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Research Article

Positive Periodic Solutions for a Class of Strongly Coupled Differential Systems with Singular Nonlinearities

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This article studies the existence of positive periodic solutions for a class of strongly coupled differential systems. By applying the fixed point theory, several existence results are established. Our main findings generalize and complement those in the literature studies.

1. Introduction

In this paper, we are concerned with the existence of positive periodic solutions of the strongly coupled differential systems:

$$L_i x_i = f_i(t, x_1, x_2) + e_i(t), \quad i = 1, 2, \tag{1}$$

where $L_i x_i = x_i'' + p_i(t)x_i' + q_i(t)x_i$ is a linear differential operator with $p_i, q_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$. In addition, we assume $e_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and $f_i \in \mathrm{Car}(\mathbb{R}/T\mathbb{Z} \times (0, \infty) \times (0, \infty), \mathbb{R})$, that is, $f_i|_{[0,T]}$: $[0,T] \times (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$ is a L^1-Caratheodory function, and it is singular at $(x_1, x_2) = (0, 0)$.

During the past few decades, the fixed point theory has been widely adopted to investigate the nonperiodic coupled differential systems, and researchers have mainly concentrated on the existence and multiplicity of positive solutions [1–3]. Meanwhile, the periodic equations and systems with singular nonlinearities have been dealt via some classical fixed point theorems, such as Schauder's fixed point theorem and fixed point theorems in cones [4–12]. What is worth mentioning is the results obtained in [5, 6, 11, 12], where the authors show, under some circumstances, weak singularities are helpful to seek out periodic solutions for not only singular equations [5] but also singular coupled systems [11]. Especially, in [10], Li and Zhang considered the singular equation

$$x'' + a(t)x = f(t, x) + c(t), \tag{2}$$

where $a, c \in L^1[0, T]$ and $f \in \mathrm{Car}([0, T] \times (0, \infty), \mathbb{R})$. By employing a fixed point theorem in cones, they established several existence theorems under the following basic assumption.

($\widetilde{H}$) There exist $b>0, \widehat{b}>0$, and $\lambda > 0$ such that

$$0 \leq \frac{\widehat{b}(t)}{x^\lambda} \leq f(t, x) \leq \frac{b(t)}{x^\lambda}, \quad \forall x > 0, \text{a.e. } t \in [0, T], \tag{3}$$

and pointed out they have not limited themselves to the weak singularities; see [10], Section 3, for more details. Besides, the case $a(t) \equiv 0$ (the resonant case) has also been studied in [10], Theorem 4.1. For other research works related to the resonant case of (2), one may refer to [13–15] and references therein.

To our knowledge, however, so far, the existing results on strongly coupled periodic singular systems are relatively few. Therefore, motivated by the aforementioned papers, we shall establish the existence of positive periodic solutions of system (1) in the present paper to further improve and complement those in the literature studies. To demonstrate our new results, we choose the following differential systems:

$$\begin{cases} x_1'' + p_1(t)x_1' + q_1(t)x_1 = \dfrac{1}{(x_1+x_2)^{\alpha_1}} + e_1(t), \\ x_2'' + p_2(t)x_2' + q_2(t)x_2 = \dfrac{1}{(x_1+x_2)^{\alpha_2}} + e_2(t), \end{cases} \tag{4}$$

where $p_i, q_i, e_i \in L^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and $\alpha_i > 0, i = 1, 2$. Here, $\alpha_i > 0$ means we need not restrict ourselves to the weak force conditions in our results.

The rest of the paper is arranged as follows. In Section 2, we give some required preliminaries and notations. In Section 3, we shall state and prove the existence results for (1) in the nonresonant case. Finally, in Section 4, an existence theorem will be proved for (1) in the resonant case $L_i x_i = x_i''$, $i = 1, 2$.

2. Preliminaries

The linear boundary value problem

$$x'' + p(t)x' + q(t)x = 0, \tag{5}$$

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{6}$$

is called nonresonant if its unique solution is the trivial one. When (5) and (6) are nonresonant, the well-known Fredholm's alternative ensures the nonhomogeneous equation

$$x'' + p(t)x' + q(t)x = l(t) \tag{7}$$

admits a unique T-periodic solution, which can be expressed as

$$x(t) = \int_0^T K(t,s)l(s)\mathrm{d}s, \tag{8}$$

where $K(t,s)$ is Green's function associated to (5) and (6).

For given $\xi \in L^1[0,T]$, we denote by ξ_* and ξ^*, respectively, the essential infimum and supremum of ξ. $\xi \succ 0$ means $\xi \geq 0$ for a.e. $t \in [0,T]$, and it is positive on a set with positive measures. Moreover, if (7) has a unique periodic solution x_l for any $l \in C(\mathbb{R}/T\mathbb{Z})$ and x_l is positive on $[0,T]$ when $l \succ 0$, then we say (5) satisfies the antimaximum principle. Recently, Hakl and Torres [16] established an explicit criterion to guarantee the antimaximum principle holds for (5). For the sake of convenience, set

$$\sigma(p)(t) = e^{\int_0^t p(s)\mathrm{d}s}, \qquad \sigma_1(p)(t) = \sigma(p)(T)\int_0^t \sigma(p)(s)\mathrm{d}s + \int_t^T \sigma(p)(s)\mathrm{d}s. \tag{9}$$

Lemma 1 (see [16]). *If $q \equiv 0$ and*

$$\int_0^T q(s)\sigma(p)(s)\sigma_1(-p)(s) \geq 0, \tag{10}$$

$$\sup_{0\le t\le T}\left\{\int_t^{t+T}\sigma(-p)(s)\mathrm{d}s \cdot \int_t^{t+T}[q(s)]_+\sigma(p)(s)\mathrm{d}s\right\} \leq 4, \tag{11}$$

then (5) satisfies the antimaximum principle, where $[q(s)]_+ = \max\{q(s), 0\}$, whereafter Chu et al. [17] pointed out if (5) admits the antimaximum principle, then $K(t,s) \geq 0$ on $[0,T]\times[0,T]$. In addition, they obtained the following.

Lemma 2. *If $q \equiv 0$ and (11) holds, then the distance between two consecutive zeroes of a nontrivial solution of (5) is greater than T.*

Obviously, Lemma 2 implies $K(t,s)$ does not vanish. As a consequence of Lemmas 1 and 2, Chu et al. established the following.

Lemma 3. *If $q \equiv 0$ and (10) and (11) hold, then $K(t,s) > 0$ on $[0,T]\times[0,T]$.*

Note that Lemma 3 plays an important role in the application of the classical fixed point theorems. Indeed, by Lemma 3, the positivity of some completely continuous operators could be easily obtained.

Remark 1. Clearly, if $p(t) \equiv 0$ (without damping terms), then (10) and (11) reduce, respectively, to

$$\int_0^T q(s)\mathrm{d}s > 0, \quad \left\|[q(s)]_+\right\|_1 < \frac{4}{T}, \tag{12}$$

which are conditions used to guarantee the positivity of Green's function corresponds to (5) and (6); see [18] for more details.

Throughout the paper, we always suppose $p_i, q_i, e_i \in L^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R})$ and $f_i \in \mathrm{Car}(\mathbb{R}/T\mathbb{Z}\times(0,\infty)\times(0,\infty), \mathbb{R})$. Furthermore,

(H1) The linear equation $L_i x_i = 0$ is nonresonant, and corresponding Green's function $K_i(t,s) > 0$ on $[0,T]\times[0,T]$, $i = 1, 2$.

(H2) There are $b_i \succ 0, \widehat{b}_i \succ 0$, and $\alpha_i > 0$ so that, for a.e. $t \in [0,T]$,

$$0 \leq \frac{\widehat{b}_i(t)}{(x_1+x_2)^{\alpha_i}} \leq f_i(t,x_1,x_2) \leq \frac{b_i(t)}{(x_1+x_2)^{\alpha_i}}, \quad \forall x_1 > 0, x_2 > 0, i = 1, 2. \tag{13}$$

It is not hard to see (H1) implies the antimaximum principle holds for $L_i x_i = 0$, and thus, $\sigma_i = (m_i/M_i) \in (0,1)$, where

$$m_i = \min_{0\le t,s\le T} K_i(t,s), M_i = \max_{0\le t,s\le T} K_i(t,s), \quad i = 1, 2. \tag{14}$$

Hence,

$$\sigma := \min\{\sigma_1, \sigma_2\} \in (0,1). \tag{15}$$

Set $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}, \mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$, where $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $\|\mathbf{x}\| = \sum_{i=1}^n |u_i|$. Let

$$\begin{aligned} E = \Big\{&\mathbf{x}(t) = (x_1(t), x_2(t)) \in C(\mathbb{R}, \mathbb{R}^2)\colon x_i(t+T) \\ &= x_i(t), t \in \mathbb{R}, i = 1, 2\Big\} \end{aligned} \tag{16}$$

be the Banach space equipped with the norm

$$\|\mathbf{x}\| = \|x_1\|_\infty + \|x_2\|_\infty, \quad \mathbf{x} = (x_1, x_2) \in E. \tag{17}$$

Here, $\|x_i\|_\infty = \sup_{t \in [0,T]} |x_i(t)|, i = 1, 2$.

A vector function $(x_1(t), x_2(t)) \in E$ is called a positive T-periodic solution of (1) if it satisfies (1) and $x_i > 0$ on $[0, T]$, $i = 1, 2$. Let

$$K = \Big\{\mathbf{x} = (x_1, x_2) \in E\colon x_i(t) \ge \sigma_i \|x_i\|_\infty, t \in [0,T], i = 1, 2\Big\}. \tag{18}$$

Then, it is not difficult to check K is a positive cone in E, and for any $\mathbf{x} = (x_1, x_2) \in E$, we get by (15) that

$$x_1(t) + x_2(t) \ge \sigma \|\mathbf{x}\|. \tag{19}$$

For $r > 0$, let $\Omega_r = \{\mathbf{x} \in K\colon \|\mathbf{x}\| < r\}$; then, $\partial\Omega_r = \{\mathbf{x} \in K\colon \|\mathbf{x}\| = r\}$.

Lemma 4 (see [19]). *Let E be a Banach space and $K \subseteq E$ a cone. Suppose $\Omega_1, \Omega_2 \subseteq E$ are open bounded subsets satisfying $0 \in \Omega_1, \overline{\Omega}_1 \subseteq \Omega_2$. If $\mathscr{A}\colon K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$ is completely continuous and satisfies*

there is $\psi \in K \backslash \{0\}$ such that $u \ne \mathscr{A}u + \lambda\psi$ for $u \in K \cap \partial\Omega_1$ and $\lambda > 0$,

$\|\mathscr{A}u\| \le \|u\|, \forall u \in K \cap \partial\Omega_2$,

or

there is $\psi \in K \backslash \{0\}$ such that $u \ne \mathscr{A}u + \lambda\psi$ for $u \in K \cap \partial\Omega_2$ and $\lambda > 0$,

$\|\mathscr{A}u\| \le \|u\|, \forall u \in K \cap \partial\Omega_1$,

then $\mathscr{A}$ admits a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. The Nonresonant Systems

This section is devoted to establishing the existence results for system (1) in the nonresonant case. To this end, we define

$$\begin{aligned} \gamma_i(t) &= \int_0^T K_i(t,s) e_i(s) \mathrm{d}s, \quad i = 1, 2, \\ B_i(t) &= \int_0^T K_i(t,s) b_i(s) \mathrm{d}s, \widehat{B}_i(t) \\ &= \int_0^T K_i(t,s) \widehat{b}_i(s) \mathrm{d}s, \quad i = 1, 2. \end{aligned} \tag{20}$$

Theorem 1. *Assume (H1) and (H2) hold. Let*

$$\begin{aligned} R_i &= \left(\frac{\widehat{b}_{i*}}{|e_{i*}|}\right)^{(1/\alpha_i)}, \\ R_0 &= \min\{R_1, R_2\}. \end{aligned} \tag{21}$$

(i) If $e_{i*} < 0$, $\widehat{b}_{i*} > 0$, and

$$\sum_{i=1}^2 \gamma_i^* \le R_0 - \sum_{i=1}^2 \frac{B_i^*}{(\sigma R_0)^{\alpha_i}}, \tag{22}$$

then (1) has a positive T-periodic solution.

(ii) If $e_{i*} \ge 0\,(i = 1, 2)$, then (1) has a positive T-periodic solution.

Proof

(i) Let $\mathscr{A}\colon E \longrightarrow E$ be an operator defined by $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2)$ with

$$(\mathscr{A}_i \mathbf{x})(t) = \int_0^T K_i(t,s) f_i(s, x_1(s), x_2(s)) \mathrm{d}s + \gamma_i(t), \quad i = 1, 2. \tag{23}$$

Then, $\mathscr{A}\colon E \longrightarrow E$ is completely continuous, and a T-periodic solution of (1) is equivalent to a fixed point of $\mathscr{A}$.

We shall divide the proof of the case into three steps as follows.

Step 1: if we choose $r \ll 1$, then for $\psi = (1, 1) \in K$,

$$\mathbf{x} \ne \mathscr{A}\mathbf{x} + \lambda\psi, \quad \forall \mathbf{x} \in K \cap \partial\Omega_r, \lambda > 0. \tag{24}$$

Or else, if there are $\mathbf{x}_0 = (x_{10}, x_{20}) \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$ so that $\mathbf{x}_0 = \mathscr{A}\mathbf{x}_0 + (\lambda_0, \lambda_0)$, then by the definition of K and (15), we get

$$\sigma r = \sigma \|\mathbf{x}_0\| \le x_{10} + x_{20} \le \|\mathbf{x}_0\| = r. \tag{25}$$

This together with (H2) implies

$$\begin{aligned} x_{i0}(t) &= (\mathscr{A}_i \mathbf{x}_0)(t) + \lambda_0 \\ &= \int_0^T K_i(t,s) f_i(s, x_{10}(s), x_{20}(s)) \mathrm{d}s + \gamma_i(t) + \lambda_0 \\ &\ge \int_0^T K_i(t,s) \frac{\widehat{b}_i(s)}{(x_{10}(s) + x_{20}(s))^{\alpha_i}} \mathrm{d}s + \gamma_{i*} + \lambda_0 \\ &\ge \frac{\widehat{B}_{i*}}{r^{\alpha_i}} + \gamma_{i*} + \lambda_0 > r, \quad i = 1, 2, \end{aligned} \tag{26}$$

which contradicts $\|\mathbf{x}_0\| = r$ since $r \ll 1$.

Step 2: for $R\colon = R_0 > r$, $\mathscr{A}$ maps $K \cap (\overline{\Omega}_R \backslash \Omega_r)$ into K.

Since $R = R_0$, for any $(x_1, x_2) \in (0, \infty) \times (0, \infty)$ satisfying $x_1 + x_2 \in (0, R]$ and a.e. $t \in [0, T]$, we can deduce from (H2) that

$$f_i(t,x_1,x_2)+e_i(t)\ge \frac{\widehat{b}_i(t)}{(x_1+x_2)^{\alpha_i}}+e_i(t)$$
$$\ge \frac{\widehat{b}_{i*}}{(x_1+x_2)^{\alpha_i}}+e_{i*}\ge 0,\quad i=1,2, \tag{27}$$

and subsequently, for $\mathbf{x}\in K\cap(\overline{\Omega}_R\setminus\Omega_r)$,

$$\begin{aligned}(\mathscr{A}_i\mathbf{x})(t)&=\int_0^T K_i(t,s)\left(f_i(s,x_1(s),x_2(s))+e_i(s)\right)ds\\ &\ge m_i\int_0^T\left(f_i(s,x_1(s),x_2(s))+e_i(s)\right)ds\\ &=\sigma_i\int_0^T M_i\left(f_i(s,x_1(s),x_2(s))+e_i(s)\right)ds\\ &\ge \sigma_i\left\|\mathscr{A}_i\mathbf{x}\right\|_\infty,\quad i=1,2,\end{aligned} \tag{28}$$

which means $\mathscr{A}$ maps $K\cap(\overline{\Omega}_R\setminus\Omega_r)$ into K.
Step 3: we shall show

$$\|\mathscr{A}\mathbf{x}\|\le\|\mathbf{x}\|,\quad \forall\mathbf{x}\in K\cap\partial\Omega_R. \tag{29}$$

Recall that $R=R_0$. For any $\mathbf{x}\in K\cap\partial\Omega_R$, we have $\sigma R=\sigma\|\mathbf{x}\|\le x_1+x_2\le\|\mathbf{x}\|=R$, and then by (27), (H1), and (H2), we can obtain

$$\begin{aligned}0\le(\mathscr{A}_i\mathbf{x})(t)&=\int_0^T K_i(t,s)\left(f_i(s,x_1(s),x_2(s))+e_i(s)\right)ds\\ &\le\int_0^T K_i(t,s)\frac{b_i(s)}{(x_1(s)+x_2(s))^{\alpha_i}}ds+\gamma_i^*\\ &\le\frac{B_i^*}{(\sigma R)^{\alpha_i}}+\gamma_i^*,\quad i=1,2,\end{aligned} \tag{30}$$

which yields $\|\mathscr{A}_i\mathbf{x}\|_\infty\le(B_i^*/(\sigma R)^{\alpha_i})+\gamma_i^*, i=1,2.$ This together with (22) shows

$$\|\mathscr{A}\mathbf{x}\|=\left\|\mathscr{A}_1\mathbf{x}\right\|_\infty+\left\|\mathscr{A}_2\mathbf{x}\right\|_\infty\le R=\|\mathbf{x}\|. \tag{31}$$

And accordingly, Lemma 4 ensures (1) admits a positive T-periodic solution.

(ii) To deal with this case, we only need to show (24)–(29) are still satisfied. Indeed, if we choose $r\ll 1$, then we can easily prove by (H2) that (24) holds true. Moreover, since $e_{i*}\ge 0$, $R\gg 1$ could be chosen so that (27) and (29) are all satisfied. Consequently, Lemma 4 yields (1) has a positive T-periodic solution. □

Remark 2. Obviously, (H2) reduces to $(\widetilde{H})$ under some special circumstances; hence, Theorem 1 generalizes [10], Theorem 3.1.

Remark 3. When $p_i(t)\equiv 0$, $f_1(t,x_1,x_2)\equiv f_1(t,x_2)$, and $f_2(t,x_1,x_2)\equiv f_2(t,x_1)$, system (1) becomes

$$\begin{cases}x_1''+q_1(t)x_1=f_1(t,x_2)+e_1(t),\\ x_2''+q_2(t)x_2=f_2(t,x_1)+e_2(t).\end{cases} \tag{32}$$

In [11], several existence theorems have been established for (32) via Schauder's fixed point theorem, where f_i satisfies only the weak force conditions. However, we do not restrict ourselves here to weak singularities, and Theorem 1 is still valid for (32) with strong singularities.

Theorem 2. *Assume (H1) and (H3) There are $b_i\succ 0,\widehat{b}_i\succ 0,d_i\succ 0$, and $\alpha_i,\eta_i>0$ such that, for a.e. $t\in[0,T]$,*

$$\begin{aligned}0&\le\frac{\widehat{b}_i(t)}{(x_1+x_2)^{\alpha_i}}\le f_i(t,x_1,x_2)\\ &\le\frac{b_i(t)}{(x_1+x_2)^{\alpha_i}}+d_i(t)(x_1+x_2)^{\eta_i},\quad\forall x_1,x_2>0.\end{aligned} \tag{33}$$

Then, the following results hold true:

(i) If $e_{i*}<0$, $\widehat{b}_{i*}>0$, and

$$\sum_{i=1}^2\gamma_i^*\le R_0-\sum_{i=1}^2\frac{B_i^*}{(\sigma R_0)^{\alpha_i}}-\sum_{i=1}^2 D_i^*R_0^{\eta_i}, \tag{34}$$

then (1) has a positive T-periodic solution.

(ii) If $e_{i*}\ge 0\,(i=1,2)$, then (1) has a positive T-periodic solution.

Here, R_0 is defined as in Theorem 1.

Proof

(i) Choose $r\ll 1$ and set $R=R_0$. By (H3) and an argument similar to the proof of Theorem 1 (i), we can get (24) and (27). To apply Lemma 4, we just need to show

$$\|\mathscr{A}\mathbf{x}\|\le\|\mathbf{x}\|,\quad\forall\mathbf{x}\in K\cap\partial\Omega_R. \tag{35}$$

In fact, for any $\mathbf{x}=(x_1,x_2)\in K\cap\partial\Omega_R$, it follows from (27) and (H3) that

$$\begin{aligned}0\le(\mathscr{A}_i\mathbf{x})(t)&=\int_0^T K_i(t,s)f_i(s,x_1(s),x_2(s))ds+\gamma_i(t)\\ &\le\int_0^T K_i(t,s)\frac{b_i(s)}{(x_1(s)+x_2(s))^{\alpha_i}}ds\\ &\quad+\int_0^T K_i(t,s)d_i(s)(x_1(s)+x_2(s))^{\eta_i}ds+\gamma_i^*\\ &\le\frac{B_i^*}{(\sigma R)^{\alpha_i}}+D_i^*R^{\eta_i}+\gamma_i^*,\quad i=1,2,\end{aligned} \tag{36}$$

and then (34) yields $\|\mathscr{A}\mathbf{x}\|=\|\mathscr{A}_1\mathbf{x}\|_\infty+\|\mathscr{A}_2\mathbf{x}\|_\infty\le R=\|\mathbf{x}\|$. Therefore, Lemma 4 implies (1) has a positive T-periodic solution.

(ii) Using Lemma 4 and similar to the proof of Theorem 1 (ii), we can easily get the conclusions. □

Remark 4. Jiang et al. [4] studied the singular equation

$$u'' + a(t)u = \frac{b(t)}{u^{\alpha}} + d(t)u^{\eta} + c(t), \tag{37}$$

where $\alpha, \eta > 0$ are constants, $b(t), d(t) \in C[0,T]$ are nonnegative, and $c(t) \in C[0,T]$. For the positone case, they supposed $c(t) \geq 0$, which means $c_* \geq 0$, and for the semipositone case, a strong force condition was required. See [4], Corollaries 3.2 and 4.3. Evidently, Theorem 2 generalizes [10], Theorem 3.2, and the corresponding ones in [4].

4. The Resonant Systems

Let us consider the singular systems

$$\begin{cases} x_1'' = f_1(t, x_1, x_2) + e_1(t), \\ x_2'' = f_2(t, x_1, x_2) + e_2(t). \end{cases} \tag{38}$$

Theorem 3. *Assume (H2) and (H4) There are $k_1, k_2 \in (0, (\pi/T))$ so that, for any $(x_1, x_2) \in \mathbb{R}_+^2 \backslash \{(0,0)\}$ and a.e. $t \in [0,T]$,*

$$f_i(t, x_1, x_2) + e_i(t) + k_i^2 x_i \geq 0, \quad i = 1, 2. \tag{39}$$

If $\widehat{b}_{i*} > 0$ and $e_i^* < 0$, then (38) has a positive T-periodic solution.

Proof.
Let us take into account the auxiliary systems

$$\begin{cases} x_1'' + k_1^2 x_1 = f_1(t, x_1, x_2) + e_1(t) + k_1^2 x_1, \\ x_2'' + k_2^2 x_2 = f_2(t, x_1, x_2) + e_2(t) + k_2^2 x_2, \end{cases} \tag{40}$$

where $k_1, k_2 \in (0, (\pi/T))$ are constants introduced as in (H4). Clearly, a solution of (40) is just a solution of original system (38), and vice versa. Therefore, to complete the proof, it is enough to show (40) has a positive T-periodic solution.

Let $K_i(t,s)$ be Green's function of Hill's equation $x_i'' + k_i^2 x_i = 0, i = 1, 2$. Then, a solution of (40) is equivalent to a fixed point of completely continuous operator $\mathscr{A}: E \longrightarrow E$ with components $(\mathscr{A}_1, \mathscr{A}_2)$:

$$\begin{aligned}(\mathscr{A}_i \mathbf{x})(t) = \int_0^T & K_i(t,s)\big(f_i(s, x_1(s), x_2(s)) \\ & + k_i^2 x_i(s) + e_i(s)\big) ds, \quad i = 1, 2.\end{aligned} \tag{41}$$

Moreover, let m_i and M_i denote the minimum and maximum of $K_i(t,s)$, respectively; then, $\sigma_i = (m_i/M_i) = \cos(k_i T/2) \in (0,1)$, and so, $\sigma := \min\{\sigma_1, \sigma_2\} \in (0,1)$. For $\sigma_i = \cos(k_i T/2)$, we denote again by K, introduced in (18), the positive cone in E.

Choose $R \gg (1/\sigma)$ sufficiently large and define $\Omega_R = \{\mathbf{x} \in E: \|\mathbf{x}\| < R\}$. Since $e_i^* < 0$, we have for $\mathbf{x} \in K \cap \partial\Omega_R$,

$$\int_0^T K_i(t,s)\left(\frac{b_i(s)}{(x_1 + x_2)^{\alpha_i}} + e_i(s)\right) ds \leq \int_0^T K_i(t,s)\left(\frac{b_i(s)}{(\sigma R)^{\alpha_i}} + e_i(s)\right) ds < 0, \tag{42}$$

and subsequently, by (H4),

$$\begin{aligned} 0 \leq (\mathscr{A}_i \mathbf{x})(t) &= \int_0^T K_i(t,s)\big(k_i^2 x_i(s) + f_i(s, x_1(s), x_2(s)) + e_i(s)\big) ds \\ &\leq \int_0^T K_i(t,s)\left(\frac{b_i(s)}{(x_1 + x_2)^{\alpha_i}} + e_i(s) + k_i^2 x_i(s)\right) ds \\ &< k_i^2 \|x_i\|_\infty \int_0^T K_i(t,s) ds = k_i^2 \frac{1}{k_i^2} \|x_i\|_\infty = \|x_i\|_\infty, \quad i = 1, 2, \end{aligned} \tag{43}$$

this yields $\|\mathscr{A}\mathbf{x}\| \leq \|\mathbf{x}\|, \forall \mathbf{x} \in K \cap \partial\Omega_R$.

By (H2) and $\widehat{b}_{i*} > 0$, there exists $r < R$ small enough such that, for any $\mathbf{x} = (x_1, x_2)$ satisfying $\sigma r \leq x_1 + x_2 \leq r$,

$$f_i(t, x_1, x_2) + e_i(t) \geq \frac{\widehat{b}_i(t)}{(x_1 + x_2)^{\alpha_i}} + e_i(t) \geq \frac{\widehat{b}_{i*}}{r^{\alpha_i}} + e_{i*} \geq 0, \tag{44}$$

which implies

$$f_i(t, x_1, x_2) + e_i(t) + k_i^2 x_i \geq k_i^2 x_i, \quad \sigma r \leq x_1 + x_2 \leq r. \tag{45}$$

Let $\Omega_r = \{\mathbf{x} \in E: \|\mathbf{x}\| < r\}$ and $\psi \equiv (1,1)^T \in K$; then,

$$\mathbf{x} \neq \mathscr{A}\mathbf{x} + \lambda\psi, \quad \forall \mathbf{x} \in K \cap \partial\Omega_r, \lambda > 0. \tag{46}$$

Otherwise, if there are $\mathbf{x}_0 = (x_{10}, x_{20}) \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$ such that $\mathbf{x}_0 = \mathscr{A}\mathbf{x}_0 + \lambda_0\psi$, then from (45), it follows

$$\begin{aligned} x_{i0}(t) &= (\mathscr{A}_i \mathbf{x}_0)(t) + \lambda_0 \\ &= \int_0^T K_i(t,s) f_i\big(s, x_{10}(s), x_{20}(s) + e_i(s) + k_i^2 x_{i0}(s)\big) ds + \lambda_0 \\ &\geq \int_0^T K_i(t,s) k_i^2 x_{i0}(s) ds + \lambda_0. \end{aligned} \tag{47}$$

Setting $\Upsilon = \min_{t \in [0,T]} x_{i0}(t)$, we obtain from (47) that $\Upsilon \geq \Upsilon + \lambda_0$, which contradicts $\lambda_0 > 0$.

To apply Lemma 4, it remains to verify

$$\mathscr{A}\left(K \cap (\overline{\Omega}_R \setminus \Omega_r)\right) \subseteq K. \tag{48}$$

Using (H4), we can easily get

$$\begin{aligned} \|\mathscr{A}_i \mathbf{x}\|_\infty \leq M_i \int_0^T & K_i(t,s)\big(f_i(s, x_1(s), x_2(s)) + e_i(s) \\ & + k_i^2 x_i(s)\big) ds, \quad i = 1, 2, \end{aligned} \tag{49}$$

and thus,

$$
\begin{aligned}
(\mathscr{A}_i\mathbf{x})(t) &\geq m_i \int_0^T \left(f_i(s, x_1(s), x_2(s)) + e_i(s) + k_i^2 x_i(s) \right) \mathrm{d}s \\
&= \sigma_i M_i \int_0^T \left(f_i(s, x_1(s), x_2(s)) + e_i(s) + k_i^2 x_i(s) \right) \mathrm{d}s \\
&= \sigma_i \left\| \mathscr{A}_i\mathbf{x} \right\|_\infty, \quad i = 1, 2,
\end{aligned}
\tag{50}
$$

which means (48) holds true.

Consequently, Lemma 4 ensures (40) has a positive T-periodic solution, and accordingly, system (38) admits a positive T-periodic solution. □

Remark 5. Many authors have paid their attention to the optimal control of the nonlinear systems, and a number of excellent results have been established. See, for instance, [20–23] and the references therein. For the optimal control of nonlinear system (1), we shall deal in the forthcoming paper.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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