Research Article

New Contributions in Generalization S -Metric Spaces to S^*p -Partial Metric Spaces with Some Results in Common Fixed Point Theorems

Asma Al Rwaily ¹ and A. M. Zidan ^{2,3}

¹Department of Mathematics, College of Science and Arts, Qassim University, Buraydah, Saudi Arabia

²Department of Mathematics, College of Science, King Khalid University, Abha 61413, Saudi Arabia

³Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt

Correspondence should be addressed to A. M. Zidan; ahmoahmed@kku.edu.sa

Received 12 January 2021; Revised 4 February 2021; Accepted 7 May 2021; Published 19 May 2021

Academic Editor: Ahmed Mostafa Khalil

Copyright © 2021 Asma Al Rwaily and A. M. Zidan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we introduce the notion of S^*p -partial metric spaces which is a generalization of S -metric spaces and partial-metric spaces. Also, we give some of the topological properties that are important in knowing the convergence of the sequences and Cauchy sequence. Finally, we study a new common fixed point theorems in this spaces.

1. Introduction

There are a large number of generalizations of the Banach contraction principle with different use forms of contractual terms in a variety of generalized metric spaces. Some of the circulars are obtained through contractual terms expressed in reasonable terms. Shahkoobi and Razani [1] introduced a new class of rational Geraghty contractive mappings in the setup of b -metric spaces and proved some fixed point for such mappings in ordered b -metric spaces which are investigated. One such generalization is a partial-metric space which was introduced by Matthews [2] in 1994, since, in this space, a self-distance of an arbitrary point need not be equal zero. Also, in 2007, Sedghi et al. [3] introduced D^* -metric spaces which are generalization of the notion of D -metric spaces introduced by Dhage [4]. Also, Sedghi et al. proved some basic properties in D^* -metric spaces. In 2012, Sedghi et al. [5] introduced S -metric spaces and gave some properties and fixed point theorem for a complete S -metric space.

Recently, in 2019, Z. Mustafa et al. [6] introduced the structure of S_b -metric spaces which generalization of S -metric space and gave some properties and fixed point theorem for these spaces.

2. Preliminaries and Definitions

We begin the section with some basic definitions and concepts.

Definition 1 (see [2]). Let Y be a set. A function $p: Y \times Y \rightarrow [0, \infty)$ is said to be a partial metric spaces on a nonempty set Y ; for all $w, t, r \in Y$, the following conditions are held:

(P_1) $0 \leq P(w, w) \leq P(w, t)$ (nonnegativity and small self-distances)

(P_2) If $P(w, w) = P(w, t) = P(t, t)$, then $w = t$ (indistancy implies equality)

(P_3) $P(w, t) = P(t, w)$ (symmetry)

(P_4) $P(w, t) \leq P(w, r) + P(r, t) - P(r, r)$ (triangularity)

Hence, the function P is called an P -metric on Y and the pair (Y, P) is called a partial metric space.

Example 1 (see [7, 8]). Let $Y = [0, \infty)$ and $p: Y \times Y \rightarrow [0, \infty)$ given by $p(w, t) = \max\{w, t\}$, for all $w, t \in Y$; then, (Y, p) is partial metric space.

Definition 2. Let (Y, p) be a partial metric space; then, one has the following:

- (i) A sequence $\{w_n\}$ in Y converges to a point $w \in Y$ if and only if $\lim_{n \rightarrow \infty} p(w_n, w) = p(w, w)$
- (ii) A sequence $\{w_n\}$ in Y is Cauchy if $\lim_{n, m \rightarrow \infty} p(w_n, w_m)$ exists and is finite
- (iii) Y is complete if every Cauchy sequence $\{w_n\}$ in Y converges to a point $w \in Y$, that is, $\lim_{n, m \rightarrow \infty} p(w_n, w_m) = p(w, w)$

Lemma 1 (see [2]). Let (Y, p) be a partial metric space. Then, Y is complete if and only if the metric space (Y, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(w_n, r) = 0$ if and only if

$$p(r, r) \lim_{n \rightarrow \infty} p^s(w_n, r) = \lim_{n, m \rightarrow \infty} p^s(w_n, w_m). \quad (1)$$

In [9], Romaguera introduced the definitions of 0-complete partial metric spaces and 0-Cauchy sequence. Also, he obtained a characterization of completeness for partial metric space using the notion of 0-completeness.

Mustafa and Sims [10] extended Banach principle and introduced the notation of generalized metric spaces, so-called G -metric spaces, as follows.

Definition 3. Let Y be a set. A function $G: Y \times Y \times Y \rightarrow [0, \infty)$ is said to be a generalized metric spaces on a nonempty set Y , for all $w, t, r, \eta \in Y$, and the following conditions hold:

- (G_1) $G(w, t, r) = 0 \Leftrightarrow w = t = r$
- (G_2) $G(w, t, r) > 0$ and $w \neq t$
- (G_3) $G(w, w, t) \leq G(w, t, r)$, and $t \neq r$
- (G_4) $G(w, t, r) = G(w, r, t) = G(t, r, w) = \dots$ (symmetry in all three variables)
- (G_5) $G(w, t, r) \leq G(w, \eta, \eta) + G(\eta, r, w)$

Hence, the function G is called an G -metric on Y and the pair (Y, G) is called an generalized metric space.

To find some basic properties and examples of G -metric spaces in Mustafa and Sims [10].

After that, Zand and Nezhad [11] introduced definition of G^p -metric space by generalization and unification of both partial metric space and G -metric space as follows.

Definition 4. Let Y be a set. A function $G^p: Y \times Y \times Y \rightarrow [0, \infty)$ is said to be a generalized partial metric spaces on a nonempty set Y , for all $w, t, r, \eta \in Y$, and the following conditions hold:

- (G_1^p) $w = t = r$ if $G^p(w, w, w) = G^p(t, t, t) = G^p(r, r, r)$
- (G_2^p) $0 \leq G^p(w, w, w) \leq G^p(w, w, t) \leq G^p(w, t, y)$
- (G_3^p) $G^p(w, t, r) = G^p(w, r, t) = G^p(t, r, w) = \dots$ (symmetry in all three variables)
- (G_4^p) $G^p(w, t, r) \leq G^p(w, \eta, \eta) + G^p(\eta, r, w) - G^p(\eta, \eta, \eta)$

Hence, the function G^p is called an G^p -metric on Y and the pair (Y, G^p) is called an generalized partial metric space.

Example 2 (see [11]). Let $Y = [0, \infty)$ and $G^p(w, t, r) = \max\{w, t, r\}$ for all $w, t, r \in Y$, then (Y, G^p) is a G^p -metric space. Also, (Y, G^p) is not a G -metric space.

Proposition 1. Let (Y, G^p) is a G^p -metric space, then, for any $w, t, r \in Y$ and $\eta \in Y$, it follows that

- (i) $G^p(w, t, r) \leq G^p(w, w, t) + G^p(w, w, r) - G^p(w, w, w)$
- (ii) $G^p(w, t, t) \leq 2G^p(w, w, t) - G^p(w, w, w)$
- (iii) $G^p(w, t, r) \leq G^p(w, \eta, \eta) + G^p(t, \eta, \eta) + G^p(r, \eta, \eta) - 2G^p(\eta, \eta, \eta)$

Shaban Sedghi et al. [3] introduced the notion of D^* -metric which is a modification of the definition of D -metric introduced by Dhage [4, 12], and they proved some basic properties in D^* -metric spaces.

Definition 5 (see [3]). Let Y be a set. A function $D^*: Y \times Y \times Y \rightarrow [0, \infty)$ is said to be a D^* -metric spaces on a nonempty set Y , for all $w, t, r \in Y$, and the following conditions hold:

- (D_1^*) $D^*(w, t, r) \geq 0$
- (D_2^*) $D^*(w, t, r) \Leftrightarrow w = t = r$
- (D_3^*) $D^*(w, t, r) = D^*(\aleph\{w, t, r\})$ (where $\aleph$ is a permutation function)

Hence, the function D^* is called an D^* -metric on Y , and the pair (Y, D^*) is called an generalized partial metric space.

Example 3 (see [3]). Let $Y = \mathbb{R}$. Denote $D^*(w, t, r) = |w - t| + |t - r| + |r - w|$, for all $w, t, r \in \mathbb{R}$. Since

$$B_{D^*}(w, \lambda) = \{t \in Y: D^*(w, t, t) < \lambda\}. \quad (2)$$

Hence,

$$\begin{aligned} B_{D^*}(1, 2) &= \{t \in \mathbb{R}: D^*(1, t, t) < 2\} \\ &= \{t \in \mathbb{R}: |t - 1| + |t - 1| < 2\} \\ &= \{t \in \mathbb{R}: |t - 1| < 1\} = (0, 2). \end{aligned} \quad (3)$$

On other way, Sedghi et al. introduced S -metric spaces as follows.

Definition 6 (see [5]). Let Y be a set. A function $S: Y^3 \rightarrow [0, \infty)$ is said to be a S -metric spaces on a nonempty set Y , for all $w, t, r, \eta \in Y$, and the following conditions hold:

- (S_1) $S(w, t, r) > 0$
- (S_2) $S(w, t, r) \Leftrightarrow w = t = r$
- (S_3) $S(w, t, r) \leq S(w, w, \eta) + S(t, t, \eta) + S(r, r, \eta)$ for all $w, t, r, \eta \in Y$ (rectangle inequality)

Hence, the function S is called an S -metric on Y and the pair (Y, S) is called an S -partial metric space.

Example 4 (see [5]). Let $Y = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Y ; then,

$$S(w, t, r) = \|t + r - 2w\| + \|t - r\| \quad (4)$$

is a S -metric on Y .

Remark 1. Every D^* -metric is S -metric, but in general, the converse is not true, see the following example.

Example 5. Let $Y = \mathbb{R}^n$ and $\|\cdot\|$ a norm on Y ; then, $S(w, t, r) = \|t + r - 2w\| + \|t - r\|$ is S -metric on Y , but it is not D^* -metric because it is not symmetric.

Example 6. Let $Y = \mathbb{R}^n$ and d is an ordinary metric on Y . Therefore, $S(w, t, r) = d(w, t) + d(w, r) + d(t, r)$ is an S -metric on Y . Then, if the points w, t , and r are connected by a line, hence, we have a triangle and if we choose a point a mediating this triangle, then the inequality

$$S(w, t, r) \leq s(w, w, \eta) + s(t, t, \eta) + (r, r, \eta), \quad (5)$$

for all $w, t, r, \eta \in Y$ holds.

Definition 7 (see [5]). Let (Y, S) be a S -metric space. Then, for $w_0 \in Y$ and $\lambda > 0$, the S -open ball and S -closed ball of radius λ with centered at w_0 are

$$\begin{aligned} B_S(w_0, \lambda) &= \{t \in Y: S(w_0, w_0, t) < \lambda\}, \\ \overline{B}_S(w_0, \lambda) &= \{t \in Y: S(w_0, w_0, t) \leq \lambda\}. \end{aligned} \quad (6)$$

Proposition 2 (see [5]). Let (Y, S) be a S -metric space and $A \subset Y$:

- (1) If, for every $w_0 \in A$, there exists $\lambda > 0$ such that $B_S(w_0, \lambda)$, then the subset A is called an open subset of Y .
- (2) A subset A of Y is said to be S -bounded if there exists $\lambda > 0$ such that $S(w, w, t) < \lambda$, for all $w, t \in A$.
- (3) A sequence $\{w_n\}$ in Y converges to w if and only if $S(w_n, w_n, w) \rightarrow 0$ as $n \rightarrow \infty$. That is, for each $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all n, n_0 , $S(w_n, w_n, w) < \lambda$ and we denote this by $\lim_{n \rightarrow \infty} w_n = w$.

Lemma 2 (see [5]). Let (Y, S) be an S -metric space; then, the following satisfy:

- (1) If $\lambda > 0$ and $w \in Y$, then the ball $B_S(w_0, \lambda)$ is an open subset of Y .
- (2) If the sequence $\{w_n\}$ in Y converges to w , then w is unique.
- (3) If the sequence $\{w_n\}$ in Y converges to w , then $\{w_n\}$ is a Cauchy sequence.

Definition 8 (see [10]). Two classes of the following mappings are

- (1) $\Pi = \{\pi: \pi: [0, \infty) \rightarrow [0, \infty)\}$ is nondecreasing, continuous, and $\pi^{-1}(0) = 0$
- (2) $\Gamma = \{\gamma: \gamma: [0, \infty) \rightarrow [0, \infty)\}$ is nondecreasing, lower semicontinuous, and $\pi^{-1}(0) = 0$

Definition 9. Let $(Y, \leq)$ be a partially ordered set. Two maps $\mathfrak{f}, \mathfrak{g}: Y \times Y$ are said to be weakly increasing if $\mathfrak{f}w \leq \mathfrak{g}w$ and $\mathfrak{g}w \leq \mathfrak{f}w$, for all $w \in Y$.

Barakat and Zidan [13] proved a common fixed point theorem for weak contractive maps by using the concept of G^P -metric spaces.

Theorem 1. Let $(Y, \leq)$ be a partially ordered set with $\mathfrak{f}$ and $\mathfrak{g}$ as weakly increasing self-mapping on a complete G^P -partial metric space. Suppose that there exist $\pi \in \Pi$ and $\gamma \in \Gamma$ such that

$$\pi(G^P(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t))) \leq \pi(M(w, w, t)) - \gamma(M(t, t, w)), \quad (7)$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) &= \zeta_1 G^P(w, w, t) + \zeta_2 G^P(w, w, \mathfrak{f}(w)) \\ &\quad + \zeta_3 G^P(t, t, \mathfrak{g}(t)) \\ &\quad + \zeta_4 [G^P(w, w, \mathfrak{g}(t)) + G^P(t, t, \mathfrak{f}(w))], \end{aligned} \quad (8)$$

where $\zeta_i > 0$, for $i = \{1, 2, 3, 4\}$, with $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \geq 1$.

Assume that one of the following cases is satisfied:

- (a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \leq r$ for all $n \in \mathbb{N}$
- (b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

3. S^{*P} -Partial Metric Spaces and Some Properties

We first introduce the concept of a S^{*P} -partial metric space or (S^{*P}) .

Definition 10. Let Y be a set. A function $S^{*P}: Y^3 \rightarrow [0, \infty)$ is said to be a S^{*P} -partial metric space on a nonempty set Y , for all $w, t, r, \eta \in Y$; the following conditions hold:

$$\begin{aligned} (S^{*P})S^{*P}(w, t, r) &= S^{*P}(w, w, w) = S^{*P}(t, t, t) = S^{*P}(r, r, r); \text{ then, } w = t = r \\ (S^{*P})S^{*P}(w, t, r) &\leq S^{*P}(w, w, \eta) + S^{*P}(t, t, \eta) + S^{*P}(r, r, \eta) - S^{*P}(\eta, \eta, \eta), \text{ for all } w, t, r, \eta \in Y \text{ (rectangle inequality)} \end{aligned}$$

Hence, the function S^{*P} is called an S^{*P} -partial metric on Y and the pair (Y, S^{*P}) is called an S^{*P} -partial metric space.

Example 7. Let $Y = \mathbb{R}$ and $\|\cdot\|$ a norm on Y ; then, we have

$$S^{*P}(w, t, r) = \max\{w, t, r\} + \|w - r\| + \|t - w\|, \quad (9)$$

which is a S -partial metric on Y .

Remark 2. From Example 7, we get every S -metric is S^{*P} -metric, but the converse is not true at all, see the following example.

$$\begin{aligned} B_{S^{*P}}(w_0, \lambda) &= \{t \in Y: S^{*P}(w_0, w_0, t) < \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(t, t, w_0)\}, \\ \overline{B}_{S^{*P}}(w_0, \lambda) &= \{t \in Y: S^{*P}(w_0, w_0, t) \leq \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(t, t, w_0)\}. \end{aligned} \quad (10)$$

Proposition 3. Let (Y, S^{*P}) be a S^{*P} -partial metric space. Then, for $w_0 \in Y$ and $\lambda > 0$, the following statements are satisfied:

- (1) If $S^{*P}(w_0, w_0, w) \leq \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(w, w, w_0)$ and $S^{*P}(w_0, w_0, t) \leq \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(t, t, w_0)$, then $w, t \in B_{S^{*P}}(w_0, \lambda)$
- (2) If $t \in B_{S^{*P}}(w_0, \lambda)$, then there exist $\rho > 0$, such that $B_{S^{*P}}(t, \rho) \subseteq B_{S^{*P}}(w_0, \lambda)$

Proof

- (1) The proof is straightforward.
- (2) Let $t \in B_{S^{*P}}(w_0, \lambda)$; then, we have

$$S^{*P}(w_0, w_0, t) \leq \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(t, t, w_0). \quad (11)$$

Also, we suppose

$$\rho = \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(r, r, w_0) - 2S^{*P}(w_0, w_0, t), \quad (12)$$

and $r \in B_{S^{*P}}(t, \rho)$; then, we obtain

$$S^{*P}(t, t, r) \leq \rho + S^{*P}(t, t, t) - S^{*P}(r, r, t). \quad (13)$$

Therefore, $t \in B_{S^{*P}}(w_0, \lambda)$, and so (2) holds.

$$\begin{aligned} S^{*P}(w_0, w_0, r) &\leq S^{*P}(w_0, w_0, t) + S^{*P}(w_0, w_0, t) \\ &\quad + S^{*P}(r, r, t) - S^{*P}(t, t, t) \\ &\leq 2S^{*P}(w_0, w_0, t) + S^{*P}(r, r, t) + \lambda \\ &\quad + S^{*P}(w_0, w_0, w_0) - S^{*P}(r, r, w_0) \\ &\quad - 2S^{*P}(w_0, w_0, t) - S^{*P}(t, t, t) \\ &\leq \lambda + S^{*P}(w_0, w_0, w_0) - S^{*P}(r, r, t). \end{aligned} \quad (14)$$

Hence, $r \in B_{S^{*P}}(w_0, \lambda)$. $\square$

Definition 12. Let (Y, S^{*P}) be a S^{*P} -partial metric space and $\{w_n\}$ a sequence in Y . A point $w \in Y$ is said to be the limit of the sequence if

Definition 11. Let (Y, S^{*P}) be a S^{*P} -partial metric space. Then, for $w_0 \in Y$ and $\lambda > 0$, the S^{*P} -open ball and S^{*P} -open closed of radius λ with centered at w_0 are

$$\lim_{n, m \rightarrow \infty} S^{*P}(w, w_n, w_m) = S^{*P}(w, w, w). \quad (15)$$

Hence, the sequence $\{w_n\}$ is S^{*P} -convergent to w .

Therefore, if $\{w_n\} \rightarrow w$ in a S -partial metric space (Y, S^{*P}) , then, for any $\hbar > 0$, there exists $\mathfrak{N} \in \mathbb{N}$ such that, for all $n, m \in \mathfrak{N}$, we have

$$|S^{*P}(w, w_n, w_m) - S^{*P}(w, w, w)| < \hbar. \quad (16)$$

Definition 13. A S^{*P} -partial metric space (Y, S^{*P}) is called a S^{*P} -partial asymmetric space if

$$S^{*P}(w, w, t) = S^{*P}(t, t, w). \quad (17)$$

Lemma 3. (Y, S^{*P}) be a S^{*P} -partial metric space. If the sequence $\{w_n\}$ in Y converges to w . Therefore, we get w is unique.

Proof. Let $\{w_n\}$ converges to w and w . Therefore, for each $\lambda > 0$, there exist $n_1, n_2 \in \mathbb{N}$; then, we have

$$n \geq n_1 \Rightarrow S^{*P}(w_n, w_n, w) < \frac{\lambda}{3}, \quad (18)$$

$$n \geq n_2 \Rightarrow S^{*P}(w_n, w_n, t) < \frac{\lambda}{3}.$$

If set $n_0 = \max\{n_1, n_2\}$, then, for every $n \geq n_0$, we have a third condition of S^{*P} -partial metric:

$$\begin{aligned} S^{*P}(w, w, t) &\leq S^{*P}(w, w, w_n) + S^{*P}(w, w, w_n) \\ &\quad + S^{*P}(t, t, w_n) - S^{*P}(w_n, w_n, w_n) \\ &\leq 2S^{*P}(w, w, w_n) + S^{*P}(t, t, w_n) \\ &\quad - S^{*P}(w_n, w_n, w_n) \\ &\leq 2S^{*P}(w, w, w_n) + S^{*P}(t, t, w_n) \\ &< 2\frac{\lambda}{3} + \frac{\lambda}{3} = \lambda. \end{aligned} \quad (19)$$

Hence, $S^{*P}(w, w, t) = 0 \Rightarrow w = t$, but the converse is not necessarily true. $\square$

Definition 14. Let (Y, S^{*P}) be a S^{*P} -partial metric space. Then, for a sequence $\{w_n\} \subseteq Y$ and a point $w \in Y$, the following are equivalent:

- (1) $\{w_n\}$ is a S^{*P} convergent to w
- (2) $\lim_{n,m \rightarrow \infty} S^{*P}(w_n, w_m, w) = S^{*P}(w, w, w)$

Proposition 4. Let (Y, S^{*P}) be a S^{*P} -partial metric space. If we get a sequence $\{w_n\} \in Y$ which is called a Cauchy sequence if, for each $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$S^{*P}(w_n, w_m, w_m) < \lambda + S^{*P}(w, w, w), \quad (20)$$

Proof. for each $n, m \geq 0$, if the sequence $\{w_n\}$ in Y converges to $w \in Y$. $\square$

Definition 15. Let (Y, S^{*P}) be a S^{*P} -partial metric space. Then, (Y, S^{*P}) is said to be complete if every Cauchy sequence is convergent.

Definition 16. Let (Y, S^{*P}) , (Y_1, S_1^{*P}) , and (Y_2, S_2^{*P}) be two S^{*P} -partial metric space. Also, we suppose a function

$$\mathfrak{F}: (Y_1, S_1^{*P}) \longrightarrow (Y_2, S_2^{*P}), \quad (21)$$

and then, $\mathfrak{F}$ is said to be S^{*P} -continuous at a point; $x \in Y_1$ if and only if, for given $h > 0$, there exists $\lambda > 0$ such that $w, t \in Y_1$ and the inequality

$$S_1^{*P}(x, w, t) < h + S_1^{*P}(w, w, w). \quad (22)$$

This is an indication:

$$S_2^{*P}(\mathfrak{F}(x), \mathfrak{F}(w), \mathfrak{F}(t)) < \lambda + S_2^{*P}(\mathfrak{F}(w), \mathfrak{F}(w), \mathfrak{F}(w)). \quad (23)$$

Hence, a function $\mathfrak{F}$ is S^{*P} -continuous on Y_1 if and only if it is S^{*P} -continuous at all $x \in Y_1$.

4. A Generalization of Common Point Theorems in S^{*P} -Partial Metric Spaces

Theorem 2. Let $(Y, \preceq)$ be a partially ordered set with $\mathfrak{f}$ and $\mathfrak{g}$ as weakly increasing self-mapping on a complete S^{*P} -partial metric space. Suppose that there exist $\pi \in \Pi$ and $\gamma \in \Gamma$ such that

$$\pi(G^P(\mathfrak{f}(w), \mathfrak{f}(w), \mathfrak{g}(t))) \leq \pi(M(w, w, t)) - \gamma(M(t, t, w)), \quad (24)$$

for all $w, t \in Y$, where

$$M(w, w, t) = \max\{S^{*P}(w, w, t), S^{*P}(w, w, \mathfrak{f}(w)), S^{*P}(t, t, \mathfrak{g}(t)), [S^{*P}(w, w, \mathfrak{g}(t)) + S^{*P}(t, t, \mathfrak{f}(w))]\}. \quad (25)$$

Assume that one of the following cases is satisfied:

- (a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \preceq r$ for all $n \in \mathbb{N}$
- (b) $\mathfrak{f}$ or $\mathfrak{g}$ is continuous

Therefore, the maps $\mathfrak{f}$ or $\mathfrak{g}$ have a common fixed point.

Proof. Suppose that h is a fixed point of $\mathfrak{f}$ and $S^{*P}(h, h, \mathfrak{g}(h)) > 0$. From (24), with $w = t = h$, we have

$$\begin{aligned} \pi(S^{*P}(h, h, \mathfrak{g}(h))) &= \pi(S^{*P}(\mathfrak{f}(h), \mathfrak{f}(h), \mathfrak{g}(h))) \\ &\leq \pi(M(h, h, h)) - \gamma(M(h, h, h)), \end{aligned} \quad (26)$$

where

$$\begin{aligned} M(h, h, h) &= \max\{S^{*P}(h, h, h), S^{*P}(h, h, \mathfrak{f}(h)), S^{*P}(h, h, \mathfrak{g}(h)), \\ &\quad [S^{*P}(h, h, \mathfrak{g}(h)) + S^{*P}(h, h, \mathfrak{f}(h))]\} \\ &= \max\{S^{*P}(h, h, h), S^{*P}(h, h, h), S^{*P}(h, h, \mathfrak{g}(h)), \\ &\quad [S^{*P}(h, h, \mathfrak{g}(h)) + S^{*P}(h, h, h)]\} \\ &= \max\{S^{*P}(h, h, h), S^{*P}(h, h, \mathfrak{g}(h))\} \\ &= S^{*P}(h, h, \mathfrak{g}(h)). \end{aligned} \quad (27)$$

Hence, we have

$$\begin{aligned} \pi(S^{*P}(h, h, \mathfrak{g}(h))) &= \pi(S^{*P}(\mathfrak{f}(h), \mathfrak{f}(h), \mathfrak{g}(h))) \\ &\leq \pi(S^{*P}(\mathfrak{f}(h), \mathfrak{f}(h), \mathfrak{g}(h))) \\ &\quad - \gamma(S^{*P}(\mathfrak{f}(h), \mathfrak{f}(h), \mathfrak{g}(h))) \\ &\Rightarrow \gamma(S^{*P}(h, h, \mathfrak{g}(h))) \leq 0, \end{aligned} \quad (28)$$

which is a contradiction. Therefore, $S^{*P}(\mathfrak{f}(h), \mathfrak{f}(h), \mathfrak{g}(h)) = 0$. So, h is common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$. Similarly, if h is a fixed point of h , then one can deduce that h is also fixed point of $\mathfrak{f}$.

If we let w_0 be an arbitrary point of Y with $\mathfrak{f}w_0 = w_0$, then the proof is finished, so we assume that $\mathfrak{f}w_0 \neq w_0$.

Now, one can construct a sequence $\{w_n\} \subseteq Y$ as follows:

$$\begin{aligned} w_1 &= \mathfrak{f}w_0 \preceq \mathfrak{g}\mathfrak{f}w_0 = \mathfrak{g}w_1 = w_2 \\ w_2 &= \mathfrak{f}w_1 \preceq \mathfrak{g}\mathfrak{f}w_1 = \mathfrak{g}w_1 = w_3 \\ &\vdots \\ w_n &\preceq w_{n+1}. \end{aligned} \quad (29)$$

Since w_{2n} and w_{2n+1} are comparable, then we may assume that $S^{*P}(w_{2n}, w_{2n}, w_{2n+1})$, for every $n \in \mathbb{N}$. If not, then $w_{2n} = w_{2n+1}$, for some n . For all those n , using (24), we obtain

$$\begin{aligned}
\pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) &= \pi(S^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}w_{2n+1})) \\
&\leq \pi(M(w_{2n}, w_{2n}, w_{2n+1})) \\
&\quad - \gamma(M(w_{2n}, w_{2n}, \mathfrak{g}w_{2n+1})), \\
M(w_{2n}, w_{2n}, w_{2n+1}) &= \max\{S^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S^{*P}(w_{2n}, w_{2n}, \mathfrak{f}w_{2n}), S^{*P}(w_{2n+1}, w_{2n+1}, \mathfrak{g}w_{2n+1}), \\
&\quad \left[\frac{S^{*P}(w_{2n}, w_{2n}, \mathfrak{g}w_{2n+1}), S^{*P}(w_{2n+1}, w_{2n+1}, \mathfrak{f}w_{2n})}{2} \right] \} \\
&= \max\{S^{*P}(w_{2n}, w_{2n}, w_{2n+1}), S^{*P}(w_{2n}, w_{2n}, w_{2n+1}), \\
&\quad S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}), \left[\frac{S^{*P}(w_{2n}, w_{2n}, w_{2n+2}), S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1})}{2} \right] \} \\
&\leq \max\left\{S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}), \frac{1}{2} \left[S^{*P}(w_{2n}, w_{2n}, w_{2n+1}) \right. \right. \\
&\quad \left. \left. + S^{*P}(w_{2n}, w_{2n}, w_{2n+1}) + S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1}) \right. \right. \\
&\quad \left. \left. - S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1}) + S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+1}) \right] \right\} \\
&= S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}).
\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}
\pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) &\leq \pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) \\
&\quad - \gamma(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})).
\end{aligned} \tag{31}$$

It means that $\gamma(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) = 0$ and $w_{2n+1} = w_{2n+2}$. Following the similar arguments, we obtain $w_{2n+2} = w_{2n+3}$, and hence, w_{2n} becomes a common fixed point of $\mathfrak{f}$ and $\mathfrak{f}$.

By taking $S^{*P}(w_{2n}, w_{2n}, w_{2n+1}) > 0$ for $n = 1, 2, 3, \dots$, now, we consider

$$\begin{aligned}
\pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) &= \pi(S^{*P}(\mathfrak{f}w_{2n}, \mathfrak{f}w_{2n}, \mathfrak{g}w_{2n+1})) \\
&\leq \pi(M(w_{2n}, w_{2n}, w_{2n+1})) \\
&\quad - \gamma(M(w_{2n}, w_{2n}, w_{2n+1})), \\
M(w_{2n}, w_{2n}, w_{2n+1}) &= \max\{S^{*P}(w_{2n}, w_{2n}, w_{2n+1})\} \\
&\quad \{S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})\}.
\end{aligned} \tag{32}$$

Now, if $S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}) \geq S^{*P}(w_{2n}, w_{2n}, w_{2n+1})$, for some $n = 0, 1, 2, \dots$, then $M(w_{2n}, w_{2n}, w_{2n+1}) = S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})$, and from (32), we have

$$\begin{aligned}
\pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) &\leq \pi(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})) \\
&\quad - \gamma(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2})),
\end{aligned} \tag{33}$$

which implies that $\gamma(S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}))$ is a contradiction. Therefore, for all $n \geq 0$,

$$S^{*P}(w_{2n+1}, w_{2n+1}, w_{2n+2}) \leq S^{*P}(w_{2n}, w_{2n}, w_{2n+1}). \tag{34}$$

Similarly, we have

$$S^{*P}(w_{2n}, w_{2n}, w_{2n+1}) \leq S^{*P}(w_{2n-1}, w_{2n-1}, w_{2n}), \tag{35}$$

for all $n \geq 0$. Hence, we obtain

$$S^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) \leq S^{*P}(w_n, w_n, w_{n+1}). \tag{36}$$

Also, $\{S^{*P}(w_{n+1}, w_{n+1}, w_{n+2})\}$ is a nonincreasing sequence; then, there exists $\mathfrak{L}$ such that

$$\lim_{n \rightarrow \infty} S^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) = \mathfrak{L}. \tag{37}$$

Hence, by the lower semicontinuity of γ ,

$$\gamma(\mathfrak{L}) \leq \liminf_{n \rightarrow \infty} \gamma(S^{*P}(w_n, w_n, w_{n+1})). \tag{38}$$

Now, we claim that $\mathfrak{L} = 0$. By lower semicontinuity of γ , taking the upper limit as $n \rightarrow \infty$ on either side of

$$\begin{aligned}
\pi(S^{*P}(w_{n+1}, w_{n+1}, w_{n+2})) &\leq \pi(M(w_n, w_n, w_{n+1})) \\
&\quad - \gamma(M(w_n, w_n, w_{n+1})),
\end{aligned} \tag{39}$$

then we obtain

$$\pi(\mathfrak{L}) \leq \pi(\mathfrak{L}) - \liminf_{n \rightarrow \infty} \gamma(M(w_n, w_n, w_{2n+1})) \leq \pi(\mathfrak{L}) - \gamma(\mathfrak{L}), \tag{40}$$

and this implies that $\gamma(\mathfrak{L}) \leq 0 \Rightarrow \gamma(\mathfrak{L}) = 0$; then, we have

$$\lim_{n \rightarrow \infty} S^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) = 0. \quad (41)$$

To show that $\{w_{n+1}\}$ is a S^{*P} -Cauchy sequence for each n, m , and $n, m > N$, we have

$$\begin{aligned} S^{*P}(w_n, w_n, w_m) &\leq 2S^{*P}(w_n, w_n, w_{n+1}) + 2S^{*P}(w_{n+1}, w_{n+1}, w_{n+2}) \\ &\quad + 2S^{*P}(w_{n+2}, w_{n+2}, w_{n+3}) + \cdots + 2S^{*P}(w_m, w_m, w_{m+1}) \\ &\quad - \{+2S^{*P}(w_{n+1}, w_{n+1}, w_{n+1}) + \cdots + 2S^{*P}(w_{m+1}, w_{m+1}, w_{m+1})\}. \end{aligned} \quad (42)$$

By taking the limit as $n, m \rightarrow \infty$ on both sides of the above inequality and from (41), we have

$$\lim_{n, m \rightarrow \infty} S^{*P}(w_n, w_n, w_m) = 0. \quad (43)$$

It follows that $\{w_n\}$ is a S^{*P} -Cauchy sequence and by S^{*P} -completeness of Y , so there exist $r \in Y$ such that $\{w_n\}$ converges to r as $n \rightarrow \infty$. Now, we will distinguish the cases (a) and (a) of this theorem.

(a) Suppose g is continuous; since $w_{2n+1} \rightarrow r$, we obtain that

$$w_{2n+1} = g(w_{2n+1}) = g(r). \quad (44)$$

However, $w_{2n+1} \rightarrow r$, as a subsequence of $\{w_n\}$. It follows that $g(r) = r$, and from the beginning of the prove, we get $g(r) = r = f(r)$. The proof, assuming that f is continuous, is similar to above.

(b) Suppose that $S^{*P}(r, r, g(r))$, and for $\{w_n\}$ and a nondecreasing sequence with $w_n \rightarrow r$ in Y , we indicate that $w_{2n+1} \leq r, \forall n \in N$. Therefore, from (24), we have

$$\begin{aligned} \pi(S^{*P}(w_{2n+1}, w_{2n+1}, g(r))) &= \pi(S^{*P}(fw_{2n}, fw_{2n}, g(r))) \\ &\leq \pi(M(w_{2n}, w_{2n}, r)) - \gamma(M(w_{2n}, w_{2n}, r)), \end{aligned} \quad (45)$$

where

$$\begin{aligned} M(w_{2n}, w_{2n}, r) &= \max\{S^{*P}(w_{2n}, w_{2n}, r), S^{*P}(w_{2n}, w_{2n}, fw_{2n}), \\ &\quad S^{*P}(r, r, gr), \frac{1}{2}[S^{*P}(w_{2n}, w_{2n}, gr) + S^{*P}(r, r, fw_{2n})]\} \\ &= \max\{S^{*P}(w_{2n}, w_{2n}, r), S^{*P}(w_{2n}, w_{2n}, w_{2n+1}), \\ &\quad S^{*P}(r, r, gr), \frac{1}{2}[S^{*P}(w_{2n}, w_{2n}, gr) + S^{*P}(r, r, w_{2n+1})]\}. \end{aligned} \quad (46)$$

Taking limit as $n \rightarrow \infty$ implies

$$\lim_{n \rightarrow \infty} M(w_{2n}, w_{2n}, r) = S^{*P}(r, r, gr). \quad (47)$$

Hence,

$$\begin{aligned} \pi(S^{*P}(r, r, gr)) &= \lim_{n \rightarrow \infty} \sup \pi(S^{*P}(fw_{2n}, fw_{2n}, gr)) \\ &\leq \lim_{n \rightarrow \infty} \sup [\pi(M(w_{2n}, w_{2n}, r)) \\ &\quad - \gamma(M(w_{2n}, w_{2n}, r))]. \end{aligned} \quad (48)$$

This a contradiction. Thus, we have

$$S^{*P}(r, r, gr) = 0 \Rightarrow r = fr = gr. \quad (49)$$

□

Corollary 1. Let $(Y, \leq)$ be a partially ordered set with f and g as weakly increasing self-mapping on a complete S^{*P} -partial metric space. Suppose that there exists $\gamma \in \Gamma$ such that

$$S^{*P}(f(w), f(w), g(t)) \leq M(w, w, t) - \gamma(M(t, t, w)), \quad (50)$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) &= \max\{S^{*P}(w, w, t), S^{*P}(w, w, f(w)), S^{*P}(t, t, g(t))\} \\ &\quad + \frac{1}{2}[S^{*P}(w, w, g(t)) + S^{*P}(t, t, f(w))]. \end{aligned} \quad (51)$$

Assume that one of the following cases is satisfied:

- (a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \leq r$ for all $n \in \mathbb{N}$
- (b) $\mathbf{f}$ or $\mathbf{g}$ is continuous

Therefore, the maps $\mathbf{f}$ or $\mathbf{g}$ have a common fixed point.

Proof. Put $\pi(t) = t$ in Theorem 2. $\square$

Corollary 2. Let $(Y, \leq)$ be a partially ordered set with $\mathbf{f}$ and $\mathbf{g}$ as weakly increasing self-mapping on a complete S^{*P} -partial metric space. Suppose that there exist $\pi \in \Pi$ and $\gamma \in \Gamma$ such that

$$\pi(S^{*P}(\mathbf{f}(w), \mathbf{f}(w), \mathbf{g}(t))) \leq \pi(M(w, w, t)) - \gamma(M(t, t, w)), \quad (52)$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) &= \zeta_1 S^{*P}(w, w, t) + \zeta_2 S^{*P}(w, w, \mathbf{f}(w)) \\ &\quad + \zeta_3 S^{*P}(t, t, \mathbf{g}(t)) \\ &\quad + \zeta_4 [S^{*P}(w, w, \mathbf{g}(t)) + S^{*P}(t, t, \mathbf{f}(w))], \end{aligned} \quad (53)$$

where $\zeta_i > 0$ for $i = \{1, 2, 3, 4\}$ with $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \geq 1$.

Assume that one of the following cases is satisfied:

- (a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \leq r$ for all $n \in \mathbb{N}$
- (b) $\mathbf{f}$ or $\mathbf{g}$ is continuous

Therefore, the maps $\mathbf{f}$ or $\mathbf{g}$ have a common fixed point.

Proof. A corollary is S^{*P} -partial metric spaces version of Theorem 1. $\square$

Corollary 3. Let $(Y, \leq)$ be a partially ordered set with $\mathbf{f}$ and $\mathbf{g}$ as weakly increasing self-mapping on a complete S^{*P} -metric space. Suppose that there exist $\gamma \in \Gamma$ such that

$$S^{*P}(\mathbf{f}(w), \mathbf{f}(w), \mathbf{g}(t)) \leq M(w, w, t) - \gamma(M(t, t, w)), \quad (54)$$

for all $w, t \in Y$, where

$$\begin{aligned} M(w, w, t) &= \zeta_1 S^{*P}(w, w, t) + \zeta_2 S^{*P}(w, w, \mathbf{f}(w)) \\ &\quad + \zeta_3 S^{*P}(t, t, \mathbf{g}(t)) \\ &\quad + \zeta_4 [S^{*P}(w, w, \mathbf{g}(t)) + S^{*P}(t, t, \mathbf{f}(w))], \end{aligned} \quad (55)$$

where $\zeta_i > 0$ for $i = \{1, 2, 3, 4\}$ with $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \geq 1$.

Assume that one of the following cases is satisfied:

- (a) If a nondecreasing sequence $\{w_n\}$ converges to $r \in Y$ implies $\{w_n\} \leq r$ for all $n \in \mathbb{N}$
- (b) $\mathbf{f}$ or $\mathbf{g}$ is continuous

Therefore, the maps $\mathbf{f}$ or $\mathbf{g}$ have a common fixed point [14].

Proof. Put $\pi(t) = t$ in Corollary 2. $\square$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

Researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding publication of this project.

References

- [1] R. J. Shahkoobi and A. Razani, "Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces," *Journal of Inequalities and Applications* Volume, vol. 2014, p. 373, 2014.
- [2] S. G. Matthews, "Partial metric topology," *Annals of the New York Academy of Sciences*, vol. 728, no. 1, pp. 183–197, 1994.
- [3] S. Sedghi, N. Shobe, and H. Zhou, "A common fixed point theorem in D-metric spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 27906, 13 pages, 2007.
- [4] B. C. Dhage, "Generalized metric spaces mappings with fixed point," *Bulletin of the Calcutta Mathematical Society*, vol. 84, no. 4, pp. 329–336, 1992.
- [5] S. Sedghi, N. Shobe, and A. Aliouche, "A generalization of fixed point theorems in S-metric spaces," *Matematicki Vesnik*, vol. 64, no. 5, pp. 258–266, 2012.
- [6] Z. Mustafa, R. J. Shahkoobi, V. Parvaneh, Z. Kadelburg, and M. M. M. Jaradat, "Ordered S_p -metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems," *Fixed Point Theory and Applications*, vol. 2019, 2019.
- [7] H. Aydi, M. Abbas, and C. Vetro, "Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces," *Topology and its Applications*, vol. 159, no. 14, pp. 3234–3242, 2012.
- [8] L. Ćirić, B. Samet, H. Aydi, and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2398–2406, 2011.
- [9] S. Romaguera, "A Kirk type characterization of completeness for partial metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 493298, 7 pages, 2010.
- [10] Z. Mustafa and S. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–2663, 2006.
- [11] M. R. A. Zhan and A. D. Nezhad, "A generalization of partial metric spaces," *Journal of Contemporary Applied Mathematics*, vol. 24, pp. 86–93, 2011.
- [12] B. C. Dhage, "A common fixed point principle in D-metric spaces," *Bulletin of the Calcutta Mathematical Society*, vol. 91, no. 6, pp. 475–480, 1999.
- [13] M. A. Barakat and A. M. Zidan, "A common fixed point theorem for weak contractive maps in G^p -metric spaces," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 2, pp. 309–314, 2015.