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Research Article

Topological Approaches for Rough Continuous Functions with Applications

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In this paper, we purposed further study on rough functions and introduced some concepts based on it. We introduced and investigated the concepts of topological lower and upper approximations of near-open sets and studied their basic properties. We defined and studied new topological neighborhood approach of rough functions. We generalized rough functions to topological rough continuous functions by different topological structures. In addition, topological approximations of a function as a relation were defined and studied. Finally, we applied our approach of rough functions in finding the images of patient classification data using rough continuous functions.

1. Introduction

Many studies have appeared recently and dealt with generalizations of topological near-open sets [1, 2] and the possibility of using them in many life applications, including their use in data reduction and reaching some new decisions and conclusions. Rough set theory is a modern approach for reasoning about data [3–7]. This theory depends on a certain topological structure that achieved great success in many areas of real-life applications [8–14]. Now, the general topologists can say, “rough sets theory is a topological bridge from real-life problems to computer science” [15, 16].

Rough set theory was introduced as a novel approach to processing of incomplete data. Among the aims of the rough set theory is a description of imprecise concepts. Suppose we are given a finite nonempty set U of elements, called universe. Each element of U is characterized by a description, for example, a set of attribute values. In rough sets formulated by Pawlak, an equivalence relation on the universe of elements is determined based on their attribute values. In particular, this equivalence relation is initiated using the equality relation on the attribute values. Many real-world applications have both nominal and continuous attributes [17–19]. It was early recognized that standard rough set model based on the indiscernibility relation is well suited in the case of nominal attributes.

Several procedures were made to overcome limitations of this approach and many authors presented interesting extensions of the initial model (see, for example, [20–24]). It was noted that considering a similarity relation instead of an indiscernibility relation is quite relevant. A binary relation forming classes of objects, which are identical or at least not noticeably different in terms of the available description, can represent the similarities between objects [25–29]. More recent approaches of rough set with its applications can be found in [30–32]. Other rough set theory applications in computer science (field of information retrievals) using topological generalizations can be found in [33–40].

In this paper, we purpose further study on rough functions and introduce new concepts based on rough functions. In Section 2, we give more details regarding the fundamentals of near-open sets. The goal of Section 3 is to introduce the concepts of topological lower and upper approximations of near-open sets and discuss their basic properties. We spotlight on rough numbers in Section 4. We aim in Section 5 to define and study new topological neighborhood approach of rough functions. Section 6 is devoted to generalize the concept of rough function to topological rough function by using different topological structures. Topological approximations of a function as a relation are defined and studied in Section 7. In Section 8, we suggest some applications of rough functions to information systems and give some applications of them in data retrieval. Finally, conclusions of the work are given in Section 9.

2. Basic Concepts of Topological Near-Open Sets

In this part, we recall the definitions of some near-open subsets of a topological space which are useful in the sequel.

A subfamily τ of the power set of U is called a topology if it contains $\varnothing, U$ as well as it is closed under arbitrary union and finite intersection. The pair (U, τ) is called a topological space; elements in τ are called open sets, and their complements are called closed sets.

For a subset of U, $\overline{A}$, A^o, and A^c denote respectively the closure, interior, and complement of A in U, respectively.

A subset A of (U, τ) is called,

(1) Semi-open (resp., pre-open, open) set if $A \subseteq \overline{(A^o)}$ (resp., $A \subseteq (\overline{A})^o$, $A \subseteq (\overline{(A^o)})^o$) and its complement is called a semi-closed (resp., pre-closed, closed) set if $(\overline{A})^o \subseteq A$ (resp., $\overline{(A^o)} \subseteq A$, $\overline{((\overline{A})^o)} \subseteq A$). A subset which is both semi-open and semi-closed is called semi-regular

(2) Semi-pre-open set (or open set) if $A \subseteq \overline{(\overline{A})^o}$ and it is called a semi-pre-closed set (or β closed set) if $(\overline{(A^o)})^o \subseteq A$

(3) Regular-open set if $A \subseteq (\overline{A})^o$ and it is called a regular-closed set if $\overline{(A^o)} = A$

(4) δ-closed set if $A = \overline{\delta(A)}$, where $\overline{\delta(A)} = \{x \in U\colon (\overline{G})^o \cap A \neq \varphi, x \in G, G \in \tau\}$

The α-closure (resp. semi-closure, semi-pre-closure) of a subset A of (U, τ) is the intersection of all α–closed (resp. semi-closed, semi-pre-closed) sets that contain and is denoted by $\alpha(\overline{A})$ (resp., $S(\overline{A})$, $sp(\overline{A})$). The semi-interior of A, denoted by $s(A^o)$, is the union of all semi-open subsets of U.

A subset A of a topological space (U, τ) is called

(1) Generalized closed set if $\overline{A} \subseteq G$ whenever $A \subseteq G$ and $G \in \tau$.

(2) Semi-generalized closed (briefly, sg-closed) set if $s(\overline{A}) \subseteq G$ whenever $A \subseteq G$ and G is semi-open set. Its complement is called a sg-open set.

(3) Generalized semi-closed set if $s(\overline{A}) \subseteq G$ whenever $A \subseteq G$ and $G \in \tau$.

(4) α-Generalized closed set if $\alpha(\overline{A}) \subseteq G$ whenever $A \subseteq G$ and $G \in \tau$.

(5) Generalized α-closed set if $\alpha(\overline{A}) \subseteq G$ whenever $A \subseteq G$ and G is α- open.

(6) $g\alpha^{**}$-closed set if $\overline{A} \subseteq \overline{(G^o)}$ whenever $A \subseteq G$ and G is α- open.

3. Topological Near-Open Approach of Rough Approximations

In this section, we introduce and investigate the concepts of topological lower and upper approximations of near-open sets and study their basic properties.

Let (U, τ) be a topological space. If $X \subseteq U$, then

(1) Semi-lower approximation of $X \subseteq U$, $\underline{X}_s = \cup\{G\colon G \in \mathrm{Semi}(U), G \subseteq X\}$, where $\mathrm{Semi}(U)$ is the family of all semi-open sets in (U, τ).

If we replace the family of all semi-open sets $\mathrm{Semi}(U)$ given in (1) above by a family of all pre-open sets $\mathrm{Pre}(U)$ (resp., a family of all α- open sets $\alpha(U)$, a family of all β-open sets $\beta(U)$, a family of all regular-open sets $\mathrm{Reg}(U)$, and a family of semi-regular-closed sets $\mathrm{SReg}(U)$), we obtain pre-lower approximation (resp., α-lower approximation, β-lower approximation, regular-lower approximation, and semi-regular-lower approximation).

(2) Semi-upper approximation of $X \subseteq U, \overline{X}_s = \cap\{F\colon F \in \mathrm{CSemi}(U), F \cap X \neq \phi\}$, where $\mathrm{CSemi}(U)$ is the set of all semi-closed sets in (U, τ).

If we replace the family of all semi-closed sets $\mathrm{CSemi}(U)$ given in (2) above by a family of all pre-closed sets $\mathrm{CPre}(U)$ (resp., a family of all α-closed sets $C\alpha(U)$, a family of all β-closed sets $C\beta(U)$, a family of all regular-closed sets $\mathrm{CReg}(U)$, and a family of semi-regular-open sets $\mathrm{SReg}(U)$), we obtain pre-upper approximation (resp., α-upper approximation, β-upper approximation, regular-upper approximation, and semi-regular-upper approximation).

Motivation for topological rough set theory has come from the need to represent subsets of a universe in terms of topological classes of the topological base generated by the general binary relation defined on the universe. That base characterizes a topological space, called topological approximation space, $\mathrm{App}_\tau = (U, R, \tau_R)$. The topological classes of R are also known as the topological granules, topological elementary sets, or topological blocks; we will use $G_{xm} \in \tau$ to denote the topological class containing $x \in U$. In the topological approximation space, we consider two operators $\underline{R}_m(X) = \{x \in U\colon G_{xm} \subseteq X\}$ and $\overline{R}_m(X) = \{x \in U\colon G_{xm} \cap X \neq \phi\}$ called the topological lower approximation and topological upper approximation of $X \subseteq U$, respectively. Also, let $\mathrm{POS}_m(X) = \underline{R}_m(X)$ denote the topological positive region of $X \subseteq U$, $\mathrm{NEG}_m(X) = U - \overline{R}_m(X)$ denotes the topological negative region of $X \subseteq U$, and $\mathrm{BON}_m(X) = \overline{R}_m(X) - \underline{R}_m(X)$ denotes the topological borderline region of $X \subseteq U$.

The degree of topological completeness characterizes by the topological accuracy measure, in which $|X|$ represents the cardinality of set $X \subseteq U$ as follows:

$$\alpha_m(X) = \frac{|\underline{R}_m(X)|}{|\overline{R}_m(X)|}, \quad X \neq \varphi. \tag{1}$$

We define here the semi-rough pairs as an example of topological rough sets and we study their properties. You can use any type of the abovementioned near-open sets as another example.

The semi-topological class on a topological approximation space $\mathrm{App}_\tau = (U, R, \tau_R)$ is determined by $(\underline{X}_s, \overline{X}_s) = \{A \subset U \colon \underline{X}_s \subset A \subset \overline{X}_s\}$. A subset $X \subset U$ is said to be semi-dense (semi-co-dense) if $\overline{X}_s = U$ $(\underline{X}_s = \varphi)$. By semi-rough pair on $\mathrm{App}_\tau = (U, R, \tau_R)$, we mean any pair (P, Q) where $P, Q \subseteq U$ satisfies the conditions:

(Semi-1) P is the semi-open set in τ_R.

(Semi-2) Q is the semi-closed set in τ_R.

(Semi-3) $P \subset Q$.

(Semi-4) there is a subset $S \subset U$ such that

(1) $S^o_s = \varphi$,
(2) $S \subset Q - \overline{P}_s$,
(3) $Q - \overline{P}_s \subset \overline{S}_s$.

Lemma 1. *For any subset $A \subset U$ in the topological approximation space $App_\tau = (U, R, \tau_R)$, the pair $(\underline{A}_s, \overline{A}_s)$ is a semi-rough pair on $App_\tau = (U, R, \tau_R)$ in which every semi-open set in U is a semi-closed set.*

Proof. Let $P = A^o_s$ and $Q = \overline{A}_s$. Then, the conditions from (Semi-1) to (Semi-3) are directly satisfied. Now, we need to prove condition (Semi-4). Define $S = A - \overline{P}_s$, then we have

(1) If $O \subseteq S$ is a semi-open set, hence $O \subseteq A$ that gives $O \cap P = \varphi$ which is a contradiction; hence, O is not contained in A. Then, it must be $S = \varphi$ which gives $S^o_s = \varphi$.

(2) Since $S = A - \overline{P}_s$, $A \subseteq \overline{A}_s$, then $A \subset Q$. Then, we have $S \subset Q - \overline{P}_s$.

(3) Let $x \in Q - \overline{P}_s$, $Q = \overline{A}_s$, this means that $x \in A$ or $x \notin A$. If $x \in A$, then for every semi-open set O and $x \in O$ such that $O \cap A \neq \varphi$ implies that $O \cap S \neq \varphi$ and we have $x \in \overline{S}_s$, then $Q - \overline{P}_s \subset \overline{S}_s$. If $x \notin A$, then there is a semi-open set O' and $x \in O'$. Now, $O' - \overline{P}_s = O' \cap [\overline{P}_s]^c$ is a semi-open set which contains x, and $x \in \overline{A}_s$, then there exists a point $y \in A$ such that $y \in O' - \overline{P}_s$, hence $y \in O' \cap S$; therefore, $O' \cap S \neq \varphi$, hence $x \in \overline{S}_s$. Then, we have the result $Q - \overline{P}_s \subset \overline{S}_s$. □

Lemma 2. *For any semi-rough pair (P, Q) in $App_\tau = (U, R, \tau_R)$ in which every semi-open subset is semi-closed, there are subsets $A, B \subseteq U$ such that $P = \overline{A}_s$ and $Q = \overline{B}_s$.*

Proof. Let (P, Q) be a semi-rough pair and let S be any subset, satisfying condition (Semi-4). Define $A = P \cup S$, then $P \subset A$, hence $P \subset A^o_s$. If $O \subset A$ is a semi-open set, then $O - \overline{P}_s = O \cap [\overline{P}_s]^c$ is another semi-open set contained in A. Since $O \subset A = P \cup S$, $P \subset \overline{P}_s$, then $O \subset \overline{P}_s \cup S$, and we have $O - \overline{P}_s \subset [(\overline{P}_s \cup S) - \overline{P}_s] = S$. Therefore, $O - \overline{P}_s$ is a semi-open set contained in S which means $O \subset \overline{P}_s$. Since $O \subset P \cup S$, it follows that $O \subset P$ and this proves that $P = \overline{A}_s$.

Now, we have $\overline{P}_s \cup \overline{S}_s^{\ \overline{P}_s \cup [Q - \overline{P}_s] = Q}$. Also, $B = P \cup S^{\overline{P}_s \cup (Q - \overline{P}_s) = Q}$ and hence $\overline{B}_s \subset \overline{Q}_s = Q$. Then, we have $Q = \overline{B}_s$. □

Theorem 1. *For any topological subspace $(X, \tau^*), X \subseteq U$ of the topological approximation space $App_\tau = (U, R, \tau_R)$, the function $f \colon (X, \tau^*) \longrightarrow (U, \tau_R)$ that defined by $f(A) = (A^o_s, \overline{A}_s), A \in \tau^*$ is bijection.*

Proof. First, we will prove that the function is onto as follows: for any semi-rough pair $(A^o_s, \overline{A}_s)$ in $\mathrm{App}_\tau = (U, R, \tau_R)$, then there exists $A \in \tau^*$ such that $f(A) = (A^o_s, \overline{A}_s)$. Second, for the proof that f is one to one, if $f(A_1) = f(A_2)$, then $(A^o_{1s}, \overline{A}_{1s}) = (A^o_{2s}, \overline{A}_{2s})$ which implies to $A^o_{1s} = A^o_{2s}$ and $\overline{A}_{1s} = \overline{A}_{2s}$ and $A_1 \approx A_2$. □

4. Topological Neighborhood Approach of Rough Continuity

Let X and Y be two subsets of a universe U, and let $\mathrm{Appr}(X) = (X, S)$ and $\mathrm{Appr}(Y) = (Y, P)$ be two approximation spaces, where S and P are binary relations on X and Y, respectively. We define two subsets $S_r(x) = \{y \in X \colon (x, y) \in S\}$ and $S_l(x) = \{y \in X \colon (y, x) \in S\}$ of X (also two subsets $P_r(x) = \{y \in Y \colon (x, y) \in P\}$ and $P_r(x) = \{y \in Y \colon (y, x) \in P\}$ of Y) which are called right and left neighborhoods of an element $x \in X$. We define now two topologies on X and on Y, respectively, using the intersection of the right and left neighborhoods $S_{r \cap l}(x) = S_r(x) \cap S_l(x)$ and $P_{r \cap l}(x) = P_r(x) \cap P_l(x)$ as follows:

$$\begin{aligned} \tau_X &= \{A \subseteq X \colon \forall a \in A, S_{r \cap l}(a) \subseteq A\}, \\ \tau_Y &= \{B \subseteq Y \colon \forall b \in B, P_{r \cap l}(b) \subseteq B\}. \end{aligned} \quad (2)$$

The rough approximations using these topologies are defined as follows:

$$\begin{aligned} \underline{P}_{\tau_Y}(B) &= \cup \{G' \in \tau_Y \colon G' \subseteq B\}, \\ \underline{S}_{\tau_X}(A) &= \cup \{G \in \tau_X \colon G \subseteq A\}, \\ \overline{S}_{\tau_X}(A) &= \cap \{F \colon F^c \in \tau_X \colon A \subseteq F\}, \\ \overline{P}_{\tau_Y}(B) &= \cap \{F' \colon F'^c \in \tau_Y \colon B \subseteq F'\}. \end{aligned} \quad (3)$$

The function $f \colon (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is called a rough function on X if the image of each rough set in X is rough in Y.

Namely, the function f is totally rough iff all subsets $A \subset X, A \neq \varphi$, such that $\underline{S}_{\tau_X}(A) \neq \overline{S}_{\tau_X}(A)$, then $\underline{P}_{\tau_Y}(f(\underline{S}_{\tau_X}(A))) \neq \overline{P}_{\tau_Y}(f(\overline{S}_{\tau_X}(A)))$ in Y.

The function f is possibly rough iff some subsets $A \subset X, A \neq \varphi$, such that $\underline{S}_{\tau_X}(A) \neq \overline{S}_{\tau_X}(A)$, then $\underline{P}_{\tau_Y}(f(\underline{S}_{\tau_X}(A))) \neq \overline{P}_{\tau_Y}(f(\overline{S}_{\tau_X}(A)))$ in Y.

Finally, the function f is exact iff all subsets $A \subset X, A \neq \varphi$, such that $\underline{S}_{\tau_X}(A) = \overline{S}_{\tau_X}(A)$, then $\underline{P}_{\tau_Y}(f(\underline{S}_{\tau_X}(A))) = \overline{P}_{\tau_Y}(f(\overline{S}_{\tau_X}(A)))$ in Y.

The function $f \colon (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is a topological rough, continuous function on X as the following:

(1) The function f is topological, totally rough, continuous if for all subsets $A \subset Y, A \neq \varphi$; if $(A)^o_{\tau_Y} \subseteq \overline{(A)}_{\tau_Y}$, then $(f^{-1}(\overline{(A)}_{\tau_Y}))^o_{\tau_X} \subseteq \overline{(f^{-1}((A)^o_{\tau_Y}))}_{\tau_X}$ in X.

(2) The function f is topological, possibly rough, continuous if for some subsets $A \subset Y, A \neq \varphi$; if $(A)^o_{\tau_Y} \subseteq \overline{(A)}_{\tau_Y}$, then $(f^{-1}(\overline{(A)}_{\tau_Y}))^o_{\tau_X} \subseteq \overline{(f^{-1}((A)^o_{\tau_Y}))}_{\tau_X}$ in X.

(3) Finally, the function f is topological exact continuous if for all subsets $A \subset Y, A \neq \varphi$; if $(A)^o_{\tau_Y} = \overline{(A)}_{\tau_Y}$, then $(f^{-1}(\overline{(A)}_{\tau_Y}))^o_{\tau_X} = \overline{(f^{-1}((A)^o_{\tau_Y}))}_{\tau_X}$ in X.

Example 1. Let (X, τ_X) and (Y, τ_Y) be topological spaces, where $X = \{a, b, c\}$ and $\tau_X = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and $Y = \{1, 2, 3\}$, $\tau_Y = \{Y, \varphi, \{1\}, \{2\}, \{1, 2\}\}$. Let $f\colon X \longrightarrow Y$ be a map defined by $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$, then our results are given in Table 1.

Then, according to Table 1, the function f is a topological totally rough continuous function.

Table 1: Calculations of topological rough continuous functions.

Subsets of Y/our measures	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	Y
$(A)^o_{\tau_Y}$	$\{1\}$	$\{2\}$	φ	$\{1, 2\}$	$\{1\}$	$\{2\}$	Y
$\overline{(A)}_{\tau_Y}$	$\{1, 3\}$	$\{2, 3\}$	$\{3\}$	Y	$\{1, 3\}$	$\{2, 3\}$	Y
$f^{-1}((A)^o_{\tau_Y})$	$\{a\}$	$\{b\}$	φ	$\{a, b\}$	$\{a\}$	$\{b\}$	X
$f^{-1}(\overline{(A)}_{\tau_Y})$	$\{a, c\}$	$\{b, c\}$	$\{c\}$	X	$\{a, c\}$	$\{b, c\}$	X
$(f^{-1}(\overline{(A)}_{\tau_Y}))^o_{\tau_Y}$	$\{a\}$	$\{b\}$	φ	X	$\{a\}$	$\{b\}$	X
$\overline{(f^{-1}(A)^o_{\tau_Y})}_{\tau_X}$	$\{a, c\}$	$\{b, c\}$	φ	X	$\{a, c\}$	$\{b, c\}$	X

Proposition 1. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f\colon X \longrightarrow Y$ be a function. The following are equivalent:*

(1) f is rough continuous.

(2) For every $F \subseteq Y$, $f^{-1}(\overline{(F)}_{\tau_Y}) \in \tau^c_X$.

(3) For every $x \in X$, f is a rough continuous at x.

(4) For every $A \subseteq X$, $f(\overline{A}_{\tau_X}) \subseteq \overline{f(A)}_{\tau_Y}$.

Proof. We will use the sequence (3) implying (1) implying (4) implying (2) implying (3) to prove the equivalence of the proposition.

(3) implying (1): suppose a nonempty open set $V \in \tau_Y$, for a fixed point $x \in f^{-1}(V)$, we have $f(x) \in V$. But since f is rough continuous at x, then there exists an open set $G_x \subseteq X$ such that $f(G_x) \subset V$ and $(f(G_x))^o_{\tau_Y} \subseteq \overline{(f(G_x))}_{\tau_Y}$, then we have $(f^{-1}(\overline{f(G_x)}_{\tau_Y}))^o_{\tau_X} \subseteq \overline{(f^{-1}((f(G_x))^o_{\tau_Y}))}_{\tau_X}$ and $G_x \in f^{-1}(V)$; this gives that f is rough continuous.

(1) implying (4): suppose that f is rough continuous and let $A \subseteq X$. Let $x \in \overline{A}_{\tau_X}$. Let an open set $V \in \tau_Y$ such that $x \in f^{-1}(V)$. Then, by the definition of rough upper approximation $f^{-1}(V) \cap A \neq \varphi$. Let $x' \in f^{-1}(V) \cap A$, then $f(x') \in V \cap f(A)$. Then, we have $V \cap f(A) \neq \varphi$. Then, we have $f(\overline{A}_{\tau_X}) \subseteq \overline{f(A)}_{\tau_Y}$.

(4) implying (2): fix a closed subset $F \subseteq Y$; let $A = f^{-1}(F)$; we will prove that $A = \overline{A}_{\tau_X}$. But each subset is contained in its upper approximation, $A \subset \overline{A}_{\tau_X}$. Now, we will prove that $\overline{A}_{\tau_X} \subset A$. Let $x \in \overline{A}_{\tau_X}$, then using (4), we have $f(x) \in f(\overline{A}_{\tau_X}) \subseteq \overline{f(A)}_{\tau_Y} \subseteq \overline{F}_{\tau_Y} = F$; hence $f(x) \in F$ or $x \in f^{-1}(F) = A$. Then, we have $f^{-1}(\overline{(F)}_{\tau_Y}) \in \tau^c_X$.

(2) implying (3): let $x \in X$ and $V \in \tau_Y$ be an open set containing $f(x)$. Then, $Y - V$ is a closed set and $f^{-1}(Y - V)$ is a closed set in X which does not contain the point x. But $x \in X - (f^{-1}(Y - V))$. Then, there exists an open set G containing x such that $x \in G \subseteq X - (f^{-1}(Y - V))$, then $f(G) \subseteq f(X - (f^{-1}(Y - V))) = f(X) - (Y - V) \subseteq V$. Then, f is rough continuous at x. □

Theorem 2. *Suppose that $(\tau_i)_X, i = 1, 2, 3, \ldots$ be a family of topologies defined on X. Let $f\colon X \longrightarrow Y$ be a rough continuous function for every $\tau_i, \forall i$ where (Y, τ_Y) is a topological space. Then, f is a rough continuous function with respect to the topology $\tau_X = (\cap_i \tau_i)_X$.*

Proof. Let $G \in \tau_Y$, then $(G)^o_{\tau_Y} \subseteq \overline{(G)}_{\tau_Y}$, since f is a rough continuous function for every $\tau_i, \forall i$, then $(f^{-1}(\overline{(G)}_{\tau_Y}))^o_{\tau_i} \subseteq \overline{(f^{-1}((G)^o_{\tau_Y}))}_{\tau_i}, \forall i$. Then, we have $(f^{-1}(\overline{(G)}_{\tau_Y}))^o_{\tau_X} \subseteq \overline{(f^{-1}((G)^o_{\tau_Y}))}_{\tau_X}$ in $\tau_X = (\cap_i \tau_i)_X$, hence f is a rough continuous function with respect to the topology $\tau_X = (\cap_i \tau_i)_X$. □

Theorem 3. *Let $f_i\colon X \longrightarrow (Y_i, \tau_i)$ be a family of functions. Suppose that τ_X is the topology on X generated by the class $\beta = \bigcup_i \{f_i^{-1}(G)\colon G \in \tau_i\}$, then*

(1) f_i is rough continuous for each i.

(2) If τ^*_X is the intersection of all topologies on X such that f_i is rough continuous for each i; then, $\tau_X = \tau^*_X$.

(3) τ_X is the coarser topology on X which gives that f_i is rough continuous for each i.

(4) The class $\beta = \cup_i \{f_i^{-1}(G)\colon G \in \tau_i\}$ is a sub-base of τ_X.

(5) The function $g\colon Y \longrightarrow X$ is rough continuous if and only if $f_i \circ g$ is rough continuous.

Proof

Part (1): for each function $f_i\colon X \longrightarrow (Y_i, \tau_i)$ if $F \in \tau_i$ then $(F)^o_{\tau_i} \subseteq \overline{(F)}_{\tau_i}$ and $f_i^{-1}(F) \in \beta$. But $\beta \subseteq \tau_i$, then $f_i^{-1}(F) \in \tau_i$, hence $(f^{-1}(\overline{(F)}_{\tau_i}))^o_{\tau_X} \subseteq \overline{(f^{-1}((F)^o_{\tau_i}))}_{\tau_X}$; then, we have the result.

Part (2): we can easily prove that $\beta \subseteq \tau^*_X$, but the topology τ_X is generated by β, then $\tau_X \subseteq \tau^*_X$. Otherwise, τ_X is one of the topologies that make the functions f_i which are rough continuous. Then, we have $\tau^*_X \subseteq \tau_X$, hence $\tau_X = \tau^*_X$.

Part (3): it is obvious by proof of Part (2).

Part (4): since any collection of subsets of X is a sub-base of a topology on X, then β is a sub-base of the topology τ_X.

Part (5): if the function $g\colon Y \longrightarrow X$ is rough continuous, then all functions $f_i \circ g$ are rough continuous. Otherwise, let $f_i \circ g$ be rough continuous and let $G \in \beta$, then there exists a subset $H \in \tau_i$ such that $G = f_i^{-1}(H)$. But $g^{-1}(G) = g^{-1}(f_i^{-1}(H)) = (f_i \circ g)^{-1}(H)$. Now, we have $(H)^o_{\tau_i} \subseteq \overline{(H)}_{\tau_i}$, then $((f_i \circ g)^{-1}(\overline{(H)}_{\tau_i}))^o_{\tau_X} \subseteq \overline{((f_i \circ g)^{-1}((H)^o_{\tau_i}))}_{\tau_X}$. Then, $f_i \circ g$ is rough continuous. □

5. Minimal Neighborhood Approach for Rough Continuity

We generalize the concept of rough function to topological rough function by using topological structures. The topological spaces with rough sets are very useful in the field of digital topology which is widely applied in the image processing in computer sciences.

Let (X, τ) be a topological space and $x \in X$. Then, we define

$N_{\min}(x) = \cap\{N \subseteq X\colon x \in G \subseteq N, \forall G \in \tau\}$ which is called the minimal neighborhood containing the point x with respect to the topology τ on X. Let (X, τ) be a topological space, for any element $x \in X$; we define the subset $\overline{N}_{\min}(x)$ which is the closure of $N_{\min}(x)$ in (X, τ).

If $f\colon (X, \tau) \longrightarrow (Y, \tau^*)$ is a function between two topological spaces (X, τ) and (Y, τ^*), we define the functions $f_{\min}\colon (X, \tau) \longrightarrow (Y, \tau^*)$ by $f_{\min}(x) = \cap\{M \subseteq Y\colon f(x) \in G' \subseteq M, \forall G' \in \tau^*\}$ for every $x \in X$.

Let $f\colon (X, \tau) \longrightarrow (Y, \tau^*)$ be a function, where X and Y are topological spaces. The function f is called a topological rough function on X if and only if $(N_{\min}(x))^o_\tau \neq \overline{(N_{\min}(x))}_\tau$ for every $x \in X$. Also, f is a topological rough function on Y if $(f_{\min}(x))^o_{\tau^*} \neq \overline{(f_{\min}(x))}_{\tau^*}$ for every point $f(x)$ in Y.

Example 2. Let (X, τ) and (Y, τ^*) be topological spaces, where $X = \{a, b, c\}$ and $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ and $Y = \{1, 2, 3\}$, $\tau^* = \{Y, \varphi, \{1\}, \{2\}, \{1, 2\}\}$. Let $f\colon X \longrightarrow Y$ be a map defined by $f(a) = 2$, $f(b) = 1$ and $f(c) = 3$, then

$$\begin{aligned} N_{\min}(a) &= \{a\}, \\ N_{\min}(b) &= \{a, b\}, \\ N_{\min}(c) &= X, \\ f_{\min}(a) &= \{2\}, \\ f_{\min}(b) &= \{1\}, \\ f_{\min}(c) &= Y. \end{aligned} \tag{4}$$

Then, we have

$$\begin{aligned} (N_{\min}(a))^o_\tau &= \{a\}, \\ \overline{(N_{\min}(a))}_\tau &= X, \\ (N_{\min}(b))^o_\tau &= \{a, b\}, \\ \overline{(N_{\min}(b))}_\tau &= X, \\ (N_{\min}(c))^o_\tau &= X, \\ \overline{(N_{\min}(c))}_\tau &= X. \end{aligned} \tag{5}$$

Also,

$$\begin{aligned} (f_{\min}(a))^o_{\tau^*} &= \{2\}, \\ \overline{(f_{\min}(a))}_{\tau^*} &= \{2, 3\}, \\ (f_{\min}(b))^o_{\tau^*} &= \{1\}, \\ \overline{(f_{\min}(b))}_{\tau^*} &= \{1, 3\}, \\ (f_{\min}(c))^o_{\tau^*} &= \{3\}, \\ \overline{(f_{\min}(c))}_{\tau^*} &= \{3\}. \end{aligned} \tag{6}$$

Then, the function f is not a topological rough function on$X$ and Y.

A function $f\colon (X, \tau) \longrightarrow (Y, \tau^*)$ is said to be topological roughly continuous at the point $x \in X$ if and only if $f^{-1}(N_{\min}(f(x))) \subseteq N_{\min}(x)$, and it is topological roughly continuous on X if it is topological roughly continuous at every point $x \in X$.

Example 3. Let$f\colon (X, \tau) \longrightarrow (Y, \tau^*)$ be a function defined by $f(a) = 2f(b) = f(d) = 3$ and $f(c) = 4$, where $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$ with

$$\tau = \{X, \varphi, \{a\}, \{a,b\}, \{a,b,c\}\},$$
$$\tau^* = \{Y, \varphi, \{1\}, \{2\}, \{1,2\}, \{2,3,4\}\}. \quad (7)$$

Then, f is a topological rough function on X and

$$\begin{aligned}
&N_{\min}(a) = \{a\}, \text{but } N_{\min}(f(a)) = N_{\min}(2) = \{2\} \text{ then } f^{-1}(N_{\min}(2)) = \{a\},\\
&N_{\min}(b) = \{a,b\}, \text{but } N_{\min}(f(b)) = N_{\min}(3) = \{2,3,4\} \text{ then } f^{-1}(N_{\min}(3)) = X,\\
&N_{\min}(c) = \{a,b,c\}, \text{but } N_{\min}(f(c)) = N_{\min}(4) = \{2,3,4\} \text{ then } f^{-1}(N_{\min}(4)) = X,\\
&N_{\min}(d) = X, \text{but } N_{\min}(f(d)) = N_{\min}(3) = \{2,3,4\} \text{ then } f^{-1}(N_{\min}(3)) = X,
\end{aligned} \quad (8)$$

then $f^{-1}(N_{\min}(f(x))) \subseteq N_{\min}(x)$ for every $x \in X$, and then f is a topological rough continuous function on X.

6. Topological Approximations of a Function as a Relation

The function $f\colon X \longrightarrow Y$ is a relation from X to Y when it satisfies the conditions:

(i) $\mathrm{Dom}(f) = X$,

(ii) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

If $X = Y$, we say f is a function on X. By this way, any function $f\colon X \longrightarrow Y$ can completely be represented by its graph $G(f) = \{(x, f(x))\colon x \in X\}$.

Let $f\colon (U_1, R_1) \longrightarrow (U_2, R_2)$ be any function, where $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ are approximation spaces, such that R_1 and R_2 are any binary relations on U_1 and U_2, respectively. We define the relation $R = R_1 \times R_2$ such that $R(x) = R_1(x) \times R_2(x)$ is the blocks of $U_1 \times U_2$. For the function, $G(f) = \{(x, f(x))\colon x \in U_1\}$, we define the approximations

$$\underline{R}(G(f)) = \cup\{G \subseteq R(x)\colon G \in G(f)\},$$
$$\overline{R}(G(f)) = \cap\{G \subseteq R(x)\colon G \cap G(f) = \varphi\}. \quad (9)$$

A function $f\colon U_1 \longrightarrow U_2$ is said to be rough in the approximation space $A = (U, R)$, where $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ are approximation spaces and $A = A_1 \times A_2, U = U_1 \times U_2$ if $\underline{R}(G(f)) = \overline{R}(G(f))$; otherwise, f is an exact function.

Example 4. Let $U_1 = \{a, b, c, d, e\}$ and $U_2 = \{1, 2, 3, 4, 5, 6\}$ be two universes; we define the function $f\colon U_1 \longrightarrow U_2$, by its graph $G(f) = \{(a,1), (a,6), (b,6), (c,5), (c,6), (e,6)\}$. Consider the blocks of the binary relations R_1 and R_2 as follows:

$$R_1(x) = \{\{a,c\}, \{a,b\}, \{d,e\}\},$$
$$R_2(x) = \{\{1,3\}, \{2,4,5\}, \{3,4\}, \{6\}\}. \quad (10)$$

Then,

$$\begin{aligned}
R(x) = R_1(x) \times R_2(x)\\
= \{&(a,1), (a,3), (c,1), (c,3)\},\\
&\{(a,2), (a,4), (a,5), (c,2), (c,4), (c,5)\},\\
&\{(a,3), (a,4), (c,3), (c,4)\},\\
&\{(a,6), (c,6)\},\\
&\{(a,1), (a,3), (b,1), (b,3)\},\\
&\{(a,2), (a,4), (a,5), (b,2), (b,4), (b,5)\},\\
&\{(a,3), (a,4), (b,3), (b,4)\},\\
&\{(a,6), (b,6)\}, \{(d,1), (d,3), (e,1), (e,3)\},\\
&\{(d,2), (d,4), (d,5), (e,2), (e,4), (e,5)\},\\
&\{(d,3), (d,4), (e,3), (e,4)\}, \{(d,6), (e,6)\}.
\end{aligned} \quad (11)$$

Then, we have

$$\underline{R}(G(f)) = \{(a,6), (b,6), (c,6)\},$$
$$\overline{R}(G(f)) = \{(a,1), (a,3), (c,1), (c,3), (a,6), (b,6), (c,6), (a,2), (a,4), (a,5), (c,2), (c,4), (c,5), (d,6), (e,6)\}. \quad (12)$$

Therefore, the function f is a rough function such that $\underline{R}(G(f)) \neq \overline{R}(G(f))$.

When we have two approximation spaces defined by two equivalence relations, we have the following proposition that governs the product space.

Proposition 2. *Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two arbitrary approximation spaces. Then, we have $(U_1 \times U_2)/R_1 \times R_2 = (U_1/R_1) \times (U_2/R_2)$.*

Proof. Suppose that $u_1, u_2 \in U_1$, and $v_1, v_2 \in U_2$, then we have

$$((u_1, v_1), (u_2, v_2)) \in R_1 \times R_2, \quad \text{iff } (u_1, u_2) \in R_1, (v_1, v_2) \in R_2. \tag{13}$$

Suppose again that $[(u_1, v_1)]_{R_1 \times R_2} \in (U_1 \times U_2)/R_1 \times R_2$. Then, we have

$$\begin{aligned}[(u_1, v_1)]_{R_1 \times R_2} &= \{(u_2, v_2): ((u_1, v_1), (u_2, v_2)) \in R_1 \times R_2\} \\ &= \{(u_2, v_2): (u_1, u_2) \in R_1, (v_1, v_2) \in R_2\} \\ &= \{u_2: (u_1, u_2) \in R_1\} \times \{v_2: (v_1, v_2) \in R_2\} \\ &= [u_1]_{R_1} \times [v_1]_{R_2}.\end{aligned} \tag{14}$$

Then, we have the result as $(U_1 \times U_2)/R_1 \times R_2 = (U_1/R_1) \times (U_2/R_2)$.

Let $f: (U_1, R_1) \longrightarrow (U_2, R_2)$ be any function, where $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ are arbitrary approximation spaces. We define the relation $G(f) = \{(x, f(x)): x \in U_1\}$ to be the graph of the function f. The rough approximations of $G(f)$ are defined as follows:

$$\begin{aligned}\underline{R}(G(f)) &= \{(u_1, u_2) \in U_1 \times U_2: [(u_1, u_2)]_R \subseteq G(f), R = R_1 \times R_2\}, \\ \overline{R}(G(f)) &= \{(u_1, u_2) \in U_1 \times U_2: [(u_1, u_2)]_R \cap G(f) \neq \varphi, R = R_1 \times R_2\}.\end{aligned} \tag{15}$$

Accordingly, the function f is rough if $\underline{R}(G(f)) = \overline{R}(G(f))$; otherwise, f is an exact function. The pair $(\underline{R}(G(f)), \overline{R}(G(f)))$ is called a rough pair of relations.

The following theorems give the conditions on approximation spaces that give exact functions, one-to-one, surjective, and continuous functions. □

Theorem 4. *The function $f: U_1 \longrightarrow U_2$ is an exact function for any selective approximation spaces $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$.*

Proof. The selective approximation space property means that $[(u, v)]_R = \{(u, v)\}, u \in U_1, v \in U_2, R = R_1 \times R_2$. Then, we have $\underline{R}(G(f)) = \overline{R}(G(f))$, which yields to that the function f is an exact function. □

Theorem 5. *The function $f: U_1 \longrightarrow U_2$ is one-to-one function for any selective approximation spaces $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ if and only if both $\underline{R}(G(f))$ and $\overline{R}(G(f))$ are one-to-one functions.*

Proof. The proof is directly using the definition of selective approximation space and using the technology in Theorem 1. □

Theorem 6. *The function $f: U_1 \longrightarrow U_2$ is a surjective function for any selective approximation space $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ if and only if both $\underline{R}(G(f))$ and $\overline{R}(G(f))$ are surjective functions.*

Proof. One can prove the theorem using similar technique given in Theorem 1. □

Theorem 7. *The function $f: U_1 \longrightarrow U_2$ is a continuous function for any selective approximation space $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ if and only if both $\underline{R}(G(f))$ and $\overline{R}(G(f))$ are continuous functions.*

Proof. As in the technique used in Theorem 5, when we have two topological spaces, generated using two bases β_{R_1}, β_{R_2}, where $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ are two approximation spaces, then we have the following proposition that governs the product topology. □

Proposition 3. *Let $T_1 = (U_1, \tau_1)$ and $T_2 = (U_2, \tau_2)$ be two arbitrary topological spaces. Then, we have $(U_1 \times U_2)/\beta_{R_1} \times \beta_{R_2} = (U_1/\beta_{R_1}) \times (U_2/\beta_{R_2})$.*

Proof. Similar to the proof of Proposition 2, the rough pairs of relations satisfied the following two important theorems. □

Theorem 8. *For the quasidiscrete product topological space $(U_1 \times U_2, \tau)$, if $(\underline{R}(G(f)), \overline{R}(G(f)))$ is a rough pair of relations, and (Q, τ') is a subspace of $(U_1 \times U_2, \tau)$ such that Q is closed in τ, then $(\underline{R}(G(f)) \cap Q, \overline{R}(G(f)) \cap Q)$ is a relative rough pair of relations when $\underline{R}(G(f)), \overline{R}(G(f)), Q \subset U_1 \times U_2, \underline{R}(G(f)) \subset Q$.*

Proof. The pair $(\underline{R}(G(f)), \overline{R}(G(f)))$ is a rough pair of relations in $(U_1 \times U_2, \tau)$, if the following condition satisfied the following:

(1) $\underline{R}(G(f))$ is an open relation in $(U_1 \times U_2, \tau)$.
(2) $\overline{R}(G(f))$ is a closed relation in $(U_1 \times U_2, \tau)$.
(3) $\underline{R}(G(f)) \subset \overline{R}(G(f))$.
(4) The relation $\underline{R}(G(f)) - \overline{\overline{R}(G(f))}_\tau$ contains a relation S of $U_1 \times U_2$ such that $S^o_\tau = \varphi$ and $\overline{R}(G(f)) - \overline{(\underline{R}(G(f)))}_\tau \subset \overline{S}_\tau$.

Only we need to prove that $(\underline{R}(G(f)) \cap Q, \overline{R}(G(f)) \cap Q)$ is a rough pair of relations in (Q, τ'); the proof will end by

(1) Since $\underline{R}(G(f))$ is an open relation in $(U_1 \times U_2, \tau)$, and (Q, τ') is a subspace of $(U_1 \times U_2, \tau)$, then $\underline{R}(G(f)) \cap Q$ is an open relation in (Q, τ').
(2) Since $\overline{R}(G(f))$ is a closed relation in $(U_1 \times U_2, \tau)$, then there is an open relation S, such that $\overline{R}(G(f)) = U_1 \times U_2 - S$, then $\overline{R}(G(f)) \cap Q = (U_1 \times U_2) \cap Q - S \cap Q = Q - S \cap Q$, but $S \cap Q$ is an open relationship with (Q, τ'), then $\overline{R}(G(f)) \cap Q$ is a closed relation in the subspace (Q, τ').
(3) Since $\underline{R}(G(f)) \subset \overline{R}(G(f))$, then $\underline{R}(G(f)) \cap Q \subset \overline{R}(G(f)) \cap Q$.
(4) By selecting $S = \overline{R \cap Q}, R = R_1 \times R_2$, then the relation $\overline{R}(G(f)) \cap Q - \overline{\underline{R}(G(f))}_\tau$ contains the relation S, and we need to prove the two subconditions:

   (a) $S^o_\tau = \varphi$,
   (b) $\overline{R}(G(f)) - \overline{(\underline{R}(G(f)))}_\tau \subset \overline{S}_\tau$.

For the proof of Part (a) $S^o_\tau = \varphi$, suppose that $S^o_\tau \neq \varphi$, then there is an τ-open relation $G \subset Q$ such that $G \subset S$ but $S = R \cap Q$, i.e., $G \subset R$, but $G = G' \cap Q$ such that G' is an open relation in $(U_1 \times U_2, \tau)$, then $G' \cap Q \subset R$; hence, $(G' \cap Q)^o_\tau \subset R^o_\tau$, but $R^o_\tau = \varphi$, which gives contradiction; then, it must be $S^o_\tau = \varphi$.

For the proof of Part (b), $\overline{R}(G(f)) - \overline{(\underline{R}(G(f)))}_\tau \subset \overline{S}_\tau$.

Since $(\underline{R}(G(f)), \overline{R}(G(f)))$ is a rough pair in $(U_1 \times U_2, \tau)$, then there is a relation $R' \subset U_1 \times U_2$, such that $\underline{R}(G(f)) = R'^o_\tau$ and $\overline{R}(G(f)) = \overline{R'}_\tau$; since $R \subset \overline{R}(G(f)) - \overline{(\underline{R}(G(f)))}_\tau$, we have $S = R \cap Q = R' \cap Q - \overline{(\underline{R}(G(f)))}_\tau$.

Now, let $(u, v) \in \overline{R}(G(f)) \cap Q - \overline{(\underline{R}(G(f))}_\tau$, then $(u, v) \in \overline{R}(G(f)) \cap Q$ and $(u, v) \notin \overline{(\underline{R}(G(f))}_\tau$.

Now, if $(u, v) \in \overline{R}(G(f)) \cap Q$, then $(u, v) \in S$ and $(u, v) \in \overline{S}_\tau$.

Finally, if $(u, v) \in R' \cap Q$ and $(u, v) \in \overline{R}(G(f)) \cap Q$ and $(u, v) \notin \overline{(\underline{R}(G(f))}_\tau$, hence $(u, v) \in \overline{R}(G(f))$ and $(u, v) \in Q$. Now, $(u, v) \in \overline{R'}_\tau$, then there is an open relation G in τ such that $(u, v) \in G$ and $G \cap R' \neq \varphi$, but $(u, v) \notin \overline{(\underline{R}(G(f))}_\tau$, then $(u, v) \in G - \overline{(\underline{R}(G(f))}_\tau = G \cap [\overline{(\underline{R}(G(f)))}_\tau]^c$. But since $\underline{R}(G(f)) = R'^o_\tau$ is an open relation in τ, and $\underline{R}(G(f)) = \underline{R}(G(f)) \cap Q$ is an open relation in τ', then $\overline{(\underline{R}(G(f))}_\tau$ is a closed relation in τ, hence $[\overline{(\underline{R}(G(f))}_\tau]^c$ is an open relation in τ, hence $G \cap [\overline{(\underline{R}(G(f))}_\tau]^c$ is an open relation containing (u, v), then $G \cap [\overline{(\underline{R}(G(f))}_\tau]^c \cap R' \neq \varphi$. This yields to $G \cap [R' - \overline{(\underline{R}(G(f))}_\tau] \neq \varphi$, i.e., $G \cap [R' - \overline{(\underline{R}(G(f))}_\tau] \cap Q \neq \varphi$, such that $(u, v) \in Q$. Hence, $G \cap [R' \cap Q - \overline{(\underline{R}(G(f))}_\tau \cap Q] \neq \varphi$, but $\underline{R}(G(f)) \subset Q$, then $G \cap [R' \cap Q - \overline{(\underline{R}(G(f))}_\tau] \neq \varphi$. But we have $R = R' - \overline{(\underline{R}(G(f))}_\tau$, hence $R \cap Q = R' \cap Q - \overline{(\underline{R}(G(f))}_\tau$ implies that $S = R' \cap Q - \overline{(\underline{R}(G(f))}_\tau$. Then, $G \cap S \neq \varphi$, but $(u, v) \in Q$, hence $(G \cap Q) \cap S \neq \varphi$. But $G \cap Q = G'$ and G' is an open relation in τ' that contains (u, v), then $G \cap S \neq \varphi$, hence $(u, v) \in \overline{S}_\tau$. □

Theorem 9. *Let $(\underline{R}(G(f)), \overline{R}(G(f)))$ be a rough pair of relations in the product topological space $(U_1 \times U_2, \tau)$, and let (Q, τ') be a subspace of $(U_1 \times U_2, \tau)$ such that Q is any relation of $U_1 \times U_2$. Then, there is a relation $P \subset Q$ such that $\underline{R}(G(f)) \cap Q = P^o_\tau$ and $\overline{R}(G(f)) \cap Q = \overline{P}_\tau$.*

Proof. We can define $P = \underline{R}(G(f)) \cap Q$, then $P \subset Q$, and then $(\underline{R}(G(f)) \cap Q)^o_\tau = P^o_\tau$, but $\underline{R}(G(f)) \cap Q$ is an open relation in (Q, τ'), i.e., $(\underline{R}(G(f)) \cap Q)^o_\tau = \underline{R}(G(f)) \cap Q$, hence $\underline{R}(G(f)) \cap Q = P^o_\tau$.

Finally, for $\overline{R}(G(f)) \cap Q = \overline{P}_\tau$, since $(\underline{R}(G(f)), \overline{R}(G(f)))$ is a rough pair in $(U_1 \times U_2, \tau)$, then $\underline{R}(G(f)) \cap Q \subset \overline{R}(G(f)) \cap Q$, i.e., $P \subset \overline{R}(G(f)) \cap Q$ implies that $\overline{P}_\tau \subset \overline{\overline{R}(G(f)) \cap Q}_\tau$. But $\overline{R}(G(f)) \cap Q$ is a closed relation, i.e., $\overline{\overline{R}(G(f)) \cap Q}_\tau = \overline{R}(G(f)) \cap Q$, hence $\overline{P}_\tau \subset \overline{R}(G(f)) \cap Q$.

For $\overline{R}(G(f)) \cap Q \subset \overline{P}_\tau$, let $(u, v) \in \overline{R}(G(f)) \cap Q$, then $(u, v) \in \overline{R}(G(f))$ and $(u, v) \in Q$, but since $\underline{R}(G(f)) \cap Q = P$, then $Q \subset [\underline{R}(G(f)) - P]^c = U_1 \times U_2 - [\underline{R}(G(f)) - P]$, hence $(u, v) \in U_1 \times U_2 - [\underline{R}(G(f)) - P]$, i.e., $(u, v) \in P$, hence $(u, v) \in \overline{P}_\tau$, then $\overline{R}(G(f)) \cap Q \subset \overline{P}_\tau$. Then, we have $\overline{R}(G(f)) \cap Q = \overline{P}_\tau$.

Let $(U_1 \times U_2, \tau)$ be a product space. For any relation $Q \subset U_1 \times U_2$, define the subspace (Q, τ') of $(U_1 \times U_2, \tau)$. We define the equivalence relation $E(\tau')$ on the power set $P(Q)$ by $(R_1, R_2) \in E(\tau') \Leftrightarrow (R_1)^o_{\tau'} = (R_2)^o_{\tau'}, (\overline{R_1})_{\tau'} = (\overline{R_2})_{\tau'}$ for any $R_1, R_2 \in P(Q)$. The set $P(Q)/E(\tau')$ is a partition of $P(Q)$ and any class $\eta \in P(Q)/E(\tau')$ is called a relative topological rough relations. □

Theorem 10. *For any product topological space $(U_1 \times U_2, \tau)$ and for any subspace (Q, τ') of it, the function $f\colon P(Q)/E(\tau') \longrightarrow \eta'$, defined by $f(R) = ((R)^o_{\tau'}, (\overline{R})_{\tau'}), R \in \eta'$, is bijection, where η' is the set of all relative rough pairs.*

Proof. The proof is directly by Theorems 5 and 6.

Let (U_1, τ_1) and (U_2, τ_2) be any two topological spaces, where β_1 and β_2 are any two bases for τ_1 and τ_2. Then, we define the base $\beta = \beta_1 \times \beta_2$ of the topology $\tau = \tau_1 \times \tau_2$.

We define the approximations for any subset $H \subseteq U_1 \times U_2$:

$$\begin{aligned} (H)^o_\tau &= \cup\{G \subseteq \beta\colon G \in H\}, \\ \overline{(H)}_\tau &= \cap\{G \subseteq \beta\colon G \cap H = \varphi\}. \end{aligned} \tag{16}$$

The function f on $U_1 \times U_2$ is called a topological rough continuous function at the point $(x, y) \in U = U_1 \times U_2$ if $f^{-1}(V(f(x, y))) \subseteq \tau$ for all open sets $V(f(x, y)) \in \tau$. The function f is topological rough continuous on $U_1 \times U_2$ if it is topological roughly continuous at every point of $U_1 \times U_2$. □

Example 5. Consider the topology $\tau_1 = \{U_1, \varphi, \{a\}, \{b, c, d\}\}$, on $U_1 = \{a, b, c\}$ and the topology $\tau_2 = \{U_2, \varphi, \{3\}, \{1, 2, 4\}$ on $U_2 = \{1, 2, 3, 4\}$. The bases $\beta_1 = \{\{a\}, \{b, c, d\}\}$ and $\beta_2 = \{\{3\}, \{1, 2, 4\}\}$ are of τ_1 and τ_2, respectively.

We defined the function $f\colon U_1 \times U_2 \longrightarrow U_1 \times U_2$ as follows:

$$f(x, y) = (a, 3). \tag{17}$$

Then, we have

$$\beta = \beta_1 \times \beta_2 = \{\{a\}, \{b, c, d\}\} \times \{\{3\}, \{1, 2, 4\}\}$$
$$= \{\{a\} \times \{3\}, \{a\} \times \{1, 2, 4\}, \{b, c, d\} \times \{3\}, \{b, c, d\} \times \{1, 2, 4\}\},$$
$$\tau = \tau_1 \times \tau_2 = \{U_1, \varphi, \{a\}, \{b, c, d\}\} \times \{U_2, \varphi, \{3\}, \{1, 2, 4\}\} \quad (18)$$
$$= \{U_1 \times U_2, U_1 \times \varphi, U_1 \times \{3\}, U_1 \times \{1, 2, 4\}, \varphi \times U_2, \varphi \times \varphi, \varphi \times \{3\}, \varphi \times \{1, 2, 4\},$$
$$\{a\} \times U_2, \{a\} \times \varphi, \{a\} \times \{3\}, \{a\} \times \{1, 2, 4\}, \{b, c, d\} \times U_2, \{b, c, d\} \times \varphi, \{b, c, d\} \times \{3\}, \{b, c, d\} \times \{1, 2, 4\}\}.$$

Then, for any point $(x, y) \in U_1 \times U_2$, we have $f(x, y) = (a, 3)$, then all open sets containing $(a, 3)$ are

$$V_1 = \{a\} \times U_2 = \{(a, 1), (a, 2), (a, 3), (a, 4)\},$$
$$V_2 = \{a\} \times \{3\} = \{(a, 3)\}. \quad (19)$$

Then, the inverse function of these open sets is

$$f^{-1}(V_1) = U_1 \times U_2,$$
$$f^{-1}(V_2) = U_1 \times U_2. \quad (20)$$

Then, the function f is topological rough continuous at every point of $U_1 \times U_2$.

7. Future Applications of Topological Rough Functions on Information Systems

In this section, we will define a function between two information systems and give all needed conditions for them. Functions of an information system can produce the reductions, and the core of this system by the projection of the system on subsystems. We will define the image of rough set using some types of these functions. Finally, we define the topological rough functions of information systems and study some of their properties.

The reader can review about information systems in [7, 18] to know about the structure and the types and the different methods of reduction of information systems.

Suppose an information system $T = (U, C, D)$ where U is the set of objects of this system (patients, plants, etc.). C is the condition attributes of these objects (temperature, muscle pain, etc.). D is the expert decisions about the condition attribute that objects suffer from.

We define the projection (restriction) function f_c: $P(C) \times P(C) \longrightarrow P(C)$, where$P(C)$ is the power set of the condition attributes as follows:

$$f_c((B, B')) = \begin{cases} C, & \text{if } \mathrm{POS}_B(D) \neq POS_{B'}(D),\ \forall B' \subseteq C, \\ B', & \text{if } \mathrm{POS}_B(D) = \mathrm{POS}_{B'}(D),\ \forall B' \subset B. \end{cases} \quad (21)$$

Figure 1 gives an example for a projection function on information system. The core of such systems is given by taking the intersection of all these projection functions on that system.

The topological rough continuous functions of information systems can be defined as follows:

The function f: $(U, C, D) \longrightarrow (U, C', D)$ is topological roughly continuous at the object $x \in U$ if $f^{-1}(D_{(-a)}(x)) = D_C(x)$, where $D_C(x) = \{y \in U\colon D_C(x) = D_C(y)\}$. The function f is topological roughly continuous on U if it is topological roughly continuous for every object of U.

By a discernibility matrix of information system T, denoted $M(T)$, we will mean $n \times n$ matrix defined as follows: $M(T) = \{m_{ij} = 1, 2, 3, n\}$, where

$$m_{ij} \begin{cases} \{a \in C\colon a(x_i) \neq a(x_j)\}, & \text{if } \exists b \in D, b(x_i) \neq b(x_j), \\ \lambda, & \text{if } \forall b \in D, b(x_i) \neq b(x_j), \end{cases} \quad (22)$$

such that $a(x_i)$ or $a(x_j)$ belongs to the C-positive region of D; m_{ij} is the set of all conditions attributed that classify objects $a(x_i)$ and $a(x_j)$ into different classes; $m_{ij} = \lambda$ denotes that this case does not need to be considered.

The discernibility function $f\colon T = (U, C, D) \longrightarrow M(T)$ of an information system is defined as follows.

For any object, $x_1 \in U$: $f_T(x_i) = \wedge_j \{\vee m_{ij}\colon i \neq j \in \{i, 2, \ldots, n\}\}$, where $\vee m_{ij}$ is the disjunction of all variables $b \in m_{ij}$, when $m_{ij} \neq \varphi$ and $\vee m_{ij}| = 0$, when $m_{ij} = \varphi$ and $\vee m_{ij} = 1$, when $m_{ij} = \lambda$.

Figure 2 gives an example for a discernibility function on information system.

According to Figure 2, the function f transfers the system $T = (U, C, E)$ into the discernibility $M(T)$ and the reduction of this system can be obtained as follows:

$$f_T(x_i) = f_T(a, b, c, d) = b \wedge (a \vee b) \wedge (c \vee d) \wedge (b \vee d) \wedge (a \vee b \vee c) \wedge (a \vee b \vee c \vee d). \quad (23)$$

Then, we have

$$f_T(x_i) = b \wedge (c \vee d). \quad (24)$$

Accordingly, the system $T = (U, C, E)$ has two reductions, namely, $R_1 = \{b, c\}$ and $R_2 = \{b, d\}$ with core $\mathrm{CORE}(T) = \{b\}$.

U	Headache	Muscle pain	Temp.	Flu
U_1	Yes	Yes	Normal	No
U_2	Yes	Yes	High	Yes
U_3	Yes	Yes	Very-high	Yes
U_4	No	Yes	Normal	No
U_5	No	No	High	No
U_6	No	Yes	Very-high	Yes

$T = (U, C, D)$

$fc \longrightarrow$

U	Headache	Temp.	Flu
U_1	Yes	Normal	No
U_2	Yes	High	Yes
U_3	Yes	Very-high	Yes
U_4	No	Normal	No
U_5	No	High	No
U_6	No	Very-high	Yes

$T = (U, C - \{a\}, D)$

FIGURE 1: Some reductions of information system by projection function.

	a	b	c	d	E
u_1	1	0	2	1	1
u_2	1	0	2	0	1
u_3	1	2	0	0	2
u_4	1	2	2	1	0
u_5	2	1	0	0	2
u_6	2	1	1	0	2
u_7	2	1	2	1	1

$T = (U, C, E)$

$f \longrightarrow$

	u_1	u_2	u_3	u_4	u_5	u_6
u_2	λ					
u_3	b, c, d	b, c				
u_4	b	b, d	c, d			
u_5	a, b, c, d	a, b, c	λ	a, b, c, d		
u_6	a, b, c, d	a, b, c	λ	a, b, c, d	λ	
u_7	λ	λ	a, b, c, d	a, b	c, d	c

$M(T)$

FIGURE 2: Discernibility function of information system.

8. Predictions of Patients Classification Data Using Rough Continuous Functions

Our aim in this application problem, which will give in this section, is to find recommendations for patients that combine treatment and exercise by explaining the function of each symptom, whether positive or negative.

In this application problem, the decision according to the medical reports is the continuation of taking all medicine and doing medical tests. In fact, it is a painful decision. Our role is to analyze the medical data using the notion of reduction which will help us to determine which of the patients can stop taking medicine as well as expect the required period of time to do that.

The structure $S = (U, At, \{V_a: a \in At\}, f_a, \{R_P: P \subseteq At\})$ is the mathematical style of information system of our patients problem. The set U is the system universe that we selected to be a set of five patients. The set At is the set of attributes of these patients with respect to tests functions such as liver, kidney, and heart functions. The set V_a is values of each attribute $a \in At$. Finally, $f_a: U \longrightarrow V_a$ is the information function such that $f_a(x) \in V_a$.

For any subset $B \in At$, we define the relation $R_P = \{(x, y): |f_a(x) - f_a(y)| < \alpha, a \in P, x, y \in U, \alpha \in R^+\}$; for $a \in At$, we define the class A_{R_a} as follows: $A_{R_a} = \{R_a(x): x \in U\}$, where $R_a(x) = \{y: xR_a y\}$.

The structure $DS = (U, \{At, D\}, \{V_a: a \in At\}, f_a, \{R_P: P \subseteq At\})$ is a decision table, where D is the set of decisions that represents for each patient if he needs surgery or enough drugs.

We define the relation of the decision attribute D by

$$R_D = \{(x, y): f_a(x) = f_a(y), a \in D, x, y \in U\}. \quad (25)$$

The class of this relation is $R_D(x) = \{y: xR_D y\}$. The set of all classes is $A_{R_a} = \{R_a(x): x \in U\}$. We define the set $P \subseteq At$ to be a reduct of At, if $\tau_D \subseteq \tau_P$ and P is minimal.

Basic data of five patients before the surgery are given in Table 2 (the decision system of patients). Each patient will measure these medical functions periodically every three months. After a period of time, we need to predict the results of the medical tests of patients at any time and accordingly they can stop drugs. Therefore, we defined the prediction function $f_P: DS \longrightarrow \overline{\overline{DS}}$, where $\overline{\overline{DS}}$ is the decision system of patients over time t (dynamic decision system of patients).

Now, if we choose for the liver function attributes $P_1 = LF = \{A_1, A_2\}$, the threshold $\alpha_1 = 4$, then $R_{P_1}(U) = \{\{X_3\}, \{X_5\}, \{X_1, X_4, X_5\}, \{X_3, X_5\}, \{X_2, X_3, X_4\}\}$. The topology generated by R_{P_1} is given by

$$\tau_{P_1} = \{U, \varphi, \{X_3\}, \{X_5\}, \{X_1, X_4, X_5\}, \{X_3, X_5\}, \{X_2, X_3, X_4\}, \{X_1, X_3, X_4, X_5\}, \{X_2, X_3, X_4, X_5\}\}. \quad (26)$$

For kidney functions, we can choose $\alpha_1 = 2.5$ for $P_2 = KF = \{A_3\}$, then $R_{P_2}(U) = \{\{X_4\}, \{X_1, X_4\}, \{X_1, X_2, X_3, X_5\}\}$,

$$\tau_{P_2} = \{U, \varphi, \{X_4\}, \{X_1, X_4\}, \{X_1, X_2, X_3, X_5\}\}. \quad (27)$$

For the heart efficiency attribute $P_3 = HE = \{A_4\}$, we can choose $\alpha_3 = 20$, and then $R_{P_3}(U) = \{\{X_4, X_5\}, \{X_1, X_2, X_3, X_5\}, \{X_1, X_2, X_3, X_4\}\}$,

Table 2: The infection information system of patients.

Patients' ID	Liver functions		Kidney functions	Heart efficiency	Decision
	A_1	A_2	A_3	A_4	D
X_1	35	45	6.8	412	Need Surgery
X_2	45	44	4.2	420	Need drugs
X_3	42	38	5.8	430	Need Surgery
X_4	30	44	9.7	480	Need Surgery
X_5	36	32	5.4	450	Need drugs

$$\tau_{P_3} = \{\{X_4, X_5\}, \{X_1, X_2, X_3\}, \{X_1, X_2, X_3, X_5\}, \{X_1, X_2, X_3, X_4\}\}. \tag{28}$$

For the decision attributes, we have $R_D(U) = \{\{X_1, X_3, X_4\}, \{X_2, X_5\}\}$, then the topology of decisions is $\tau_D = \{U, \varphi, \{X_1, X_3, X_4\}, \{X_2, X_5\}\}$.

The condition attributes are exactly three attributes, namely, $At = \{\text{LF}, \text{KF}, \text{HE}\}$. The numbers of nontrivial subsets of the set of all condition attributes are seven subsets, namely, $\{P_1, P_2, P_3, \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_1, P_2, P_3\}\}$.

Now, we will calculate the classes of the residue subsets by taking the intersections as follows:

$R_{P_1,P_2}(U) = R_{P_1}(U) \cap R_{P_2}(U) = \varphi$, with topology $\tau_{P_1,P_2} = \{U, \varphi\}$.

$R_{P_1,P_3}(U) = R_{P_1}(U) \cap R_{P_3}(U) = \varphi$, with topology $\tau_{P_1,P_3} = \{U, \varphi\}$.

$R_{P_2,P_3}(U) = R_{P_2}(U) \cap R_{P_3}(U) = \{\{X1, X2, X3, X5\}\}$, with topology $\tau_{P_2,P_3} = \{U, \varphi, \{X1, X2, X3, X5\}\}$.

The covering class of universe using all condition attributes is given as follows:

$R_{P_1,P_2,P_3}(U) = R_{P_1}(U) \cap R_{P_2}(U) \cap R_{P_3}(U) = \varphi$, with topology $\tau_{P_1,P_3} = \{U, \varphi\}$. Then, the system given in Table 2 has no topological reductase.

Now, we define the function $f_P\colon \text{DS} \longrightarrow \overline{\overline{\overline{\text{DS}}}}, P \subseteq At$ by $f_P(Xi) = Xi$, $i = 1, 2, 3, 4, 5$. Then, according the function $f_P(Xi)$, the image of Table 2 after a period of three months has no topological reduces and this function is one-to-one rough continuous function.

9. Conclusion and Future Work

The emergence of topology in the construction of some rough functions will be the bridge for many applications and will discover the hidden relations between data. Topological generalizations of the concept of rough functions open the way for connecting rough continuity with the area of near continuous functions.

Applications of topological rough functions of information systems open the door about the many transformations among different types of information systems such as multivalued and single-valued information systems.

Future applications of our approach in the computer can be as follows:

In information retrieval fields, we can modify the query running online by defining a function that converted documents to weighted vectors of the words of that document. Then, we can extract the results of weights in a decision table that we can classify the documents according to the reduction of this table. The query is constructed by defining a Boolean function of all words of the reduction.

Classification and summarization of documentation using topological functions of neighboring systems are defined in documents.

Finally, we will benefit from the applications given in [41–47] to investigate new practical applications using rough sets and soft sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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