H(t)={}&A_{0,0}+A_{1,0}t^p+A_{0,1}t^q+A_{2,0}t^{2p}+A_{1,1}t^{p+q}+A_{0,2}t^{2q}+\cdots+A_{l,0}t^{lp}+A_{l-1,1}t^{(l-1)p+q}+A_{l-2,2}t^{(l-2)p+2q}\\ &+\cdots+A_{2,l-2}t^{2p+(l-2)q}+A_{1,l-1}t^{p+(l-1)q}+A_{0,l}t^{ql}+\cdots+A_{[m/2],0}t^{[m/2]p}+A_{[m/2]-1,1}t^{([m/2]-1)p+q}+A_{[m/2]-2,2}t^{([m/2]-2)p+2q}\\ &+\cdots+A_{2,[m/2]-2}t^{2p+([m/2]-2)q}+A_{1,[m/2]-1}t^{p+([m/2]-1)q}+A_{0,[m/2]}t^{[m/2]q}.\end{aligned}\tag{28}$$

So, the degree of $H(t)$ is bounded by $\lambda = [m/2]\max\{p, q\}$, and we conclude that $f^0(r)$ has at most λ positive root r. Hence, Theorem 2 is proved.

4. Proof of Theorem 3

Consider the polynomial differential system (5) with$q = $np. From equation (25), we obtain

$$f^0(r) = -\frac{r^{np(-2p+1)+p+1}}{T}\sum_{s+k=0}^{[m/2]} A_{s,k} r^{2p(s+nk)}. \tag{29}$$

The zeros positive number of $f^0(r)$ is equal to the following:

$$S(r) = \sum_{s+k=0}^{[m/2]} A_{s,k} r^{2p(s+nk)}. \tag{30}$$

We write (30) as follows:

$$\begin{aligned} S(r) = {} & A_{0,0} + \left(A_{1,0}r^{2p} + A_{0,1}r^{2pn}\right) + \left(A_{2,0}r^{4p} + A_{1,1}r^{(n+1)2p} + A_{0,2}r^{4np}\right) + \left(A_{3,0}r^{6p} + A_{2,1}r^{(n+2)2p} + A_{1,2}r^{(1+2n)2p} + A_{0,3}r^{6np}\right) \\ & \cdot \left(\left(A_{4,0}r^{8p} + A_{3,1}r^{(n+3)2p} + A_{2,2}r^{(2+2n)2p} + A_{1,3}r^{(1+3n)2p} + A_{0,4}r^{8np}\right) + \cdots \right. \\ & + \begin{bmatrix} A_{[m/2]-2,0}r^{([m/2]-2)2p} + A_{[m/2]-3,1}r^{([m/2]+n-3)2p} + A_{[m/2]-4,2}r^{([m/2]+2n-4)2p} + \cdots + A_{2,[m/2]-4}r^{(2+([m/2]-4)n)2p} \\ +A_{1,[m/2]-3}r^{(1+([m/2]-3)n)2p} + A_{0,([m/2]-2)}r^{([m/2]-2)2np} \end{bmatrix} \\ & + \begin{bmatrix} A_{[m/2]-1,0}r^{([m/2]-1)2p} + A_{[m/2]-2,1}r^{([m/2]+n-2)2p} + A_{[m/2]-3,2}r^{([m/2]+2n-3)2p} + \cdots + A_{2,[m/2]-3}r^{(2+([m/2]-3)n)2p} \\ +A_{1,[m/2]-2}r^{(1+([m/2]-2)n)2p} + A_{0,[m/2]-1}r^{([m/2]-1)2np} \end{bmatrix} \\ & + \begin{bmatrix} A_{[m/2],0}r^{[m/2]2p} + A_{[m/2]-1,1}r^{([m/2]+n-1)2p} + A_{[m/2]-2,2}r^{([m/2]+2n-2)2p} + \cdots + A_{2,[m/2]-2}r^{(2+([m/2]-2)n)2p} \\ +A_{1,[m/2]-1}r^{(1+([m/2]-1)n)2p} + A_{0,[m/2]}r^{[m/2]2np} \end{bmatrix}. \end{aligned} \tag{31}$$

Let us write (31) as

$$\begin{aligned} S(r) = {} & \left[A_{0,0} + A_{1,0}r^{2p} + A_{2,0}r^{4p} + \cdots + A_{[m/2]-2,0}r^{([m/2]-2)2p} + A_{[m/2]-1,0}r^{([m/2]-1)2p} + A_{[m/2],0}r^{[m/2]2p}\right] + \\ & \cdot \left[A_{0,1}r^{2np} + A_{1,1}r^{(n+1)2p} + A_{2,1}r^{(n+2)2p} + \cdots + A_{[m/2]-3,1}r^{(n+[m/2]-3)2p}\right. \\ & \left. + A_{[m/2]-2,1}r^{(n+[m/2]-2)2p} + A_{[m/2]-1,1}r^{(n+[m/2]-1)2p}\right] \\ & + \left[A_{0,2}r^{4np} + A_{1,2}r^{(2n+1)2p} + A_{2,2}r^{(2n+2)2p} + \cdots + A_{[m/2]-4,2}r^{(2n+[m/2]-4)2p} + A_{[m/2]-3,2}r^{(2n+[m/2]-3)2p}\right. \\ & \left. + A_{[m/2]-2,2}r^{(2n+[m/2]-2)2p}\right] \\ & + \cdots + \left[A_{0,([m/2]-2)}r^{(([m/2]-2)n)2p} + A_{1,[m/2]-2}r^{(1+([m/2]-2)n)2p} + A_{2,[m/2]-2}r^{(2+([m/2]-2)n)2p}\right] \\ & + \left[A_{0,[m/2]-1}r^{([m/2]-1)2np} + A_{1,[m/2]-1}r^{(1+([m/2]-1)n)2p}\right] \\ & + A_{0,[m/2]}r^{[m/2]n}. \end{aligned} \tag{32}$$

To find the number of positive roots of polynomials $S(r)$, we distinguish two cases.

Case 1. For $[m/2] \leq n-1$, the number term in polynomial (32) is

$$\left(\left[\frac{m}{2}\right]+1\right)+\left[\frac{m}{2}\right]+\left(\left[\frac{m}{2}\right]-1\right) + \cdots + 2 + 1 = \frac{1}{2}\left(\left[\frac{m}{2}\right]+2\right)\left(\left[\frac{m}{2}\right]+1\right). \quad (33)$$

Now, the Descartes theorem of the Appendix will be applied, and the appropriate coefficients $A_{i,j}$ can be selected for the simple positive zeros number of $S(r)$ as at most

$$\mu_1 = \frac{1}{2}\left(\left[\frac{m}{2}\right]+2\right)\left(\left[\frac{m}{2}\right]+1\right) - 1 = \frac{1}{2}\left(\left[\frac{m}{2}\right]\left(\left[\frac{m}{2}\right]+3\right)\right). \quad (34)$$

Hence, (a) of Theorem 3 is proved.

Case 2. For $[m/2] \geq n$, the number term in polynomial (32) is

$$\left(\left[\frac{m}{2}\right]+1\right)+\left[\frac{m}{2}\right]+\left(\left[\frac{m}{2}\right]-1\right)+\cdots+2+1-\left(\left[\frac{m}{2}\right]-n+1\right)-\left(\left[\frac{m}{2}\right]-n\right)-\left(\left[\frac{m}{2}\right]-n-1\right)-\cdots-2-1$$
$$= \frac{1}{2}\left[\left(\left[\frac{m}{2}\right]+2\right)\left(\left[\frac{m}{2}\right]+1\right)-\left(\left[\frac{m}{2}\right]-n+1\right)\left(\left[\frac{m}{2}\right]-n+2\right)\right] = n\left[\frac{m}{2}\right]-\frac{n(n-3)}{2}, \quad (35)$$

by Descartes theorem of the Appendix, and we can choose the appropriate coefficients $A_{i,j}$ in order that the simple positive roots number of $S(r)$ is at most

$$\mu_2 = n\left[\frac{m}{2}\right] - \frac{n(n-3)}{2} - 1 = n\left[\frac{m}{2}\right] - \frac{n(n-3)+2}{2}. \quad (36)$$

Hence, (b) of Theorem 3 is proved.

Example 1. We consider system (5), where $p = 1, q = 3$, and

$$f(u,v) = 2u^3 + u^2 v - 7.2365u^2 + 5v^2 - 0.5u + 0.7605. \quad (37)$$

In this case, $m = 3, n = 3$ and $Cs\theta$ and $Sn\theta$ are T-periodic functions with period $T = 8.413\,1$. From equation (5), we obtain

$$f^0(r) = -\frac{1}{Tr}\left(A_{0.0} + A_{1.0}r^2 + A_{0.1}r^6\right). \quad (38)$$

Upon using (16), we get

$$\alpha_{0,0} = 8.4131,$$
$$\alpha_{2,0} = 3.6276 \text{ and } \alpha_{0,2} = 2.1033. \quad (39)$$

So,

$$f^0(r) = -\frac{1}{8.4131r}\left(6.3982 - 26.251r^2 + 10.517r^6\right). \quad (40)$$

This polynomial has two positive real roots, $r_1 = 0.5$ and $r_2 = 1.2$. According to statement (a) of Theorem 3, that system has exactly two limit cycles bifurcating from the periodic orbits of the center $\dot{u} = -v, \dot{u} = v^5$, using the averaging theory of first order. Figure 1 shows the limit cycles for Example 1.

Appendix

We remember Descartes' theorem regarding the real roots number of a real polynomial (for a proof, see, for example, [20]).

Descartes theorem: consider the following real polynomial:

$$p(u) = a_{l_1}u_{l_1} + a_{l_2}u_{l_2} + \cdots + a_{l_k}u_{l_k}, \quad (A.1)$$

with $0 \leq l_1 < l_2 < \cdots < l_k$ and $a_{l_i} \neq 0$ real constants for $i \in \{1, 2, 3, \ldots, k\}$. Since $a_{l_i}a_{l_{i+1}} < 0$, it can said that a_{l_i} and $a_{l_{i+1}}$ admit a variation of sign. If the signs variations number of is n, then $p(u)$ admits at most m positive real zeros. In aditios, it is always possible to pick out the coefficients of $p(u)$, where $p(u)$ admits exactly $k-1$ positive real zero.

Data Availability

No data were used to support this study.

Conflicts of Interest

This work does not have any conflicts of interest.

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