Research Article

Sufficient Conditions for Graphs to Be $k$-Connected, Maximally Connected, and Super-Connected

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Let $G$ be a connected graph with minimum degree $\delta(G)$ and vertex-connectivity $\kappa(G)$. The graph $G$ is $k$-connected if $\kappa(G) \geq k$, maximally connected if $\kappa(G) = \delta(G)$, and super-connected if every minimum vertex-cut isolates a vertex of minimum degree. In this paper, we present sufficient conditions for a graph with given minimum degree to be $k$-connected, maximally connected, or super-connected in terms of the number of edges, the spectral radius of the graph, and its complement, respectively. Analogous results for triangle-free graphs with given minimum degree to be $k$-connected, maximally connected, or super-connected are also presented.

1. Introduction

Let $G = (V, E)$ be a simple connected undirected graph, where $V = V(G)$ is the vertex-set of $G$ and $E = E(G)$ is the edge-set of $G$. The order and size of $G$ are defined by $n = |V(G)|$ and $m = |E(G)|$, respectively; $d_G(x)$ is the degree of a vertex $x$ in $G$, that is, the number of edges incident with $x$ in $G$; $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is the minimum degree of $G$. For a subset $X \subseteq V(G)$, use $G[X]$ to denote the subgraph of $G$ induced by $X$. For two subsets $X$ and $Y$ of $V(G)$, let $[X, Y]$ be the set of edges between $X$ and $Y$. The complement of $G$ is denoted by $\overline{G}$. Let $G_1 \cup G_2$ denote the disjoint union of graphs $G_1$ and $G_2$, and let $G_1 \vee G_2$ denote the graph obtained from $G_1 \cup G_2$ by joining each vertex of $G_1$ to each vertex of $G_2$. The graph $G$ is called a triangle-free graph if $G$ contains no triangle. Denote by $\rho(G)$ the largest eigenvalue or the spectral radius of the adjacency matrix of $G$ and it is called the spectral radius of $G$. If $G$ is connected, then, by Perron-Frobenius Theorem, $\rho(G)$ is simple and there exists a unique (up to a multiple) corresponding positive eigenvector.

A vertex-cut of a connected graph $G$ is a set of vertices whose removal disconnects $G$. The vertex-connectivity or simply the connectivity $\kappa = \kappa(G)$ of a connected graph $G$ is the minimum cardinality of a vertex-cut of $G$ if $G$ is not complete, and $\kappa(G) = n - 1$ if $G$ is the complete graph $K_n$ of order $n$. A vertex-cut $S$ is a minimum vertex-cut or a $\kappa$-cut of $G$ if $|S| = \kappa(G)$. Apparently, $\kappa(G) \leq \delta(G)$ for any graph $G$. The graph $G$ is $k$-connected if $\kappa(G) \geq k$, maximally connected if $\kappa(G) = \delta(G)$, and super-connected (or super-$\kappa$) if every minimum vertex-cut isolates a vertex of minimum degree. Hence, every super-connected graph is also maximally connected. An edge-cut of a connected graph $G$ is a set of edges whose removal disconnects $G$. The edge connectivity $\lambda = \lambda(G)$ of a connected graph $G$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of $G$. An edge-cut $S$ is a minimum edge-cut if $|S| = \lambda(G)$. The inequality $\lambda(G) \leq \delta(G)$ is obvious. The graph $G$ is maximally edge-connected if $\lambda(G) = \delta(G)$, and it is super-edge-connected if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Therefore, every super-edge-connected graph is also maximally edge-connected. For graph-theoretical terminology and notation not defined here, one can refer to [1, 2].
2 Complexity

Section 2, by setting parameters as in terms of the same parameters as in graphs with fixed minimum degree to be maximally connected or super-connected, or super-edge-connected if the number of edges is large enough, and the results corresponding to triangle-free graphs were generalized to connected graphs with given clique number by Volkman[5].

On the other hand, the relationship between graph properties and eigenvalues has attracted much attention. Fiedler [6] initiated the research on the relationship between graph connectivity and graph eigenvalues, and Fiedler and Nikiforov [7] initiated the investigation on the spectral radius of the graph, or its complement, respectively. In addition, we also give sufficient conditions for a triangle-free graph to be super-connected in terms of the number of edges and the spectral radius of the graph, respectively.

2. k-Connected Graphs

Let G be a connected graph of order n, minimum degree δ, and vertex-connectivity k. If n ≤ 4 or k = 1, then k = δ. If δ = n − 1, then G = Kn, and k = δ. If δ = n − 2, then when u and v are nonadjacent, the other n − 2 vertices are all common neighbors of u and v. It is necessary to delete all common neighbors of some pair of vertices to separate the graph, so k ≥ n − 2 = δ. Therefore, we only need to consider n ≥ 5 and 2 ≤ δ ≤ 3 in the following.

Theorem 1. Let G be a connected graph of order n ≥ 5, size m, and minimum degree δ ≥ k ≥ 2.

(a) If

\[ m ≥ \frac{1}{2} n(n-1) - (δ - k + 2)(n - δ - 1), \] (1)

then G is k-connected, unless G = K_{k-1} ∨ (K_{δ-k+2} ∪ K_{n-δ-1}).

(b) If n ≥ (1/2)(k + 1)(δ - k + 2) + (δ + 2) and

\[ m ≥ \frac{1}{2} n(n-1) - \frac{1}{2} (δ - k + 2)(2n - 2δ + k - 3), \] (2)

then G is k-connected, unless G is a subgraph of K_{k-1} ∨ (K_{δ-k+2} ∪ K_{n-δ-1}).

Proof. Let k = k(G). On the contrary, suppose that G is not k-connected; that is, 1 ≤ k ≤ k − 1. Let S be an arbitrary minimum vertex-cut, and let X_0, X_1, . . . , X_k−1 be the vertex sets of the components of G − S, where |X_0| ≤ |X_1| ≤ · · · ≤ |X_k−1|. Each vertex in X_0 is adjacent to at most |X_0| − 1 vertices of X_0 and k = |S| vertices of S. Thus,

\[ δ|X_0| ≥ \sum_{x ∈ X_0} d(x) ≥ |X_0|[|X_0| + κ − 1], \] (3)

and so |X_0| ≥ δ − κ + 1. Let Y = \bigcup_{i=1}^{k-1} X_i; then |Y| = n − k − |X_0|. Therefore,

\[ δ − κ + 1 ≤ |X_0| ≤ |Y| ≤ n − δ − 1. \] (4)

Since G − S is disconnected, there are no edges between X_0 and Y in G and

\[ m ≤ \frac{1}{2} n(n-1) - |X_0| · |Y|. \] (5)

(a) Since we suppose that G is not k-connected, it suffices to prove G = K_{k-1} ∨ (K_{δ-k+2} ∪ K_{n-δ-1}). By (4) and |X_0| + |Y| = n − k, and since k ≤ k − 1, we obtain

\[ |X_0| · |Y| ≥ (δ - κ + 1)(n - δ - 1) ≥ (δ - k + 2)(n - δ - 1). \] (6)

Substituting (6) into (5), it follows that
Combining this with (1), we obtain \( m = (1/2)n \) \((n - 1) - (\delta - k + 2)(n - \delta - 1)\). Thus, all the inequalities in (6) must be equalities and so \( \kappa = k - 1 \), \( |X_0| = \delta - k + 2 \), and \( |Y| = n - \delta - 1 \). Thus, \( G \) is obtained from \( K_\kappa \) by deleting all the edges of the complete bipartite subgraph \( K_{|X_0||Y|} \) of \( K_\kappa \). That is, \( G[X_0] = K_{\delta-k+2} \), \( G[S] = K_{k-1} \), \( G[Y] = K_{n-\delta-1} \), and \( G = K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}) \).

(b) To prove that \( G \) is a subgraph of \( K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}) \), we first show that \( |X_0| = \delta - k + 2 \). Suppose that \( |X_0| > \delta - k + 3 \). Since \( |X_0| \leq |Y|, |X_0| + |Y| = n - \kappa \), and \( \kappa \leq k - 1 \), we have

\[
|X_0| \cdot |Y| \geq (\delta - k + 3)(n - \kappa - (\delta - k + 3))
\]

\[
\geq (\delta - k + 3)(n - \delta - 2).
\]

Substituting (8) into (5), it follows that

\[
m \leq \frac{1}{2} n(n - 1) - (\delta - k + 3)(n - \delta - 2).
\]

Combining this with (2), it is easy to get \( n \leq (1/2)(k + 1)(\delta - k + 2) + \delta + 2 \). By the hypothesis, we have \( n = (1/2)(k + 1)(\delta - k + 2) + \delta + 2 \). Hence, \( m = (1/2)n \) \((n - 1) - (\delta - k + 3)(n - \delta - 2)\) and all the inequalities in (8) must be equalities. Thus, \( \kappa = k - 1 \), \( |X_0| = \delta - k + 3 \), \( |Y| = n - \delta - 2 \), and \( G \) is obtained from \( K_\kappa \) by deleting all the edges of the complete bipartite subgraph \( K_{|X_0||Y|} \) of \( K_\kappa \). That is, \( G = K_{k-1} \cup (K_{\delta-k+3} \cup K_{n-\delta-2}) \). However, \( \delta(G) = \delta(K_{k-1} \cup (K_{\delta-k+3} \cup K_{n-\delta-2})) = \delta + 1 > \delta \), a contradiction. Thus, \( |X_0| \leq \delta - k + 2 \). Combining this with \( |X_0| \geq \delta - k + 1 \geq \delta - k + 2 \), we get \( |X_0| = \delta - k + 2 \). Since \( |S| = \kappa \leq k - 1 \) and \( d_G(x) \geq \delta \) for each \( x \in X_0 \), we have that each vertex of \( X_0 \) is adjacent to each vertex of \( S \) and \( |S| = \kappa - 1 \). Therefore, \( G[X_0 \cup S] = K_{\delta+1} \) and \( G \) is a subgraph of \( K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}) \).

Theorem 2. Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \geq k \geq 2 \). If

\[
\rho(G) \geq \rho(K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1})),
\]

then \( G \) is \( k \)-connected, unless \( G = K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}) \), where \( \rho(K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1})) \) is the largest root of the equation

\[
\lambda^3 - (n - 3)\lambda^2 + ((\delta - k + 2)(n - \delta - 1) - 2n + 3)\lambda + (\delta - k + 2)(n - \delta - 1)k - n + 1 = 0.
\]

Proof. Let \( \kappa = \kappa(G) \). Assume that (10) holds but \( 1 \leq \kappa \leq k - 1 \). Let \( S \) be an arbitrary minimum vertex-cut of \( G \), and let \( X_0, X_1, \ldots, X_{p-1} \) \((p \geq 2)\), denote the vertex-sets of the components of \( G - S \), where \( |X_0| \leq |X_1| \leq \cdots \leq |X_{p-1}| \). Each vertex in \( X_i \) is adjacent to at most \( |X_{i-1}| \) vertices of \( X_i \) and \( \kappa = |S| \) vertices of \( S \). Thus,

\[
\delta |X_i| \leq \sum_{x \in X_i} d(x) \leq |X_i|(|X_i| - 1 + \kappa),
\]

and so \( |X_i| \geq \delta - \kappa + 1 \) for each \( i = 0, 1, \ldots, p - 1 \). Let \( Y = \bigcup_{i=1}^{p-1} X_i \). Then, \( \delta - k + 1 \leq |X_0| \leq |Y| \leq n - \delta - 1 \) and \( |X_0| + |Y| = n - \kappa \). Since there are no edges between \( X_0 \) and \( Y \) in \( G \), \( G \) is a subgraph of \( K_{k-1} \cup (K_{|X_0|} \cup K_{|Y|}) \) and \( \rho(G) \leq \rho(K_{k-1} \cup (K_{|X_0|} \cup K_{|Y|})) \).

Next, we shall show

\[
\rho(K_{k-1} \cup (K_{|X_0|} \cup K_{|Y|})) \leq \rho(K_{k-1} \cup (K_{\delta-k+1} \cup K_{n-\delta-1}))
\]

\[
\leq \rho(K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1})).
\]

Denote \( G(a, b, \kappa) = K_{\kappa} \cup (K_a \cup K_b) \) for short, where \( b \geq a \geq \delta - k + 1 \) and \( a + b + \kappa = n \). Let \( x = (x_1, x_2, \ldots, x_n) \) be the unique positive unit eigenvector corresponding to \( \rho(G(a, b, \kappa)) \). By symmetry, let \( x = x_i \) for any \( i \in K_a; y = x_j \) for any \( j \in K_b \); \( z = x_i \) for any \( \ell \in K_\kappa \). According to \( \lambda x = \sum_{i \in E(G(ab))} x_i \) and the uniqueness of \( x \), we have that \( \rho(G(a, b, \kappa)) \) is the largest root of the following equations:

\[
\begin{align*}
\lambda x &= (a - 1)x + ky, \\
\lambda y &= ax + (\kappa - 1)y + bz, \\
\lambda z &= kx + (b - 1)z.
\end{align*}
\]

Thus, \( \rho(G(a, b, \kappa)) \) is the largest root of the equation

\[
\begin{align*}
f(\lambda; a, b, \kappa) &= \lambda^3 - (n - 3)\lambda^2 + (ab - 2n + 3)\lambda \\
&\quad + (a + b)(\kappa + 1) - n + 1 = 0.
\end{align*}
\]

Then, we have

\[
\begin{align*}
f(\lambda; a, b, \kappa) - f(\lambda; \delta - k + 1, n - \delta - 1, \kappa) &= (\lambda + \kappa + 1)(ab - (\delta - k + 1)(n - \delta - 1)) \\
&\geq 0,
\end{align*}
\]

for any \( \lambda > 0 \) and \( b \geq a \geq \delta - k + 1 \). Therefore, \( \rho(G(a, b, \kappa)) \leq \rho(G(\delta - k + 1, n - \delta - 1, \kappa)) \) for any \( b \geq a \geq \delta - k + 1 \), which means that

\[
\rho(K_{k-1} \cup (K_{\delta-k+1} \cup K_{n-\delta-1})) \leq \rho(K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1})).
\]

Hence, from the discussion above, we have

\[
\begin{align*}
\rho(G) &\leq \rho(K_{k-1} \cup (K_{|X_0|} \cup K_{|Y|})) \\
&\leq \rho(K_{k-1} \cup (K_{\delta-k+1} \cup K_{n-\delta-1})) \\
&\leq \rho(K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1})).
\end{align*}
\]

By (10), the above inequalities must be equalities. Thus, \( |X_0| = \delta - k + 2 \), \( \kappa = k - 1 \), \( |Y| = n - \delta - 1 \), and so \( G = K_{k-1} \cup (K_{\delta-k+2} \cup K_{n-\delta-1}) \). The result follows from (15).
Remark 1. In Corollary 3.5 in [16], the authors showed that if \( G \) is a connected graph of minimum degree \( \delta(G) \geq k \geq 3 \) and order \( n \geq (\delta - k + 2)(k^2 - 2k + 4) + 3 \), and \( \rho(G) \geq \rho(K_{k-1} \cup (K_{k-2} \cup K_{n-\delta-1})) \), then \( G \) is \( k \)-connected, unless \( G = \rho(K_{k-1} \cup (K_{k-2} \cup K_{n-\delta-1})) \). Apparently, without restriction on the order of graph, Theorem 2 improves Corollary 3.5 in [16].

Theorem 3. Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta \geq k \geq 3 \). If \( G \) is a subgraph of \( K_{k+1} \cap (K_{k-1} \cup K_{n-\delta-1}) \) and \( n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7) \), then

\[
\rho(G) < n - \delta + k - 3,
\]

unless \( G = K_{k-1} \cap (K_{k-1} \cup K_{n-\delta-1}) \).

Proof. Denote \( H = K_{k-1} \cap (K_{k+1} \cup K_{n-\delta-1}) \) for short. Let \( x = (x_1, x_2, \ldots, x_n)^T \) be the unique positive unit eigenvector corresponding to \( \rho(G) \). Recall that Rayleigh's principle implies that

\[
\rho(G) = x^T A(G)x = 2 \sum_{ij \in E(G)} x_i x_j.
\]

Assume that \( G \) is a proper subgraph of \( H \). Clearly, we could assume that \( G \) is obtained by omitting just one edge \( uv \) of \( H \). Let \( X, Y, Z \) be the set of vertices of \( H \) of degree \( \delta \), \( n - 1 \), \( n - \delta + k - 3 \), respectively, where \( |X| = \delta - k + 2 \), \( |Y| = k - 1 \), and \( |Z| = n - \delta - 1 \). Since \( \delta(G) = \delta(G) \) must contain all the edges between \( X \) and \( Y \). Therefore, \( \{u, v\} \subset Y \cup Z \), with three possible cases: (a) \( \{u, v\} \subset Y \); (b) \( u \in Y, v \in Z \); and (c) \( \{u, v\} \subset Z \). We shall show that case (c) yields a graph whose spectral radius is not smaller than the spectral radius of the graph in case (b) and that case (b) yields a graph whose spectral radius is not smaller than the spectral radius of the graph in case (a).

Firstly, suppose that case (a) occurs; that is, \( \{u, v\} \subset Y \). Choose a vertex \( w \in Z \). If \( x_u \geq x_v \), then by removing the edge \( uv \) and adding the edge \( uw \) we obtain a new graph \( G_1 \) which is covered by case (b). By the Rayleigh principle,

\[
\rho(G_1) - \rho(G) \geq x^T A(G_1)x - x^T A(G)x = 2x_v(x_v - x_u) \geq 0.
\]

If \( x_u > x_v \), then by removing all the edges between \( X \) and \( u \) and adding all the edges between \( X \) and \( v \) we obtain a new graph \( G_1 \) which is also covered by case (b). By the Rayleigh principle,

\[
\rho(G_1) - \rho(G) \geq x^T A(G_1)x - x^T A(G)x = 2x_u(x_u - x_v) \geq 0.
\]

Secondly, suppose that case (b) occurs; that is, \( u \in Y \), \( v \in Z \). Choose a vertex \( w \in Z \) and \( w \neq v \). If \( x_u \geq x_w \), then by removing the edge \( vw \) and adding the edge \( uv \) we obtain a new graph \( G_2 \) which is covered by case (c). By the Rayleigh principle,

\[
\rho(G_2) - \rho(G) \geq x^T A(G_2)x - x^T A(G)x = 2x_v(x_v - x_u) \geq 0.
\]

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If \( x_u > x_v \), then by removing all the edges between \( X \) and \( \{u, v\} \) and adding all the edges between \( X \) and \( \{w, v\} \) we obtain a new graph \( G_2 \) which is also covered by case (c). By the Rayleigh principle,

\[
\rho(G_2) - \rho(G) \geq x^T A(G_2)x - x^T A(G)x = 2x_u(x_u - x_v) \geq 0.
\]

Therefore, we could assume that \( \{u, v\} \subset Z \). By symmetry, let \( x = x_i \) for any \( i \in X \); \( y = x_j \) for any \( j \in Y \); \( z = x_\ell \) for any \( \ell \in Z \), \( \{u, v\} \); and \( t = x_u = x_v \). According to \( \lambda x_i = \sum_{j \in E(G)} x_j \) and the uniqueness of \( x_i \), we have that \( \rho \) is the largest root of following equations:

\[
\begin{align*}
\lambda x &= (\delta - k + 1)x + (k - 1)y, \\
\lambda y &= (\delta - k + 2)x + (k - 2)y + (n - \delta - 3)z + 2t, \\
\lambda z &= (k - 1)y + (n - \delta - 4)z + 2t, \\
\lambda t &= (k - 1)y + (n - \delta - 3)z.
\end{align*}
\]

Thus, \( \rho(G) \) is the largest root of the equation

\[
f(\lambda) = \lambda^4 - (n - 5)\lambda^3 + ((n - \delta - 1)(\delta - k - 2) - 4\delta + 7)\lambda^2
\]

\[
+ \left[(\delta k + 2\delta + 2)(n - \delta - k + 3) - (k^2 + 3)(n - 1) + 6\right] + 2((\delta - k + 1)(k^2 - 2k + 7) - 1) + (k - 1)(n - \delta - 3).
\]

By some basic calculations, we have

\[
f(n - \delta + k - 3) = 2n^2 - (\delta - k + 2)(k^2 - 2k + 7)n
\]

\[
+ (\delta - k + 2)((\delta - k + 1)(k^2 - 2k + 7) - 1).
\]

Set \( g(x) = 2x^2 - (\delta - k + 2)(k^2 - 2k + 7)x + (\delta - k + 2)((\delta - k + 1)(k^2 - 2k + 5) - 2(k - 3)) \). It is easy to see that the function \( g(x) \) is strictly increasing when \( x > (1/2)(\delta - k + 2)(k^2 - 2k + 7) > (1/4)(\delta - k)(k^2 - 2k + 7) \), we get

\[
f(n - \delta + k - 3) = g(n) \geq g\left(\frac{1}{2}(\delta - k + 2)(k^2 - 2k + 7)\right)
\]

\[
= (\delta - k + 2)((\delta - k + 1)(k^2 - 2k + 7) - 1)
\]

\[
\geq (\delta - k + 2)(k^2 - 4k + 11) > 0.
\]
\[ f(n - \delta + k - 4) = -n^2 + 4(\delta - k + 3)n^2 \]
\[ -\left(\left(k^2 - 8k + 5\delta + 23\right)(\delta - k + 2) - 5\right)n \]
\[ + \left(\left(k^2 - 5k + 2\delta + 15\right)(\delta - k + 2) + 2\right) \]
\[ = -n(n - 2(\delta - k + 3))^2 \]
\[ -\left(\left(k^2 - 4k + 2\delta + 7\right)(\delta - k + 2) + 1\right) \]
\[ \cdot (\delta - k + 2)(k - 1) \]
\[ \leq -2(3 \cdot 3 + k(1 - k)) < 0, \]
\[ (30) \]
\[ f(0) = 2((\delta - k + 1)(n - \delta - 2) - 1) + (k - 1)(n - \delta - 3)) \]
\[ \geq 2(k - 1) > 0, \]
\[ f(-2) = -2(k - 2)(\delta - k + 2) + 2 < 0, \]
\[ \text{and } f(-\infty) > 0. \]
\[
\rho(G) \geq n - \delta - k - 3, \quad \text{Lemma 1 (see [25]).} \]

Equality holds if and only if G is either a regular graph or a bipartite graph in which each vertex is of degree either \( \delta \) or \( n - 1 \).

**Theorem 4.** Let G be a connected graph of order n and minimum degree \( \delta \geq k \geq 3 \). If \( n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7) \) and

\[ \rho(G) \geq n - \delta - k - 3, \]

then G is k-connected, unless G = \( K_{\delta - 1} \cup K_{n - \delta - 1} \).

**Proof.** On the contrary, suppose that \( \kappa(G) < k \). Since G is connected and \( \rho(G) \geq n - \delta - k - 3 \), by Lemma 1, we have

\[ n - \delta - k - 3 \leq \rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2E(G) - \delta n + \frac{(\delta + 1)^2}{4}}, \]

which yields

\[ |E(G)| \geq \frac{1}{2}n(n - 1) - \frac{1}{2}(\delta - k + 2)(2n - 2\delta + k - 3). \]

Since \( n \geq (1/2)(\delta - k + 2)(k^2 - 2k + 7) \), we obtain \( n \geq (1/2)(k + 1)(\delta - k + 2) + (\delta + 2) \). By Theorem 1 (b), G is a subgraph of \( K_{\delta - 1} \cup K_{n - \delta - 1} \). Since \( \rho(G) \geq n - \delta - k - 3 \), by Theorem 3, G = \( K_{\delta - 1} \cup K_{n - \delta - 1} \). The proof is completed.

**Remark 2.** In Theorem 3.4 in [16], the authors proved that if G is a connected graph of minimum degree \( \delta(G) \geq \delta \geq k \geq 3 \) and order \( n \geq (\delta - k + 1)(k^2 - 2k + 4) + 3 \), and \( \rho(G) \geq n - \delta - k - 3 \), then G is k-connected unless G = \( K_{\delta + 1} \cup (K_{\delta - 2} \cup K_{n - \delta - 1}) \). Obviously, Theorem 4 improves Theorem 3.4 in [16] from the perspective of the restriction on the order of graph.

Another sufficient condition for graphs to be k-connected can be obtained by using the spectral radius of the complement of a graph.

**Theorem 5.** Let G be a connected graph of order \( n \geq 5 \) and minimum degree \( \delta \geq k \geq 2 \). If

\[ \rho(G) \leq \sqrt{(\delta - k + 2)(n - \delta - 1)}, \]

then G is k-connected, unless G = \( K_{\delta - 1} \cup (K_{\delta - 2} \cup K_{n - \delta - 1}) \).

**Proof.** Let \( \kappa(G) = \kappa \). Assume that (35) holds but \( 1 \leq \kappa \leq k - 1 \). Let S be an arbitrary minimum vertex-cut of G, and let \( X_0, X_1, \ldots, X_{p - 1} \) (\( p \geq 2 \)), denote the vertex-sets of the components of \( G - S \), where \( |X_0| \leq |X_1| \leq \cdots \leq |X_{p - 1}| \). Each vertex in \( X_i \) is adjacent to at most \( |X_i| - 1 \) vertices of \( X_i \) and \( \kappa = |S| \) vertices of S. Thus,

\[ |X_i| \leq \sum_{x \in X_i} d(x) \leq |X_i|(|X_i| - 1 + \kappa), \]

and so \( |X_i| \geq \delta - \kappa + 1 \) for each \( i = 0, 1, \ldots, p - 1 \). Let \( Y = \cup_{i=0}^{p-1} X_i \). Then, \( \delta - \kappa + 1 \leq |X_0| \leq |Y| \leq n - \delta - 1 \) and \( |X_0| + |Y| = n - \kappa \). Since there are no edges between \( X_0 \) and \( Y \) in G, \( K_{|X_0|,|Y|} \) is a subgraph of G. Thus,

\[ \rho(G) \geq \rho(K_{|X_0|,|Y|}) \geq \sqrt{|X_0||Y|} = \sqrt{|X_0|(|X_0| - \kappa - |X_0|)} \]

\[ \geq \sqrt{(\delta - k + 1)(n - \delta - 1)} \geq \sqrt{(\delta - k + 2)(n - \delta - 1)}. \]

By (35), the above inequalities must be equalities. Thus,

\[ |X_0| = \delta - k + 2, \quad \kappa = k - 1 \text{ and } G = K_{\delta - k + 2, n - \delta - 1}, \]

and so G = \( K_{\delta - 1} \cup (K_{\delta - 2} \cup K_{n - \delta - 1}) \).

**3. Maximally Connected Graphs**

If \( \kappa(G) = \delta(G) \), then G is maximally connected. Therefore, by setting \( k = \delta \) in Theorem 1, we obtain the following theorem.

**Theorem 6.** Let G be a connected graph of order \( n \geq 5 \), size \( m \), and minimum degree \( \delta \geq 2 \).

(a) If \( m \geq \left(\frac{n - 2}{2}\right) + 2\delta - 1 \), then G is maximally connected, unless G = \( K_{\delta - 1} \cup (K_2 \cup K_{n - \delta - 1}) \).
(b) If \( n \geq 2\delta + 3 \) and \( m \geq \left( \frac{n-2}{2} \right) + \delta \), then \( G \) is maximally connected, unless \( G \) is a subgraph of \( K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1}) \).

**Theorem 7.** Let \( G \) be a connected graph of order \( n \geq 5 \) and minimum degree \( \delta \geq 2 \). If

\[
\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{(n - \delta - 1)(n - 4) + \frac{(\delta + 1)^2}{4}},
\]

then \( G \) is maximally connected, unless \( G = K_{\delta-1} \cup (K_2 \cup K_2) \). 

**Proof.** On the contrary, suppose that \( \kappa(G) < \delta \). Since \( G \) is connected, by (38) and Lemma 1, we have

\[
\frac{\delta - 1}{2} + \sqrt{(n - \delta - 1)(n - 4) + \frac{(\delta + 1)^2}{4}} \leq \rho(G) \leq \frac{\delta - 1}{2} + \sqrt{\frac{2E(G)}{n} - \delta n + \frac{(\delta + 1)^2}{4}},
\]

which yields

\[
|E(G)| \geq \left( \frac{n - 2}{2} \right) + 2\delta - 1.
\]

By Theorem 6 (a), \( G = K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1}) \). To complete the proof, we only need to show \( \delta = n - 3 \).

Since \( |E(G)| = \left( \frac{n - 2}{2} \right) + 2\delta - 1 \), the equalities hold in (39). Thus, by Lemma 1, \( G \) is either a regular graph or a bidirected graph in which each vertex is of degree \( \delta \) or \( n - 1 \).

However, the vertices of \( G \) have degrees from the set \{\( \delta, n - 3, n - 1 \)\}. Therefore, \( \delta = n - 3 \) and the result follows. \( \square \)

By setting \( k = \delta \) in Theorem 2, we obtain the following result.

**Theorem 8.** Let \( G \) be a connected graph of order \( n \geq 5 \) and minimum degree \( \delta \geq 2 \). If

\[
\rho(G) \geq \rho(K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1})),
\]

then \( G \) is maximally connected, unless \( G = K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1}) \), where \( \rho(K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1})) \) is the largest root of the equation

\[
\lambda^3 - (n - 3)\lambda^2 - (2\delta - 1)\lambda + 2\delta(n - \delta - 1) - n + 1 = 0.
\]

**Theorem 9.** Let \( G \) be a connected graph of order \( n \geq 5 \) and minimum degree \( \delta \geq 2 \). If \( n \geq 2\delta^2 - 2\delta + 7 \) and

\[
\rho(G) \geq n - 3,
\]

then \( G \) is maximally connected, unless \( G = K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1}) \).

**Proof.** Set \( k = \delta \) in the proofs of Theorems 3 and 4. If \( \delta \geq 3 \), then the result follows from Theorem 4. If \( \delta = 2 \), then case (a) cannot occur in the proof of Theorems 3. In Theorem 3, by noting that \( f(n - 3) > 0, f(n - 4) < 0, f(0) > 0, f(-\sqrt{3}) = 2\sqrt{3} - 4 < 0 \), and \( f(-\infty) > 0 \), we have \( \rho(G) < n - 3 \) and so Theorem 3 holds for \( \delta = k = 2 \). Hence, Theorem 4 also holds for \( \delta = k = 2 \) and the result follows. \( \square \)

**Remark 3.** In the proof of Theorem 3, if we take \( k = \delta \geq 2 \) and \( n = \delta^2 - 2\delta + 6 \), then \( f(n - 3) = g(n) = g(\delta^2 - 2\delta + 6) = 10 - 4\delta < 0 \) when \( \delta \geq 3 \). Notice that \( f(+\infty) = +\infty \). So, the largest root of \( f(x) = 0 \) is greater than \( n - 3 \) if \( \delta \geq 3 \), and it follows that \( \rho(G) > n - 3 \). That is to say, the requirement \( n \geq \delta^2 - 2\delta + 7 \) in Theorem 9 is best possible when \( \delta \geq 3 \).

By setting \( k = \delta \) in Theorem 5, we have the following result.

**Theorem 10.** Let \( G \) be a connected graph of order \( n \geq 5 \) and minimum degree \( \delta \geq 2 \). If

\[
\rho(G) \leq \sqrt{2(n - \delta - 1)},
\]

then \( G \) is maximally connected, unless \( G = K_{\delta-1} \cup (K_2 \cup K_{n-\delta-1}) \).

**4. Super-Connected Graphs**

For any connected graph \( G \) of order \( n \), if \( 2 \leq n \leq 4 \), then \( G \) is super-\( \kappa \). Therefore, \( n \geq 5 \) is considered in this section.

**Theorem 11.** Let \( G \) be a connected graph of order \( n \geq 5 \), size \( m \), and minimum degree \( \delta \). If

\[
m \geq \left( \frac{n - 2}{2} \right) + 2\delta,
\]

then \( G \) is super-\( \kappa \), unless \( G = (K_{\delta} \cup (K_2 \cup K_{n-\delta-2})) \) and \( e \), where \( e = xy \) is an edge of \( K_{\delta} \cup (K_2 \cup K_{n-\delta-2}) \) with \( d(x) = \delta + 1 \) and \( d(y) = n - 1 \).

**Proof.** Since \( m \geq \left( \frac{n - 2}{2} \right) + 2\delta \), by Theorem 6 (a), \( \kappa(G) = \delta \). On the contrary, suppose that \( G \) is not super-\( \kappa \). Let \( S \) be an arbitrary minimum vertex-cut with \( \delta \) vertices, and let \( X_0, X_1, \ldots, X_\rho \) (\( \rho \geq 2 \)) denote the vertex-sets of the components of \( G - S \), where \( 2 \leq |X_0| \leq |X_1| \leq \cdots \leq |X_\rho| \). Denote \( Y = \bigcup_{i=1}^{\rho} X_i \). Since \( G - S \) is disconnected, there are no edges between \( X_0 \) and \( Y \) in \( G \), and

\[
m \geq \frac{1}{2} n(n - 1) - |X_0| \cdot |Y|.
\]

Thus, by \( |X_0| + |Y| = n - \delta \) and \( 2 \leq |X_0| \leq |Y| \leq n - \delta - 2 \), we have

\[Complexity]
Let $G$ be a connected graph of order $n \geq 5$ and minimum degree $\delta$. If
\[
\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{2 + (n - \delta - 1)(n - 4) + \frac{(\delta + 1)^2}{4}},
\]
then $G$ is super-$\kappa$.

**Proof.** On the contrary, suppose that $G$ is not super-$\kappa$. Since $G$ is connected, by (50) and Lemma 1, we have
\[
\sqrt{2 + (n - \delta - 1)(n - 4) + \frac{(\delta + 1)^2}{4}} \leq \rho(G) - \frac{\delta - 1}{2} \leq 2|E(G)| - \delta n + \frac{(\delta + 1)^2}{4},
\]
which yields
\[
|E(G)| \geq \left(\frac{n - 2}{2}\right) + 2\delta.
\]
By Theorem 11, $G = K_4 \vee (K_3 \cup K_{n-4}) - e$, where $e = xy$ is an edge of $K_4 \vee (K_3 \cup K_{n-4})$ with $d(x) = \delta + 1$ and $d(y) = n - 1$.

Since $|E(G)| = \left(\frac{n - 2}{2}\right) + 2\delta$, the equalities hold in (51). Thus, by Lemma 1, $G$ is either a regular graph or a bidiregular graph in which each vertex is of degree $\delta$ or $n - 1$. However, the vertices of $G$ have degree from the set $\{\delta, \delta + 1, n - 3, n - 2, n - 1\}$. Thus, $G$ cannot be a bidiregular graph, which yields a contradiction. Hence, $G$ is super-$\kappa$.

**Theorem 13.** Let $G$ be a connected graph of order $n \geq 5$ and minimum degree $\delta$. If
\[
\rho(G) \geq \rho(K_4 \vee (K_3 \cup K_{n-4})),
\]
then $G$ is super-$\kappa$, where $\rho(K_4 \vee (K_3 \cup K_{n-4}))$ is the largest root of the equation
\[
\lambda^3 - (n - 3)\lambda^2 - (2\delta + 1)\lambda + 2(\delta + 1)(n - \delta - 2) - n + 1 = 0.
\]

**Proof.** On the contrary, suppose that $G$ is not super-$\kappa$. Let $S$ be an arbitrary minimum vertex-cut with $\kappa \leq \delta$ vertices, and let $X_0, X_1, \ldots, X_{p-1}$ ($p \geq 2$) denote the vertex-sets of the components of $G - S$, where $2 \leq |X_0| \leq |X_1| \leq \cdots \leq |X_{p-1}|$. Denote $Y = \cup_{i=1}^{p-1} X_i$. Then $2 \leq |X_0| \leq |Y| \leq n - \kappa - 2$ and $|X_0| + |Y| = n - \kappa$. Since there are no edges between $X_0$ and $Y$ in $G$, $G$ is a subgraph of $K_\kappa \vee (K_{|X_0|} \cup K_{|Y|})$ and $\rho(G) \leq \rho(K_\kappa \vee (K_{|X_0|} \cup K_{|Y|}))$.

According to (15) in the proof of Theorem 2, $\rho(K_\kappa \vee (K_{|X_0|} \cup K_{|Y|}))$ is the largest root of the equation
\[
\lambda^3 - (n - 3)\lambda^2 - (2\delta + 1)\lambda + 2(\delta + 1)(n - \delta - 2) - n + 1 = 0.
\]
Then we have
\[
f(\lambda; |X_0|, |Y|, \kappa) = \lambda^3 - (n - 3)\lambda^2 + |X_0||Y| - 2n(\lambda + 1) - n + 1 = 0.
\]

(55)

5. Sufficient Conditions for Triangle-Free Graphs

Let us extend an interesting result by applying the famous theorem of Mantel [26] and Turán [27].

Theorem 15 (see [26, 27]). For any triangle-free graph \( G \) of order \( n \), we have \( |E(G)| \leq \left\lfloor \left( \frac{1}{4} \right) n^2 \right\rfloor \), with equality if and only if \( G = K_{(n/2), (n/2)} \).

Theorem 16. Let \( G \) be a connected triangle-free graph of order \( n \), size \( m \), and minimum degree \( \delta \geq k \geq 2 \). If
\[
m \geq \delta^2 + \frac{1}{4}(n - 2\delta + k - 1)^2,
\]
then \( G \) is a k-connected, unless \( V(G) = X \cup S \cup Y \) and \( S \) is a minimum vertex-cut of \( G \) with \( G[S] = K_{\delta,k} \) and \( G[X \cup S] = K_{\delta,k} \).

Proof. Let \( \kappa = \kappa(G) \). On the contrary, suppose that \( \kappa \leq k - 1 \). Let \( S \) be a minimum vertex-cut of \( G \) and let \( X, Y_1, \ldots, Y_p \) (\( p \geq 2 \)) denote the vertex-sets of the components of \( G - S \), where \( |X| \leq |Y_1| \leq \cdots \leq |Y_p| \). Denote \( Y = \bigcup_{i=1}^{p} Y_i \). Then \( |X| \leq |Y| \) and \( |X| + |Y| = n - \kappa \). By Theorem 15, we deduce that
\[
|E(G[X \cup S])| \leq \left\lfloor \frac{(|X| + |S|)^2}{4} \right\rfloor,
\]

(62)

with equalities if and only if \( G[X \cup S] = K_{\lfloor (|X| + |S|)/2 \rfloor, \lfloor (|X| + |S|)/2 \rfloor} \).

\[
|E(G[Y \cup S])| \leq \left\lfloor \frac{(|Y| + |S|)^2}{4} \right\rfloor,
\]

(63)

If \( x \in X \), then \( \delta \leq d_G(x) \leq |X| + |S| \). The assumption \( \kappa \leq k - 1 \leq \delta - 1 \) implies that \( x \) has at least one neighbor \( y \in Y \). Since \( G \) is triangle-free, we deduce that \( N_G(x) \cap N_G(y) = \emptyset \), where \( N_G(x) \) is the neighbor set of \( x \). As \( N_G(x) \cup N_G(y) \leq X \cup S \), it follows that
\[
|X| + |S| = |X \cup S| \geq |N_G(x) \cup N_G(y)| = |N_G(x)| + |N_G(y)| \geq 2\delta,
\]

(64)

and thus \( |X| \geq 2\delta - \kappa \). Therefore, we arrive at
\[
2\delta - \kappa \leq |X| \leq |Y| \leq n - 2\delta.
\]

(65)

Together with \( |X| + |Y| = n - \kappa \) and (62), it leads to

\[
\delta(\beta(G) - K_{\kappa,k+2}) = \delta + 1 > \delta, \quad \text{a contradiction.}
\]

This completes the proof.
\[
m = |E(G[X \cup S])| + |E(G[Y \cup S])| - |E(G[S])|
\]
\[
\leq \frac{1}{4}(|X| + |S|)^2 + \frac{1}{4}(|Y| + |S|)^2 - |E(G[S])| \quad \text{(by (62))}
\]
\[
\leq \frac{1}{4}(|X| + |S|)^2 + \frac{1}{4}(|Y| + |S|)^2
\]
\[
= \frac{1}{4}(|X| + |Y| + |S|)^2 + |S| - |X| \cdot |Y|
\]
\[
\leq \frac{n^2 + \kappa^2}{2} - \frac{(2\delta - \kappa)(n - 2\delta)}{2} \quad \text{(by (65))}
\]
\[
= \delta^2 + \frac{1}{4}(n - 2\delta + \kappa)^2
\]
\[
\leq \delta^2 + \frac{1}{4}(n - 2\delta + k - 1)^2, \quad \text{(by } \kappa \leq k - 1)\).
\]

Combining this with (61), we have \(m = \delta^2 + \left(\frac{1}{4}(n - 2\delta + k - 1)^2\right)\), and so \(|S| = \kappa = k - 1, |X| = 2\delta - \kappa + 1, |Y| = n - 2\delta, |E(G[S])| = 0, |E(G[X \cup S])| = \delta^2, \text{ and } |E(G[Y \cup S])| = \left(\frac{1}{4}(n - 2\delta + k - 1)^2\right)\). Therefore, \(G[S] = K_{K-1}, G[X \cup S] = K_{\delta}, \text{and } G[Y \cup S] = K_{(n-2\delta+k-1)/2}, (n-2\delta+k-1)/2\). This completes the proof. \(\square\)

**Theorem 17.** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\). If
\[
\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{\frac{1}{4}(n - 2\delta + k - 1)^2 - \delta(n - 2\delta) + \frac{(\delta + 1)^2}{4}}.
\]
then \(G\) is \(k\)-connected.

**Proof.** On the contrary, suppose that \(k(G) < k\). Since \(G\) is connected, by (67) and Lemma 1, we have
\[
\sqrt{\frac{1}{4}(n - 2\delta + k - 1)^2} - \delta(n - 2\delta) + \frac{(\delta + 1)^2}{4}
\]
\[
\leq \rho(G) - \frac{\delta - 1}{2} \leq \sqrt{|E(G)|} - \delta n + \frac{(\delta + 1)^2}{4},
\]
which yields
\[
|E(G)| \geq \delta^2 + \frac{1}{4}(n - 2\delta + k - 1)^2.
\] By setting \(k = \delta\) in Theorems 16 and 17, we obtain the two following theorems. \(\square\)

**Theorem 18.** Let \(G\) be a connected triangle-free graph of order \(n\), size \(m\), and minimum degree \(\delta \geq 2\). If
\[
m \geq \delta^2 + \frac{1}{4}(n - \delta - 1)^2,
\]
then \(G\) is maximally connected, unless \(V(G) = X \cup S \cup Y\) and \(S\) is a minimum vertex-cut of \(G\) with \(G[S] = K_{K-1}, G[X \cup S] = K_{\delta}, \text{ and } G[Y \cup S] = K_{(n-2\delta+k-1)/2}, (n-2\delta+k-1)/2\).

**Theorem 19.** Let \(G\) be a connected triangle-free graph of order \(n\) and minimum degree \(\delta \geq 2\). If
\[
\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{\frac{1}{4}(n - \delta - 1)^2 + \delta(n - 2\delta) + \frac{(\delta + 1)^2}{4}},
\]
then \(G\) is maximally connected.

For super-connected graphs, we have the following results.

**Theorem 20.** Let \(G\) be a connected triangle-free graph of order \(n\), size \(m\), and minimum degree \(\delta \geq 2\). If
\[
m \geq \delta^2 + \frac{1}{4}(n - \delta)^2,
\]
then \(G\) is super-\(k\).

**Proof.** Let \(k = k(G)\). On the contrary, suppose that \(G\) is not super-\(k\). Since \(m \geq \delta^2 + \left(\frac{1}{4}(n - \delta)^2\right)\), by Theorem 18, \(\kappa = \delta\). Let \(S\) be a minimum vertex-cut of \(G\) with \(\delta\) vertices, and let \(X, Y_1, \ldots, Y_{p+1}\) denote the vertex-sets of the components of \(G - S\), where \(2 \leq |X| \leq |Y_1| \leq \cdots \leq |Y_{p+1}|\). Set \(Y = \bigcup_{i=1}^{p+1} Y_i\) then \(|Y| \geq |X| \geq 2\). Therefore, with the same proceeding of the proof of Theorem 16 (from (62) to (65)), we arrive at
\[
\delta \leq |X| \leq |Y| \leq n - 2\delta.
\]
Together with \(|X| + |Y| = n - \delta\) and (62), it leads to
\[
m = |E(G[X \cup S])| + |E(G[Y \cup S])| - |E(G[S])|
\]
\[
\leq \frac{1}{4}(|X| + |S|)^2 + \frac{1}{4}(|Y| + |S|)^2 - |E(G[S])|
\]
\[
\leq \frac{1}{4}(|X| + |S|)^2 + \frac{1}{4}(|Y| + |S|)^2
\]
\[
= \frac{1}{4}(|X| + |Y| + |S|)^2 + |S| - |X| \cdot |Y|
\]
\[
= \frac{n^2 + \delta^2 - \delta(n - 2\delta)}{2}
\]
\[
= \delta^2 + \frac{1}{4}(n - \delta)^2.
\]
Combining this with (72), we have \( m = \delta^2 + \left\lfloor \frac{1}{4} (n - \delta^2) \right\rfloor \), and so \(|X| = |S| = \delta, |Y| = n - 2\delta, |E(G[S])| = 0, |E(G[X \cup S])| = \delta^2, \) and \(|E(G[Y \cup S])| = \left\lfloor \frac{1}{4} (n - \delta^2) \right\rfloor \). Therefore, \( G[S] = K_{\delta}, G[X \cup S] = K_{\delta, \delta}, \) and \( G[Y \cup S] = K_{n - 2\delta |(n - \delta^2)|} \). Thus, \( G[X] = K_{\delta} \), which contradicts the fact that \( G[X] \) is a component of \( G \) with at least two vertices. The result follows. \( \square \)

**Theorem 21.** Let \( G \) be a connected triangle-free graph of order \( n \) and minimum degree \( \delta \geq 2 \). If

\[
\rho(G) \geq \frac{\delta - 1}{2} + \sqrt{\frac{1}{4} (n - \delta^2) - \delta (n - 2\delta)} + \frac{(\delta + 1)^2}{4},
\]

then \( G \) is super-\( \kappa \).

**Proof.** Since \( G \) is connected, by (75) and Lemma 1, we have

\[
\sqrt{\frac{1}{4} (n - \delta^2) - \delta (n - 2\delta)} + \frac{(\delta + 1)^2}{4} \leq \rho(G) - \frac{\delta - 1}{2} \leq 2|E(G)| - \delta n + \frac{(\delta + 1)^2}{4},
\]

which yields

\[
|E(G)| \geq \delta^2 + \frac{1}{4} (n - \delta^2) .
\]

By Theorem 20, \( G \) is super-\( \kappa \). \( \square \)

**Remark 4.** The lower bound on \( m \) given in Theorem 20 is sharp. For example, let \( n = 3\delta + 3, V(G) = X \cup S \cup Y, G[X] = K_{\delta}, G[Y] = K_{1, \delta+1} \) and \( S \) is a minimum vertex-cut of \( G \) with \( G[S] = K_\delta, G[X \cup S] = K_{\delta, \delta+1}, \) and \( G[Y \cup S] = K_{\delta+1, \delta+1} \). It is easy to check that

\[
|E(G)| = \delta (\delta + 1) + (\delta + 1)^2 = \delta^2 + \frac{1}{4} (n - \delta^2) - 1.
\]

However, \( G - S = K_{1, \delta} \cup K_{1, \delta+1} \), which yields that \( G \) is not super-connected.

**Data Availability**

No data were used to support the findings of the study.

**Conflicts of Interest**

The authors declare no conflicts of interest.

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