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Research Article

Partition Dimension of Generalized Petersen Graph

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Let $G = (V_G, E_G)$ be the connected graph. For any vertex $i \in V_G$ and a subset $B \subseteq V_G$, the distance between i and B is $d(i; B) = \min\{d(i, j) | j \in B\}$. The ordered k-partition of V_G is $\Pi = \{B_1, B_2, \ldots, B_k\}$. The representation of vertex i with respect to Π is the k-vector, that is, $r(i|\Pi) = (d(i, B_1), d(i, B_2), \ldots, d(i, B_k))$. The partition Π is called the resolving (distinguishing) partition if $r(i|\Pi) \neq r(j|\Pi)$, for all distinct $i, j \in V_G$. The minimum cardinality of the resolving partition is called the partition dimension, denoted as $\mathrm{pd}(G)$. In this paper, we consider the upper bound for the partition dimension of the generalized Petersen graph in terms of the cardinalities of its partite sets.

1. Introduction

The graphs considered are simple, undirected, and without loops. The classical distance $d(i, j)$ between the two vertices $i, j V_G$ is the length of the shortest path between them. For an ordered set $R = \{\ell_1, \ell_2, \ldots, \ell_k\} \subseteq V_G$, the ordered k-tuple,

$$r(i|R) = (d(i, \ell_1), d(i, \ell_2), \ldots, d(i, \ell_k)), \tag{1}$$

is known as metric representation of i with respect to R. When the vertices of the graph G have distinct representation, the set R is termed as the resolving (distinguishing) set which contains minimum number of vertices. The minimum cardinality of such a resolving set is the metric dimension of the G, denoted as $\dim(G)$. Slater introduced this concept in 1975 [1], after which Melter and Harary independently renamed it the resolving set [2]. In the graph's theoretical study, this concept is called a metric basis or basis set. The concept of metric dimension does find applications in chemistry [3], else problem concerning pattern recognition and image processing, some involves the utilization of hierarchical data structures [4].

For an ordered-partition $\Pi = B_1, B_2, \ldots, B_k \subseteq V_G$, the partition representation of the vertex $i \in V_G$ with respect to Π is

$$r(i|\Pi) = (d(i, B_1), d(i, B_2), \ldots, d(i, B_k)), \tag{2}$$

where $d(i; B_h)\,(1 \leq h \leq k)$ denotes the distance between the vertex i and B_h, that is,

$$d(i; B_h) = \min\{d(i, j) | j \in B_h\}. \tag{3}$$

We can affirm that Π is the resolving partition of the graph G if different vertices have unique partition representation, that is, $r(i|\Pi) \neq r(j|\Pi)$, where $i, j \in V_G$. The partition dimension of the graph G is than the minimum number of resolving partition set in G. It is denoted as $pd(G)$. This concept was introduced by Chartrand et al. in 2000 [5].

The concept of resolving partition sets and partition dimensions extensively appeared in the literature. For example, the graph with partition dimension $|V| - 3$ is discussed [6] and the graph obtained by graph operations and its corresponding partition dimension is studied in [7]. The

bounds on the partition dimension for convex polytopes are studied in [8–10] and bounds of partition on the circulant and multipartite discussed in [11, 12]. The partition dimension of chemical structure fullerenes graphs is studied in [13], and on the bounded partition, dimension of the Cartesian product of graphs are studied in [14]. Furthermore, in [15], yielded bounds for the subdivision of different graphs, than in [16] presented the bounds on tree graphs. The bounds of unicyclic graphs in the form of subgraphs are considered in [17]. For more recent literature and results, we refer to see [18–24]. The graph obtained by graph operations and its corresponding partition dimension is considered in [7]. The relation between the metric dimension and partition dimension in a connected graph G is represented as follows.

Theorem 1 (see [3]). *For a nontrivial connected graph G,*

$$\dim(G) \leq \mathrm{pd}(G) + 1. \tag{4}$$

Some of the known results are related to the parameter of partition dimension of the graphs.

Theorem 2 (see [5]). *Let Π be a partition resolving set of $V(G)$ and $i, j \in V(G)$. If $d(i,m) = d(j,m)$, for all vertices $m \in V(G)\backslash(i, j)$, then i, j belongs to different subsets of G.*

Theorem 3 (see [5]). *Let G be a simple and connected graph; then,*

(i) $pd(G)$ is two iff G is the only path graph

(ii) $pd(G)$ is $|G|$ iff G is a complete graph

2. Generalized Petersen Graph $P_{n,k}$

Generalized Petersen graph $P_{n,k}$, where $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, is a graph having the vertex set

$$V(P_{n,k}) = \{\alpha_j, \beta_j \colon j = 1, 2, \ldots, n\}, \tag{5}$$

and the edge set

$$E(P_{n,k}) = \{\alpha_j\alpha_{j+1}, \beta_j\beta_{j+k}, \alpha_j\beta_j \colon j = 1, 2, \ldots, n\}. \tag{6}$$

Petersen graph is a cubic graph, and it has three types of edges according to the definition of generalized Petersen graphs; the edges between the vertices α_j and α_{j+1} are called outer edges, the edges produced by the vertices β_j and β_{j+k} are called inner edges, and the edges created by α_j and β_j are called spokes. The vertices α_j are called outer vertices and β_j are known as the inner vertices. The generalized Petersen graph $P_{12,4}$ is shown in Figure 1. The study on the resolving set and metric dimension of Petersen and generalized Petersen graphs, and we refer the readers to [25–27]. Motivated by these results on the metric dimension of the generalized Petersen graphs, we study the problem of partition dimension and have shown the sharp upper bound.

Let $m > 0$ be the positive integer called the modulus. We can say that two integers x and y are congruent modulo m if $y - x$ is divisible by m. It can be written as follows.

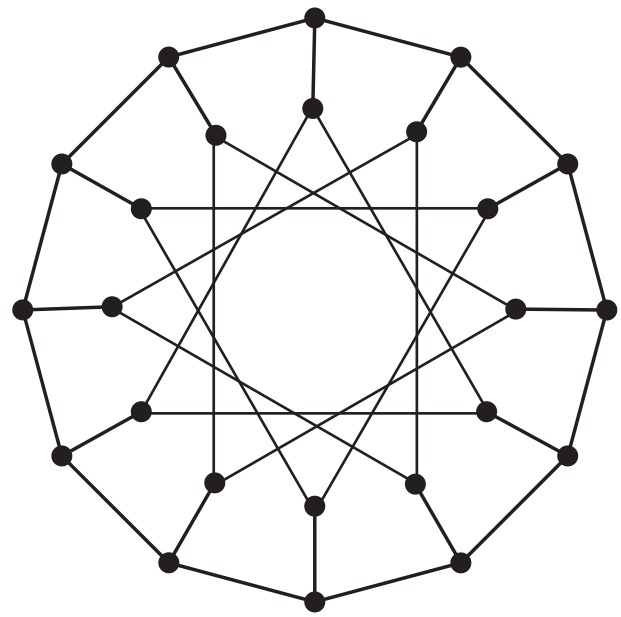

Figure 1: Generalized Petersen graph $P_{12,4}$.

$x = y \pmod m \Leftrightarrow x - y = m.k$, for some positive integer k. The symbol $y(\mathrm{MOD}\, m)$ denotes the smallest positive integer x.

$x \equiv y \pmod m$, where $y(\mathrm{MOD}\, m)$ is the remainder when y is divided by m as many times as possible.

The difference between $(\mathrm{MOD}\, m)$ and $(\mathrm{mod}\, m)$ is the first one is the equality relation, while the second one is the equivalence relation.

In this paper, we employed the equivalence relation because it can have many solutions for x. The partition dimension of the generalized Petersen graph is solved while taking the (mod 4), that is, for $n \equiv 0, 1, 2, 3 \pmod 4$.

3. Results on the Generalized Petersen Graph $P_{n,k}$

In this section, we provide sharp bounds on the partition dimension of the generalized Petersen graph $P_{n,k}$.

In the following section, $g = \lfloor j/2 \rfloor$, $\kappa = \lceil \ell/2 \rceil$, and p is 1 if the representation of the vertices belongs to the $\Pi\backslash B_4$ or B_1, B_2, B_3, otherwise 0.

Theorem 4. *Let $P_{n,k}$ is a generalized Petersen graph of order $n \geq 20$, when $n \equiv 0 \pmod 4$. Then, $pd(P_{n,k}) \leq 4$.*

Proof. Let $n = 4j$, where $j \geq 7 \in \mathbb{Z}^+$. The proof is exhibited by considering two cases. □

Case 1. When j is odd.

Assume the partition resolving set $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{n-2}\}$, and $B_4 = V(P_{n,k})\backslash \{\beta_1, \alpha_3, \beta_{n-2}\}$. Now, we exhibit the representations of vertices of $P_{n,k}$ with respect to Π.

The representations of the outer vertices; $r(\alpha_{4\ell+1}|\Pi) = (\ell+1, 2, \ell+3, p)$ if $\ell = 0, 1$; $r(\alpha_{4\ell+1}|\Pi) = (\ell+1, \ell+3, \ell+3, p)$ if $\ell = 2, \ldots, g-1$; $r(\alpha_{4\ell+1}|\Pi) = (\ell+1, \ell+3, \ell+2, p)$ if

$\ell = g$; $r(\alpha_{4\ell+1}|\Pi) = (j-\ell+1, j-\ell+4,, j-\ell+1, p)$ if $\ell = g+1, \ldots, j-1$:

$$r(\alpha_{4\ell+2}|\Pi) = \begin{cases} (\ell+2, 2\ell+1, \ell+2, 0), & \text{if } \ell = 0, 1, \\ (\ell+2, \ell+3, \ell+1, 0), & \text{if } \ell = 2, \ldots, g-1, \\ (\ell+2, \ell+3, \ell+1, 0), & \text{if } \ell = g, \\ (j-\ell+2, j-\ell+3, j-\ell, 0), & \text{if } \ell = g+1, \ldots, j-1, \end{cases}$$
$$r(\alpha_{4\ell+3}|\Pi) = \begin{cases} (\ell+3, \ell+2, \ell+3, 0), & \text{if } \ell = 1, \ldots, g-1, \\ (\ell+3, \ell+2, \ell+2, 0), & \text{if } \ell = g, \\ (j-\ell+2, j-\ell+2, j-\ell+1, 0), & \text{if } \ell = g+1, \ldots, j-1. \end{cases} \tag{7}$$

Further representation of outer vertices are $r(\alpha_{4\ell+4}|\Pi) = (3, 1, 4, p)$ if $\ell = 0$, $r(\alpha_{4\ell+4}|\Pi) = (\ell+3, \ell+3, \ell+4, p)$ if $\ell = 1, 2, \ldots, g-1$, $r(\alpha_{4\ell+4}|\Pi) = (\ell+2, \ell+3, \ell+2, p)$ if $\ell = g$, $r(\alpha_{4\ell+4}|\Pi) = (j-\ell+1, j-\ell+3, j-\ell+1, p)$ if $\ell = g+1, \ldots, j-2$, and $r(\alpha_{4\ell+4}|\Pi) = (2, 3, 3, p)$ if $\ell = j-1$.

The above representations indicate that there exist two no vertices having the exact representation in the outer cycle.

Now, the representations of the inner vertices are shown:

$$r(\beta_{4\ell+1}|\Pi) = \begin{cases} (\ell, \ell+2, \ell+4, p), & \text{if } \ell = 0, 1, \ldots, g-1, \\ (\ell, \ell+2, \ell+3, 0), & \text{if } \ell = g, \\ (j-\ell, j-\ell+3, j-\ell+2, 0), & \text{if } \ell = g+1, \ldots, j-1, \end{cases}$$
$$r(\beta_{4\ell+2}|\Pi) = \begin{cases} (\ell+3, \ell+2, \ell+1, 0), & \text{if } \ell = 0, 1, \ldots, g-1, \\ (\ell+3, \ell+2, \ell, 0), & \text{if } \ell = g, \\ (j-\ell+3, j-\ell+2, j-\ell-1, p), & \text{if } \ell = g+1, \ldots, j-1, \end{cases}$$
$$r(\beta_{4\ell+3}|\Pi) = \begin{cases} (\ell+4, \ell+1, \ell+4, p), & \text{if } \ell = 0, 1, \ldots, g-1, \\ (\ell+4, \ell+1, \ell+3, p), & \text{if } \ell = g, \\ (j-\ell+3, j-\ell+1, j-\ell+2, p), & \text{if } \ell = g+1, \ldots, j-1, \end{cases} \tag{8}$$
$$r(\beta_{4\ell+4}|\Pi) = \begin{cases} (\ell+4, \ell+2, \ell+5, p), & \text{if } \ell = 0, 1, \ldots, g-1, \\ (\ell+3, \ell+2, \ell+3, p), & \text{if } \ell = g, \\ (j-\ell+2, j-\ell+2, j-\ell+2, p), & \text{if } \ell = g+1, \ldots, j-2, \\ (3, 3, 4, p), & \text{if } \ell = j-1. \end{cases}$$

From the representation of inner vertices, it is evident that no two vertices have the exact representation in the inner cycle.

Case 2. When j is even.

Let the partition resolving set be $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_7\}$, $B_3 = \{\beta_{2j+1}\}$, and

$B_4 = V(P_{n,k}) \backslash \{\beta_1, \alpha_7, \beta_{2j+1}\}$; the following are the representations of the entire vertex set of $P_{n,k}$ with respect to Π.

The representation of the outer vertices:

$$r(\alpha_{4\ell+1}|\Pi) = \begin{cases} \left(1, 5, \frac{j+2}{2}, 0\right), & \text{if } \ell = 0, \\ \left(\ell+1, 2, \frac{j-2\ell+2}{2}, 0\right), & \text{if } \ell = 1, 2, \\ \left(\ell+1, \ell+2, \frac{j-2\ell+2}{2}, 0\right), & \text{if } \ell = 3, \ldots, \frac{j}{2}, \\ (\ell-1, \ell+2, 2, 0), & \text{if } \ell = \frac{j+2}{2}, \\ \left(j-\ell+1, j-\ell+5, \frac{2\ell-j+2}{2}, 0\right), & \text{if } \ell = \frac{j+4}{2}, \ldots, j-1, \end{cases}$$

$$r(\alpha_{4\ell+2}|\Pi) = \begin{cases} \left(2, 4, \frac{j+4}{2}, 0\right), & \text{if } \ell = 0, \\ \left(\ell+2, 2\ell-1, \frac{j-2\ell+4}{2}, 0\right), & \text{if } \ell = 1, 2, \\ \left(\ell+2, \ell+2, \frac{j-2\ell+4}{2}, 0\right), & \text{if } \ell = 3, \ldots, \frac{j}{2}, \\ \left(j-\ell+2, j-\ell+4, \frac{2\ell-j+4}{2}, 0\right), & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1, \end{cases} \tag{9}$$

$$r(\alpha_{4\ell+3}|\Pi) = \begin{cases} \left(3, 3, \frac{j+4}{2}, 0\right), & \text{if } \ell = 0, \\ \left(\ell+3, \ell+1, \frac{j-2\ell+4}{2}, 0\right), & \text{if } \ell = 2, 3, \ldots, \frac{j-2}{2}, \\ (\ell+2, \ell+1, 3, 0), & \text{if } \ell = \frac{j}{2}, \\ \left(j-\ell+2, j-\ell+3, \frac{2\ell-j+6}{2}, 0\right) & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1, \end{cases}$$

$$r(\alpha_{4\ell+4}|\Pi) = \begin{cases} \left(\ell+3, 3-2\ell, \frac{j-2\ell+2}{2}, 0\right), & \text{if } \ell = 0, 1, \\ \left(\ell+3, \ell+2, \frac{j-2\ell+2}{2}, 0\right), & \text{if } \ell = 2, \ldots, \frac{j-2}{2}, \\ (\ell+1, \ell+2, 3, 0), & \text{if } \ell = \frac{j}{2}, \\ \left(j-\ell+1, j-\ell+4, \frac{2\ell-j+6}{2}, 0\right), & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1. \end{cases}$$

The representation of outer vertices within themselves is unique, with no two vertices consisting of the exact representation.

Now, we exhibit the representations of the inner vertices:

$$r(\beta_{4\ell+1}|\Pi) = \begin{cases} \left(1, 3, \frac{j-2}{2}, p\right), & \text{if } \ell = 1, \\ \left(\ell, \ell+1, \frac{j-2\ell}{2}, p\right), & \text{if } \ell = 2, \ldots, \frac{j-2}{2}, \\ (j-\ell, \ell+1, 1, p), & \text{if } \ell = \frac{j+2}{2}, \\ \left(j-\ell, j-\ell+4, \frac{2\ell-j}{2}, p\right), & \text{if } \ell = \frac{j+4}{2}, \ldots, j-1, \end{cases}$$

$$r(\beta_{4\ell+2}|\Pi) = \begin{cases} \left(3, 3, \frac{j+6}{2}, p\right), & \text{if } \ell = 0, \\ \left(\ell+3, \ell+1, \frac{j-2\ell+6}{2}, p\right), & \text{if } \ell = 1, 2, \ldots, \frac{j}{2}, \\ \left(j-\ell+3, j-\ell+3, \frac{2\ell-j+6}{2}, p\right), & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1, \end{cases}$$

$$r(\beta_{4\ell+3}|\Pi) = \begin{cases} \left(4, 2, \frac{j+6}{2}, p\right), & \text{if } \ell = 0, \\ \left(\ell+4, \ell, \frac{j-2\ell+6}{2}, p\right), & \text{if } \ell = 1, 2, 3, \ldots, \frac{j-2}{2}, \\ (\ell+3, \ell, 4, p), & \text{if } \ell = \frac{j}{2}, \\ \left(j-\ell+3, j-\ell+2, \frac{2\ell-j+8}{2}, p\right), & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1, \end{cases} \tag{10}$$

$$r(\beta_{4\ell+4}|\Pi) = \begin{cases} \left(4, 3, \frac{j+4}{2}, p\right), & \text{if } \ell = 0, \\ \left(\ell+4, \ell+1, \frac{j-2\ell+4}{2}, p\right), & \text{if } \ell = 1, 2, 3, \ldots, \frac{j-2}{2}, \\ (\ell+2, \ell+1, 4, p), & \text{if } \ell = \frac{j}{2}, \\ \left(j-\ell+2, j-\ell+3, \frac{2\ell-j+8}{2}, p\right), & \text{if } \ell = \frac{j+2}{2}, \ldots, j-1. \end{cases}$$

The representations indicate that no two inner vertices among them have the exact representation.

The outer and inner vertices in the graph of $P_{n,k}$ have distinct representations. Thus, $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{n-2}\}$, and $B_4 = V(P_{n,k})\backslash \{\beta_1, \alpha_3, \beta_{n-2}\}$ is the partition resolving set for odd j. Moreover, $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_7\}$, $B_3 = \{\beta_{2j+1}\}$, and $B_4 = V(P_{n,k})\backslash\{\beta_1, \alpha_7, \beta_{2j+1}\}$ is the partition resolving for even j. Hence,

$$\mathrm{pd}\left(P_{n,k}\right) \leq 4. \tag{11}$$

From the following theorem p is 1, if the representation of the vertices belongs to the $\Pi\backslash B_5$ or B_1, B_2, B_3, B_4, otherwise $p = 0$.

Theorem 5. *Let $P_{n,k}$ is a generalized Petersen graph with $n \geq 17$, $n \equiv 1 \pmod 4$. Then, pd $(P_{n,k}) \leq 5$.*

Proof. Let $n = 4j + 1$, $j \geq 4, \in \mathbb{Z}^+$. □

Case 3. Let the partition resolving set be $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_2\}$, $B_3 = \{\beta_3\}$, $B_4 = \{\alpha_5\}$, and $B_5 = V(P_{n,k})\backslash\{\beta_1, \alpha_2, \beta_3, \alpha_5\}$; the following are the representations of the entire vertex set of $P_{n,k}$ with respect to Π.

The representation of the outer vertices:

$$\begin{aligned}
r\left(\alpha_{4\ell+1}|\Pi\right) &= \begin{cases} (1,1,3,3,0), & \text{if } \ell = 0, \\ (\ell+1, \ell+3, \ell+2, \ell+1, 0), & \text{if } \ell = 2, \ldots, g, \\ (j-\ell+2, j-\ell+4, j-\ell+3, 2\ell-g, 0), & \text{if } \ell = g+1, \\ (j-\ell+2, j-\ell+4, j-\ell+3, j-\ell+4, 0), & \text{if } \ell = g+2, \ldots, j-1, \\ (2,2,3,4,0), & \text{if } \ell = j, \end{cases} \\
r\left(\alpha_{4\ell+2}|\Pi\right) &= \begin{cases} (3,3,3,1,0), & \text{if } \ell = 1, \\ (\ell+2, \ell+2, \ell+2, \ell+2, 0), & \text{if } \ell = 2, 3, \ldots, g-1, \\ (j-\ell+1, 2\ell-g+2, 2\ell-g+2, 2\ell-g+2, 0), & \text{if } \ell = g, \\ (j-\ell+1, j-\ell+3, j-\ell+3, j-\ell+3, 0), & \text{if } \ell = g+1, \ldots, j-1, \end{cases} \\
r\left(\alpha_{4\ell+3}|\Pi\right) &= \begin{cases} (3+\ell, 3\ell+1, \ell+1, 2, 0), & \text{if } \ell = 0, 1, \\ (\ell+3, \ell+3, \ell+1, \ell+3, 0), & \text{if } \ell = 2, \ldots, g-1, \\ (j-\ell+2, j-\ell+2, 2\ell-g+1, 2\ell-g+3, 0), & \text{if } \ell = g, \\ (j-\ell+2, j-\ell+2, j-\ell+2, j-\ell+4, 0), & \text{if } \ell = g+1, \ldots, j-1, \end{cases} \\
r\left(\alpha_{4\ell+4}|\Pi\right) &= \begin{cases} (3+\ell, 3\ell+2, \ell+2, 2\ell+1, 0), & \text{if } \ell = 0, 1, \\ (\ell+3, \ell+4, \ell+2, \ell+3, 0), & \text{if } \ell = 2, \ldots, g-1, \\ (j-\ell+2, j-\ell+3, j-\ell+1, 2\ell-g+3, 0), & \text{if } \ell = g, \\ (j-\ell+2, j-\ell+3, j-\ell+1, j-\ell+4, 0), & \text{if } \ell = g+1, \ldots, j-2, \\ (3,3,2,5,p), & \text{if } \ell = j-1. \end{cases}
\end{aligned} \tag{12}$$

The representation of outer vertices between them is distinct concerning Π.

The representation of the inner vertices:

$$
\begin{aligned}
r(\beta_{4\ell+1}|\Pi) &= \begin{cases} (\ell, \ell+2, \ell+3, \ell, p), & \text{if } \ell = 1, 2, 3, \ldots, g, \\ (2\ell - g - 1, j - \ell + 3, j - \ell + 4, 2\ell - g - 1, p), & \text{if } \ell = g + 1, \\ (j - \ell + 3, j - \ell + 3, j - \ell + 4, j - \ell + 3, p), & \text{if } \ell = g + 2, \ldots, j, \end{cases} \\
r(\beta_{4\ell+2}|\Pi) &= \begin{cases} (3, 1, 3, 3, p), & \text{if } \ell = 0, \\ (\ell + 3, \ell + 1, \ell + 3, \ell + 1, p), & \text{if } \ell = 2, 3, \ldots, g - 1, \\ (j - \ell, 2\ell - g + 1, 2\ell - g + 3, 2\ell - g + 1, p), & \text{if } \ell = g, \\ (j - \ell, j - \ell + 2, j - \ell + 4, j - \ell + 2, p), & \text{if } \ell = g + 1, \ldots, j - 1, \end{cases} \\
r(\beta_{4\ell+3}|\Pi) &= \begin{cases} (4 + \ell, \ell + 2, \ell, \ell + 2, p), & \text{if } \ell = 1, 2, \ldots, g - 1, \\ (j - \ell + 3, j - \ell + 1, 2\ell - g, 2\ell - g + 2, p), & \text{if } \ell = g, \\ (j - \ell + 3, j - \ell + 1, 2\ell - g - 1, j - \ell + 3, p), & \text{if } \ell = g + 1, \\ (j - \ell + 3, j - \ell + 1, j - \ell + 3, j - \ell + 3, p), & \text{if } \ell = g + 2, \ldots, j - 1, \end{cases} \\
r(\beta_{4\ell+4}|\Pi) &= \begin{cases} (4 + \ell, \ell + 3, \ell + 3, \ell + 2, p), & \text{if } \ell = 0, 1, \ldots, g - 2, \\ (2g - \ell + 2, 2\ell - g + 4, j - \ell, 2\ell - g + 3, p), & \text{if } \ell = g - 1, \\ (j - \ell + 3, j - \ell + 2, j - \ell, 2\ell - g + 2, p), & \text{if } \ell = g, \\ (j - \ell + 3, j - \ell + 2, j - \ell, j - \ell + 3, p), & \text{if } \ell = g + 1, \ldots, j - 1. \end{cases}
\end{aligned} \tag{13}
$$

The inner vertices have unique representation with respect to Π. Moreover, the outer and inner vertices have distinct representation among them. Thus, $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_2\}$, $B_3 = \{\beta_3\}$, $B_4 = \{\alpha_5\}$, and $B_5 = V(P_{n,k}) \backslash \{\beta_1, \alpha_2, \beta_3, \alpha_5\}$, is the partition resolving set for $n = 4j + 1$. Hence,

$$\mathrm{pd}\left(P_{n,k}\right) \leq 5. \tag{14}$$

Theorem 6. *Let $P_{n,k}$ be a generalized Petersen graph with $n \geq 18$, when $n \equiv 2 \pmod 4$. Then, pd $(P_{n,k}) \leq 5$.*

Proof. Let $n = 4j + 2$, $j \geq 7 \in \mathbb{Z}^+$. The proof is split into two cases. □

Case 4. When j is odd.

Let the partition resolving set be $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{2j+1}\}$, $B_4 = \{\alpha_{2j+3}\}$, and $B_5 = V(P_{n,k}) \backslash \{\beta_1, \alpha_3, \beta_{2j+1}, \alpha_{2j+3}\}$.

The representation of the outer vertices:

$$
\begin{aligned}
r(\alpha_{4\ell+1}|\Pi) &= \begin{cases} (\ell + 1, 2, g + 2, g - \ell + 3, 0), & \text{if } \ell = 0, 1, \\ (\ell + 1, \ell + 3, g - \ell + 3, g - \ell + 3, 0), & \text{if } \ell = 2, 3, \ldots, g, \\ (j - \ell + 3, j - \ell + 3, \ell - g + 2, \ell - g + 1, 0), & \text{if } \ell = g + 1, \ldots, j - 1, \\ (3, 3, g + 2, g + 2, 0), & \text{if } \ell = j, \end{cases} \\
r(\alpha_{4\ell+2}|\Pi) &= \begin{cases} (\ell + 2, 2\ell + 1, g - \ell + 2, g - \ell + 4, 0), & \text{if } \ell = 0, 1, \\ (\ell + 2, \ell + 3, g - \ell + 2, g - \ell + 4, 0), & \text{if } \ell = 2, 3, \ldots, g - 1, \\ (g + 2, \ell + 3, \ell - g + 2, j - 2\ell + 2, 0), & \text{if } \ell = g, g + 1, \ldots, j - 1, \\ (j - \ell + 2, j - \ell + 4, \ell - g + 2, \ell - g + 2, 0), & \text{if } \ell = g + 2, \ldots, j - 1, \\ (2, 3, g + 3, g + 3, 0), & \text{if } \ell = j, \end{cases} \\
r(\alpha_{4\ell+3}|\Pi) &= \begin{cases} (\ell + 3, \ell + 2, g - \ell + 1, g - \ell + 4, p), & \text{if } \ell = 1, 2, \ldots, g - 1, \\ (j - \ell + 1, \ell + 2, \ell - g + 1, 2, p), & \text{if } \ell = g, g + 1, \\ (j - \ell + 1, j - \ell + 4, \ell - g + 1, \ell - g + 3, p), & \text{if } \ell = g + 2, \ldots, j - 1, \end{cases} \\
r(\alpha_{4\ell+4}|\Pi) &= \begin{cases} (3, 1, g + 2, g + 3, 0), & \text{if } \ell = 0, \\ (\ell + 3, \ell + 3, g - \ell + 2, g - \ell + 3, 0), & \text{if } \ell = 1, 2, \ldots, g - 1, \\ (j - \ell + 2, g + 3, \ell - g + 2, 2\ell - j + 2, 0), & \text{if } \ell = g, g + 1, \\ (j - \ell + 2, j - \ell + 3, \ell - g + 2, \ell - g + 3, 0), & \text{if } \ell = g + 2, \ldots, j - 1. \end{cases}
\end{aligned} \tag{15}
$$

The representation of the outer cycle vertices is distinct concerning Π.

Now, the representation of the inner vertices:

$$
\begin{aligned}
r(\beta_{4\ell+1}|\Pi) &= \begin{cases} (1,3,g+2,g+1,p), & \text{if } \ell=1,\\ (\ell,\ell+2,g-\ell+4,g-\ell+2,p), & \text{if } \ell=2,3,\ldots,g,\\ (\ell,j-\ell+2,\ell-g+3,\ell-g,p), & \text{if } \ell=g+1,g+2,\\ (j-\ell+4,j-\ell+2,\ell-g+3,\ell-g,p), & \text{if } \ell=g+3,g+4,\ldots,j-2,\\ (j-\ell+4,j-\ell+2,j-\kappa+1,\ell-g,p), & \text{if } \ell=j-1,j, \end{cases}\\
r(\beta_{4\ell+2}|\Pi) &= \begin{cases} (\ell+3,\ell+2,g-\ell+3,g-\ell+3,p), & \text{if } \ell=0,1,\ldots,g,\\ (j-\ell+3,j-\ell+3,\ell-g+3,\ell-g+1,p), & \text{if } \ell=g+1,\ldots,j, \end{cases}\\
r(\beta_{4\ell+3}|\Pi) &= \begin{cases} (4,1,g,g+2,p), & \text{if } \epsilon=0,\\ (\ell+4,\ell+1,g-\ell,g-\ell+3,p), & \text{if } \ell=1,2,\ldots,g-2,\\ (\ell+3,\ell+1,1,4,p), & \text{if } \ell=g-1,\\ (j-\ell,j-\ell+2,1,3,p), & \text{if } \ell=g+1,\\ (j-\ell,j-\ell+3,\ell-g,\ell-g+2,p), & \text{if } \ell=g+2,\ldots,j-1, \end{cases}\\
r(\beta_{4\ell+4}|\Pi) &= \begin{cases} (\ell+4,\ell+2,g-\ell+3,g-\ell+2,p), & \text{if } \ell=0,1,\ldots,g,\\ (j-\ell+3,j-\ell+2,\ell-g+3,\ell-g+2,p), & \text{if } \ell=g+1,g+2,\ldots,j-1. \end{cases}
\end{aligned}
\tag{16}
$$

The representations of inner cycle vertices with respect to Π are distinct. Both the outer and inner cycle vertices show distinct representations. Thus, $\Pi=\{B_1,B_2,B_3,B_4\}$, where $B_1=\{\beta_1\}$, $B_2=\{\alpha_3\}$, $B_3=\{\beta_{2j+1}\}$, $B_4=\{\alpha_{2j+3}\}$, and $B_5=V(P_{n,k})\backslash\{\beta_1,\alpha_3,\beta_{2j+1},\alpha_{2j+3}\}$, is the partition resolving set when j is odd.

Case 5. When j is even.

Let the partition resolving set be $\Pi=\{B_1,B_2,B_3,B_4\}$, where $B_1=\{\beta_1\}$, $B_2=\{\alpha_3\}$, $B_3=\{\beta_{2j+1}\}$, $B_4=\{\alpha_{2j+5}\}$, and $B_5=V(P_{n,k})\backslash\{\beta_1,\alpha_3,\beta_{2j+1},\alpha_{2j+5}\}$.

The representation of the outer vertices:

$$
r(\alpha_{4\ell+1}|\Pi)=\begin{cases} \left(\ell+1,2,\dfrac{j-2\ell+2}{2},\dfrac{j-2\ell+6}{2},0\right), & \text{if } \ell=0,1,\\[2ex] \left(\ell+1,\ell+3,\dfrac{j-2\ell+2}{2},\dfrac{j-2\ell+6}{2},0\right), & \text{if } \ell=2,3,\ldots,\dfrac{j}{2},\\[2ex] \left(j-\ell+3,j-\ell+3,\dfrac{2\ell-j+2}{2},\dfrac{2\ell-j+2}{2},0\right), & \text{if } \ell=\dfrac{j+2}{2},\ldots,j, \end{cases}
$$

$$
r(\alpha_{4\ell+2}|\Pi)=\begin{cases} \left(\ell+2,2\ell+1,\dfrac{j-2\ell+4}{2},\dfrac{j+6}{2},0\right), & \text{if } \ell=0,1,\\[2ex] \left(\ell+2,\ell+3,\dfrac{j-2\ell+4}{2},\dfrac{j-2\ell+8}{2},0\right), & \text{if } \ell=2,3,\ldots,\dfrac{j-2}{2},\\[2ex] \left(j-\ell+2,\dfrac{j+6}{2},\dfrac{2\ell-j+4}{2},j-2\ell+3,0\right), & \text{if } \ell=\dfrac{j}{2},\dfrac{j+2}{2},\\[2ex] \left(j-\ell+2,j-\ell+4,\dfrac{2\ell-j+4}{2},\dfrac{2\ell-j+4}{2},0\right), & \text{if } \ell=\dfrac{j+4}{2},\ldots,j-1,\\[2ex] \left(2,3,\dfrac{j+4}{2},\dfrac{j+4}{2},p\right), & \text{if } \ell=j, \end{cases}
$$

$$r(\alpha_{4\ell+3}|\Pi)=\begin{cases}\left(\ell+3,\ell+2,\frac{j-2\ell+4}{2},\frac{j-2\ell+8}{2},p\right), & \text{if}\,\ell=0,1,\ldots,\frac{j-2}{2},\\ \left(j-\ell+1,\ell+2,\frac{2\ell-j+6}{2},2,0\right), & \text{if }\ell=\frac{j}{2},\frac{j+2}{2},\\ \left(j-\ell+1,j-\ell+4,\frac{2\ell-j+6}{2},\frac{2\ell-j+6}{2},0\right), & \text{if }\ell=\frac{j+4}{2},\ldots,j-1,\end{cases}$$
$$r(\alpha_{4\ell+4}|\Pi)=\begin{cases}\left(3,1,\frac{j+2}{2},\frac{j+6}{2},p\right), & \text{if }\ell=0,\\ \left(\ell+3,\ell+3,\frac{j-2\ell+2}{2},\frac{j-2\ell+6}{2},0\right), & \text{if }\ell=1,\ldots,\frac{j-2}{2},\\ \left(j-\ell+2,j-\ell+3,\frac{2\ell-j+6}{2},2\ell-j+1,0\right), & \text{if }\ell=\frac{j}{2},\frac{j+2}{2},\\ \left(j-\ell+2,j-\ell+3,\frac{2\ell-j+6}{2},\frac{2\ell-j+6}{2},0\right), & \text{if }\ell=\frac{j+4}{2},\ldots,j-1.\end{cases}\tag{17}$$

The representations of the outer cycle vertices are distinct concerning Π.

Now, the representations of the inner vertices:

$$r(\beta_{4\ell+1}|\Pi)=\begin{cases}\left(\ell,\ell+2,\frac{j-2\ell}{2},\frac{j-2\ell+4}{2},p\right), & \text{if }\ell=1,\ldots,\frac{j-2}{2},\\ (\ell,\ell,1,1,p), & \text{if }\ell=\frac{j+2}{2},\\ \left(j-\ell+4,j-\ell+2,\frac{2\ell-j}{2},\frac{2\ell-j}{2},p\right), & \text{if }\ell=\frac{j+4}{2},\ldots,j,\end{cases}$$
$$r(\beta_{4\ell+2}|\Pi)=\begin{cases}\left(3,2,\frac{j+6}{2},\frac{j+4}{2},p\right), & \text{if }\ell=0,\\ \left(\ell+3,\ell+2,\frac{j-2\ell+6}{2},\frac{j-2\ell+6}{2},p\right), & \text{if }\ell=1,\ldots,\frac{j}{2},\\ \left(j-\ell+3,j-\ell+3,\frac{2\ell-j+6}{2},\frac{2\ell-j+2}{2},p\right), & \text{if }\ell=\frac{j+2}{2},\ldots,j.\end{cases}\tag{18}$$

$$r(\beta_{4\ell+3}|\Pi)=\begin{cases}\left(4,1,\frac{j+2}{2},\frac{j+2}{2},p\right), & \text{if }\ell=0,\\ \left(\ell+4,\ell+1,\frac{j-2\ell+6}{2},\frac{j-2\ell+6}{2},p\right). & \text{if }\ell=1,2,\ldots,\frac{j-4}{2},\\ (j-\ell,\ell+1, & \\ 4,\frac{j-2\ell+6}{2},p\Big), & \text{if }\ell=\frac{j-2}{2},\frac{j}{2},\\ \left(j-\ell,j-\ell+3,\frac{2\ell-j+8}{2},\frac{2\ell-j+4}{2},p\right) & \text{if }\ell=\frac{j+2}{2},\ldots,j-2,\\ \left(1,4,\frac{2\ell-j+4}{2},\frac{2\ell-j+4}{2},p\right), & \text{if }\ell=j-1.\end{cases}\tag{19}$$

$$r(\beta_{4\ell+4}|\Pi) = \begin{cases} \left(\ell+4, \ell+2, \frac{j-2\ell+4}{2}, \frac{2\ell-j+4}{2}, p\right), & \text{if } \ell = 0, 1, \ldots, \frac{j-2}{2}, \\ \left(j-\ell+3, j-\ell+2, \frac{2\ell-j+8}{2}, \frac{2\ell-j+4}{2}, p\right), & \text{if } \ell = \frac{j}{2}, \frac{j+2}{2}, \ldots, j-1. \end{cases} \tag{20}$$

The representations of the inner cycle vertices are distinct. Also, outer and inner vertices are distinct as well with respect to Π. Thus, $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{2j+1}\}$, $B_4 = \{\alpha_{2j+5}\}$, and $B_5 = V(P_{n,k}) \backslash \{\beta_1, \alpha_3, \beta_{2j+1}, \alpha_{2j+5}$, is the partition resolving set, when j is even. From above facts, we have

$$\mathrm{pd}(P_{n,k}) \leq 5. \tag{21}$$

Theorem 7. *Let $P_{n,k}$ be a generalized Petersen graph with $n \geq 19$ and $n \equiv 3 \pmod 4$. Then, pd $(P_{n,k}) \leq 5$.*

Proof. When $n \equiv 3 \pmod 4$, $n = 4j+3$, and $j \geq 7 \in \mathbb{Z}^+$. The proof is divided in two cases. □

Case 6. When j is odd.

Let the partition resolving set be $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{2j+1}\}$, $B_4 = \{\alpha_{2j+2}\}$, and $B_5 = V(P_{n,k}) \backslash \{\beta_1, \alpha_3, \beta_{2j+1}, \alpha_{2j+2}\}$. The representation of vertices with respect to Π is presented.

The representations of the outer vertices:

$$r(\alpha_{4\ell+1}|\Pi) = \begin{cases} (\ell+1, 2, g-\ell+3, g+3, 0), & \text{if } \ell = 0, 1, \\ (\ell+1, \ell+3, g-\ell+3, g-\ell+4, 0), & \text{if } \ell = 2, \ldots, g-1, \\ (\ell+1, \ell+3, 3, j-2\ell+2, 0), & \text{if } \ell = g, g+1, \\ (j-\ell+3, j-\ell+4, \ell-g+2, \ell-g+2, 0), & \text{if } \ell = g+2, \ldots, j. \end{cases} \tag{22}$$

$$r(\alpha_{4\ell+2}|\Pi) = \begin{cases} (\ell+2, 2\ell+1, g-\ell+2, g-\ell+4, p), & \text{if } \ell = 0, 1, \\ (\ell+2, \ell+3, g-\ell+2, g-\ell+4, p), & \text{if } \ell = 2, \ldots, g-1, \\ (\ell+2, g+3, \ell-g+2, 2, p), & \text{if } \ell = g, g+1, \\ (j-\ell+3, j-\ell+3, \ell-g+2, \ell-g+3, p), & \text{if } \ell = g+2, \ldots, j-1, \\ (3, 3, g+2, g+4, p), & \text{if } \ell = j. \end{cases} \tag{23}$$

$$r(\alpha_{4\ell+3}|\Pi) = \begin{cases} (\ell+3, \ell+2, g-\ell+1, g-\ell+3, 0), & \text{if } \ell = 1, 2, \ldots, g-1, \\ (j-\ell+2, \ell+2, \ell-g+1, 2\ell-j+2, 0), & \text{if } \ell = g, g+1, \\ (j-i+2, j-\ell+4, \ell-g+1, \ell-g+3, 0), & \text{if } \ell = g+2, \ldots, j-1, \\ (2, 3, g+2, g+3, 0), & \text{if } \ell = j. \end{cases} \tag{24}$$

$$r(\alpha_{4\ell+4}|\Pi) = \begin{cases} (3, 1, g+2, g+2, 0), & \text{if } \ell = 0, \\ (\ell+3, \ell+3, g-\ell+2, g-\ell+2, 0), & \text{if } \ell = 1, 2, \ldots, g-1, \\ (j-\ell+1, j-\ell+4, \ell-g+2, \ell-g+2, 0), & \text{if } \ell = g+1, \ldots, j-1. \end{cases} \tag{25}$$

The representation of the outer cycle vertices with respect to Π is distinct.

Now, the representation of the inner vertices:

$$r(\beta_{4\ell+1}|\Pi) = \begin{cases} (\ell, \ell+2, g-\ell+4, g-\ell+3, p), & \text{if } \ell = 1, 2, \ldots, g, \\ (\ell, j-\ell+3, \ell-g+3, \ell-g+1, p), & \text{if } \ell = g+1, g+2, \\ (j-\ell+4, j-\ell+3, \ell-g+3, \ell-g+1, p), & \text{if } \ell = g+3, g+4, \ldots, j. \end{cases} \tag{26}$$

$$r(\beta_{4\ell+2}|\Pi) = \begin{cases} (\ell+3, \ell+2, g-\ell+3, g-\ell+3, p), & \text{if } \ell = 0, 1, \ldots, g, \\ (j-\ell+4, j-\ell+2, \ell-g+3, \ell-g+2, p), & \text{if } \ell = g+1, g+2, \ldots, j-2, \\ (j-\ell+4, j-\ell+2, j-\kappa+1, \ell-g+2, p), & \text{if } \ell = j-1, j. \end{cases} \tag{27}$$

$$r\left(\beta_{4\ell+3}|\Pi\right)=\begin{cases}(\ell+4,\ell+1,g-\ell,g-\ell+2,p), & \text{if } \ell=0,1,\ldots,g-1,\\ (\ell+2,\ell+1,1,3,p), & \text{if } \ell=g+1,\\ (j-\ell+3,j-\ell+3,\ell-g,\ell-g+2,p), & \text{if } \ell=g+2,\ldots,j-1,\\ (3,3,\ell-g,\ell-g+1,p), & \text{if } \ell=j.\end{cases} \tag{28}$$

$$r\left(\beta_{4\ell+4}|\Pi\right)=\begin{cases}(4,2,g+2,g+1,p), & \text{if } \ell=0,\\ (\ell+4,\ell+2,g-\ell+3,g-\ell+1,p), & \text{if } \ell=1,2,\ldots,g-2,\\ (j-\ell,\ell+2,g-\ell+3,g-\ell+1,p), & \text{if } \ell=g-1,g,\\ (j-\ell,j-\ell+3,\ell-g+3,g-\ell+1,p), & \text{if } \ell=g+1,\ldots,j-1.\end{cases} \tag{29}$$

The representations of the inner cycle vertices are distinct with respect to Π. Also, the outer and inner vertices have distinct representations as well. Thus, $\Pi=\{B_1,B_2,B_3,B_4\}$, where $B_1=\{\beta_1\}$, $B_2=\{\alpha_3\}$, $B_3=\{\beta_{2j+1}\}$, $B_4=\{\alpha_{2j+2}\}$, and $B_5=V(P_{n,k})\backslash\{\beta_1,\alpha_3,\beta_{2j+1},\alpha_{2j+2}\}$, is the partition resolving set when j is odd.

Case 7. When j is even.

Let the partition resolving set be $\Pi=\{B_1,B_2,B_3,B_4\}$, where $B_1=\{\beta_1\}$, $B_2=\{\alpha_3\}$, $B_3=\{\beta_{2j+1}\}$, $B_4=\{\alpha_{2j+3}\}$, and $B_5=V(P_{n,k})\backslash\{\beta_1,\alpha_3,\beta_{2j+1},\alpha_{2j+3}\}$.

The representation of the outer vertices:

$$r\left(\alpha_{4\ell+1}|\Pi\right)=\begin{cases}\left(\ell+1,2,\frac{j-2\ell+2}{2},\frac{j+6}{2},0\right), & \text{if } \ell=0,1,\\ \left(\ell+1,\ell+3,\frac{j-2\ell+2}{2},\frac{j-2\ell+8}{2},0\right), & \text{if } \ell=2,\ldots,\frac{j-2}{2},\\ \left(\ell+1,\frac{j+6}{2},\frac{2\ell-j+2}{2},2,0\right), & \text{if } \ell=\frac{j}{2},\frac{j+2}{2},\\ \left(j-\ell+3,j-\ell+4,\frac{2\ell-j+2}{2},\frac{2\ell-j+6}{2},0\right), & \text{if } \ell=\frac{j+2}{2},\ldots,j.\end{cases} \tag{30}$$

$$r\left(\alpha_{4\ell+2}|\Pi\right)=\begin{cases}\left(\ell+2,2\ell+1,\frac{j-2\ell+4}{2},\frac{j-2\ell+6}{2},0\right), & \text{if } \ell=0,1,\\ \left(\ell+2,\ell+3,\frac{j-2\ell+4}{2},\frac{j-2\ell+6}{2},0\right), & \text{if } \ell=2,\ldots,\frac{j-2}{2},\\ \left(\frac{j+4}{2},j-\ell+3,\frac{2\ell-j+4}{2},2\ell-j+1,0\right), & \text{if } \ell=\frac{j}{2},\frac{j+2}{2},\\ \left(j-\ell+3,j-\ell+3,\frac{2\ell-j+4}{2},\frac{2\ell-j+6}{2},0\right), & \text{if } \ell=\frac{j+4}{2},\ldots,j.\end{cases} \tag{31}$$

$$r\left(\alpha_{4\ell+3}|\Pi\right)=\begin{cases}\left(\ell+3,\ell+2,\frac{j-2\ell+4}{2},\frac{j-2\ell+4}{2},p\right), & \text{if } \ell=1,\ldots,\frac{j-2}{2},\\ \left(j-\ell+2,j-\ell+4,\frac{2\ell-j+6}{2},\frac{2\ell-j+4}{2},p\right), & \text{if } \ell=\frac{j+2}{2},\ldots,j,\\ \left(2,3,\frac{j+4}{2},\frac{j+4}{2},p\right), & \text{if } \ell=j.\end{cases} \tag{32}$$

$$r\left(\alpha_{4\ell+4}|\Pi\right)=\begin{cases}\left(\ell+3,\ell+1,\frac{j+2}{2},\frac{j+6}{2},0\right), & \text{if } \ell=0,\\ \left(\ell+3,\ell+3,\frac{j-2\ell+2}{2},\frac{j-2\ell+6}{2},0\right), & \text{if } \ell=1,\ldots,\frac{j-4}{2},\\ \left(j-\ell+1,\ell+3,\frac{2\ell-j+6}{2},j-2\ell+1,0\right), & \text{if } \ell=\frac{j-2}{2},\frac{j}{2},\\ \left(j-\ell+1,j-\ell+4,\frac{2\ell-j+6}{2},\frac{2\ell-j+6}{2},0\right), & \text{if } \ell=\frac{j+2}{2},\ldots,j-1.\end{cases} \tag{33}$$

The representation of the outer cycle vertices with respect to Π is distinct.

Now, the representation of the inner vertices:

$$r\left(\beta_{4\ell+1}|\Pi\right)=\begin{cases}\left(\ell,\ell+2,\frac{j-2\ell}{2},\frac{j-2\ell+6}{2},p\right), & \text{if } \ell=1,\ldots,\frac{j-2}{2},\\ (\ell,\ell+1,1,3,p), & \text{if } \ell=\frac{j+2}{2},\\ (j-\ell+4,j-\ell+3, & \\ \frac{2\ell-j}{2},\frac{2\ell-j+4}{2},p\Big) & \text{if } \ell=\frac{j+4}{2},\ldots,j.\end{cases} \tag{34}$$

$$r\left(\beta_{4\ell+2}|\Pi\right)=\begin{cases}\left(\ell+3,\ell+2,\frac{j+2\ell+2}{2},\frac{j-2\ell+4}{2},p\right), & \text{if } \ell=0,1,\\ \left(\ell+3,\ell+2,\frac{j-2\ell+6}{2},\frac{j-2\ell+4}{2},p\right), & \text{if } \ell=2,3,\ldots,\frac{j}{2},\\ \left(j-\ell+4,j-\ell+2,\frac{2\ell-j+6}{2},\frac{2\ell-j+4}{2},p\right) & \text{if } \ell=\frac{j+2}{2},\ldots,j.\end{cases} \tag{35}$$

$$r\left(\beta_{4\ell+3}|\Pi\right)=\begin{cases}\left(\ell+4,\ell+1,\frac{j-2\ell+6}{2},\frac{j-2\ell+2}{2},p\right), & \text{if } \ell=0,1,\ldots,\frac{j-2}{2},\\ (\ell+3,\ell+1,4,1,p), & \text{if } \ell=\frac{j}{2},\\ \left(j-\ell+3,j-\ell+3,\frac{2\ell-j+8}{2},\frac{2\ell-j+2}{2},p\right), & \text{if } \ell=\frac{j+2}{2},\ldots,j-1,\\ \left(3,3,\frac{j+6}{2},\frac{j+2}{2},p\right), & \text{if } \ell=j.\end{cases} \tag{36}$$

$$r\left(\beta_{4\ell+4}|\Pi\right)=\begin{cases}\left(\ell+4,\ell+2,\frac{j-2\ell+4}{2},\frac{j-2\ell+4}{2},p\right), & \text{if } \ell=0,1,\ldots,\frac{j-4}{2},\\ \left(j-\ell,\ell+2,\frac{2\ell-j+8}{2},\frac{j-2\ell+4}{2},p\right), & \text{if } \ell=\frac{j-2}{2},\frac{j}{2},\\ \left(j-\ell,j-\ell+3,\frac{2\ell-j+8}{2},\frac{2\ell-j+4}{2},p\right), & \text{if } \ell=\frac{j+2}{2},\ldots,j-2,\\ \left(1,4,\frac{j+2}{2},\frac{j+2}{2},p\right), & \text{if } \ell=j-1.\end{cases} \tag{37}$$

Table 1: Representation of the generalized Petersen graph $P_{17,k}$.

Vertices	Representation
$r(\alpha_1\|\Pi)$	1, 1, 3, 3, 0
$r(\alpha_2\|\Pi)$	2, 0, 2, 3, 0
$r(\alpha_3\|\Pi)$	3, 1, 1, 2, 0
$r(\alpha_4\|\Pi)$	3, 2, 2, 1, 0
$r(\alpha_5\|\Pi)$	2, 3, 3, 0, 0
$r(\alpha_6\|\Pi)$	3, 3, 3, 1, 0
$r(\alpha_7\|\Pi)$	4, 4, 2, 2, 0
$r(\alpha_8\|\Pi)$	4, 5, 3, 3, 0
$r(\alpha_9\|\Pi)$	3, 5, 4, 3, 0
$r(\alpha_{10}\|\Pi)$	3, 4, 4, 4, 0
$r(\alpha_{11}\|\Pi)$	4, 4, 3, 5, 0
$r(\alpha_{12}\|\Pi)$	4, 5, 3, 5, 0
$r(\alpha_{13}\|\Pi)$	3, 5, 4, 4, 0
$r(\alpha_{14}\|\Pi)$	2, 4, 4, 4, 0
$r(\alpha_{15}\|\Pi)$	3, 3, 3, 5, 1
$r(\alpha_{16}\|\Pi)$	3, 3, 2, 5, 0
$r(\alpha_{17}\|\Pi)$	2, 2, 3, 4, 1
$r(\beta_1\|\Pi)$	0, 2, 4, 2, 1
$r(\beta_2\|\Pi)$	3, 1, 3, 3, 1
$r(\beta_3\|\Pi)$	4, 2, 0, 3, 1
$r(\beta_4\|\Pi)$	4, 3, 3, 2, 1
$r(\beta_5\|\Pi)$	1, 3, 4, 1, 1
$r(\beta_6\|\Pi)$	3, 2, 4, 2, 1
$r(\beta_7\|\Pi)$	5, 3, 1, 3, 1
$r(\beta_8\|\Pi)$	5, 4, 3, 3, 1
$r(\beta_9\|\Pi)$	2, 4, 5, 2, 1
$r(\beta_{10}\|\Pi)$	2, 3, 5, 3, 1
$r(\beta_{11}\|\Pi)$	5, 3, 2, 4, 1
$r(\beta_{12}\|\Pi)$	5, 4, 2, 4, 1
$r(\beta_{13}\|\Pi)$	3, 4, 5, 3, 1
$r(\beta_{14}\|\Pi)$	1, 3, 5, 3, 1
$r(\beta_{15}\|\Pi)$	4, 2, 3, 4, 1
$r(\beta_{16}\|\Pi)$	4, 3, 1, 4, 1
$r(\beta_{17}\|\Pi)$	3, 3, 4, 3, 1

The representation of the inner cycle vertices is distinct. Moreover, the outer and inner vertices are distinct as well. Thus, $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_3\}$, $B_3 = \{\beta_{2j+1}\}$, $B_4 = \{\alpha_{2j+3}\}$, and $B_5 = V(P_{n,k})\backslash\{\beta_1, \alpha_3, \beta_{2j+1}, \alpha_{2j+3}\}$, is the partition resolving set when j is even. From all these facts, we conclude that

$$\mathrm{pd}(P_{n,k}) \leq 5. \tag{38}$$

Example 1. Let $P_{17,k}$ be a generalized Petersen graph; then, $\mathrm{pd}(P_{n,k}) \leq 5$.

Proof. Let the partition resolving set be $\Pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{\beta_1\}$, $B_2 = \{\alpha_2\}$, $B_3 = \{\beta_3\}$, $B_4 = \{\alpha_5\}$, and $B_5 = V(P_{17,k})\backslash\{\beta_1, \alpha_2, \beta_3, \alpha_5\}$. The representation of the entire vertex set of $P_{17,k}$ with respect to Π is presented in Table 1. □

4. Conclusion

In this paper, authors provide the sharp bounds of the partition dimension for the generalized Petersen network $P_{n,k}$, and by using Theorem 1.3 for lower bound, we concluded that

$$3 \leq \mathrm{pd}(P_{n,k}) \leq \begin{cases} 4, & \text{if } n \geq 20 \text{ and } n \equiv 0 \,(\mathrm{mod}\, 4), \\ 5, & \text{if } n \geq 17 \text{ and } n \equiv 1, 2, 3 \,(\mathrm{mod}\, 4). \end{cases} \tag{39}$$

Theorem 4 discusses the partitioning of generalized Petersen network when $n \equiv 0 \,(\mathrm{mod}\, 4)$, and Theorems 5–7 contain the partitioning when $n \equiv 1, 2, 3 \,(\mathrm{mod}\, 4)$, respectively.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to this paper.

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References

[1] P. J. Slater, "Leaves of trees," *Congress Numerentium*, vol. 14, pp. 549–559, 1975.

[2] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.

[3] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, pp. 99–113, 2000.

[4] R. A. Melter and I. Tomescu, "Metric bases in digital geometry," *Computer Vision, Graphics, and Image Processing*, vol. 25, pp. 113–121, 1984.

[5] G. Chartrand, E. Salehi, and P. Zhang, "The partition dimension of graph," *Aequationes Mathematicae*, vol. 59, pp. 45–54, 2000.

[6] E. T. Baskoro and D. O. Haryeni, "All graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$," *Heliyon*, vol. 6, 2020.

[7] H. Vertana and T. A. Kusmayadi, "On the partition dimension of $C_m + P_n$ graph," *Journal of Physics: Conference Series*, vol. 855, 2016.

[8] Y.-M. Chu, M. F. Nadeem, M. Azeem, and M. K. Siddiqui, "On sharp bounds on partition dimension of convex polytopes," *IEEE Access*, vol. 8, pp. 224781–224790, 2020.

[9] J. B. Liu, M. F. Nadeem, and M. Azeem, "Bounds on the partition dimension of convex polytopes," *Combinatorial Chemistry and Throughput Screening*, vol. 4, 2020.

[10] Z. Hussain, S. M. Kang, M. Rafique et al., "Bounds for partition dimension of M-wheels," *Open Physics*, vol. 16, 2019.

[11] C. Grigorious, S. Stephen, B. Rajan, and M. Miller, "On the partition dimension of circulant graphs," *The Computer Journal*, vol. 60, no. 2, pp. 180–184, 2017.

[12] E. C. M. Maritz and T. Vetrik, "The partition dimension of circulant graphs," *Quaestiones Mathematicae*, vol. 41, no. 1, 2018.

[13] N. Mehreen, R. Farooq, and S. Akhter, "on partition dimension of fullerene graphs," *AIMS Mathematics*, vol. 3, no. 3, pp. 343–352, 2018.

[14] I. G. Yero, A. Juan, and J. A. Rodrłuez-Velàzquez, "A note on the partition dimension of cartesian product graphs," *Applied Mathematics and Computation*, vol. 217, no. 7, pp. 3571–3574, 2010.

[15] S. Amrullah, S. Azmi, H. Soeprianto, M. Turmuzi, and Y. S. Anwar, "The partition dimension of subdivision graph on the star," *Journal of Physics: Conference Series*, vol. 1280, pp. 22–37, 2019.

[16] J. A. Rodrłuez-Velàzquez, I. G. Yero, and M. Lemanska, "On the partition dimension of trees," *Discrete Applied Mathematics*, vol. 166, pp. 204–209, 2014.

[17] J. A. Rodrłuez-Velàzquez, I. G. Yero, and H. Fernau, "On the partition dimension of unicyclic graphs," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 57, pp. 381–391, 2014.

[18] M. C. Monica and S. Santhakumar, "Partition dimension of certain honeycomb derived networks," *International Journal of Pure and Applied Mathematics*, vol. 108, no. 4, pp. 809–818, 2016.

[19] B. Rajan, A. William, I. Rajasingh, C. Grigorious, and S. Stephen, "On certain networks with partition dimension three," in *Proceedings of the International Conference Mathematics, Engineering and Business Management*, pp. 169–172, March 2012.

[20] I. Javaid and S. Shokat, "On the partition dimension of some wheel related graphs," *Journal of Prime Research in Mathematics*, vol. 4, pp. 154–164, 2008.

[21] E. Moreno, "On the k-partition dimension of graphs," *Theoretical Computer Science*, vol. 806, pp. 42–52, 2020.

[22] D. O. Haryeni, E. T. Baskoro, S. W. Saputro, M. Bač̃a, and A. S. Fenovcikova, "On the partition dimension of two-component graphs," *Proceedings Mathematical Sciences*, vol. 127, no. 5, pp. 755–767, 2017.

[23] T. U. Rehman and N. Mehreen, "Partition dimension and strong metric dimension of chain cycle," *Jordan Journal of Mathematics and Statistics*, vol. 13, no. 2, pp. 305–325, 2020.

[24] S. Safriadi, H. Hasmawati, and L. Haryanto, "Partition dimension of complete multipartite graph," *Jurnal Matematika, Statistika dan Komputasi*, vol. 16, no. 3, 2020.

[25] S. Naz, M. Salman, U. Ali, I. Javaid, and S. A.-U.-H. Bokhary, "On the constant metric dimension of generalized petersen graphs $P(n, 4)$," *Acta Mathematica Sinica, English Series*, vol. 30, no. 7, pp. 1145–1160, 2014.

[26] S. Ahmad, M. A. Chaudhry, I. Javaid, and M. Salman, "On the metric dimension of generalized petersen graphs," *Quaestiones Mathematicae*, vol. 36, no. 3, pp. 421–435, 2013.

[27] M. Imran, M. K. Siddiqui, and R. Naeem, "On the metric dimension of generalized petersen multigraphs," *IEEE Access*, vol. 6, pp. 2169–3536, 2018.