Research Article

Power Improved Generalized Heronian Mean Operators Utilizing Hamacher Operations with Picture Fuzzy Information

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In multiple attribute decision-making (MADM), to better denote complicated preference information of decision-makers (DMs), picture fuzzy set (PFS) as an expansion of intuitionistic fuzzy set (IFS) has become a powerful tool in the recent years. Meanwhile, to remove the impact of abnormal data and capture the correlations among attributes in MADM issue, we propose the power improved generalized Heronian mean (PIGHM) operators in this paper, which have the merits of both power average (PA) operator and improved generalized Heronian mean (IGHM) operator. Additionally, Hamacher operations as a generalization of Algebraic operations and Einstein operations demonstrate good smooth approximate. Motivated by these, the main purpose is to explore PIGHM operators utilizing Hamacher operations to cope with MADM issue with picture fuzzy information. First, we introduce the Hamacher operations, the normalized hamming distance, and similarity measure of picture fuzzy numbers (PHNs). Second, based on these, two new picture fuzzy aggregating operators (AOs), the picture fuzzy Hamacher weighted power improved generalized Heronian mean (PFHWPIGHM) operator and the picture fuzzy Hamacher weighted geometric power improved generalized Heronian mean (PFHWGPIGHM) operator, are put forward, and some properties and special instances of proposed AOs are also investigated. Third, a new MADM model in terms of the PIGHM AOs is developed. Eventually, a practical MADM example, together with sensitivity analysis and comparative analysis, is conducted to verify the credibility and superiority of the new MADM model.

1. Introduction

Intuitionistic fuzzy set (IFS), which was firstly defined by Atanassov [1] and is an extended form of fuzzy set (FS) proposed by Zadeh [2], can better depict the truth, the falsity, and indeterminacy memberships, where the degree of indeterminacy membership is dependent on the membership degrees of truth and falsity. However, in actual environment, the indeterminacy degree should be given by decision-makers (DMs) according to real situation. For instance, in a voting, sixty percent are for it, ten percent remain neutral, twenty percent are against it, and ten percent refuse to vote. Clearly, this circumstance is beyond the capability of IFS. Thus, picture fuzzy set (PFS) defined by Cuong [3], including the degrees of positive, neutral, and negative, can convey multiple responses of DMs.

Since PFS can express different attitudes of DMs in complex situations, some studies regarding PFS have been conducted [4–9], concentrating on the aggregating operators (AOs), distance measure, cross entropy, and correlation coefficient. AOs, as an interesting direction owing to the diversity correlations among attributes, are an efficient tool for information fusion in MADM problem. Power average (PA) operator, which can decrease the impact of irrational data provided by DMs, was firstly defined by Yager [10]. Then, the PA operator has been applied to cope with different fuzzy information. Rani and Garg [11] applied PA operator and its extension to complex intuitionistic fuzzy set. Jiang et al. [12] proposed PA operator based on entropy and applied it to IFS. Ju et al. [13] extended PA AOs to q-rung orthopair fuzzy set (qROFS). Yang and Li [14] explored PA AOs to multivalued neutrosophic linguistic set

(MVNLS). Garg and Kumar [15] constructed a MADM model utilizing PA AOs for linguistic IFS. Liu and Luo [16] extended PA AOs to simplified neutrosophic set (SNS). Garg and Nancy [17] developed PA AOs to linguistic single-valued neutrosophic set (LSVNS). Liu et al. [7] applied PA AOs to complex PFS. Considering the interrelationships among aggregating values, Heronian mean (HM) operator was initially introduced by Beliakov et al. [18]. Later, Sykora [19] proposed the generalized HM (GHHM) operator. HM operator and GHHM operator mentioned above do not consider the weights of input argument. Then, Yu and Wu [20] extended the GHHM to interval-valued IFS (IVIFS) and established generalized weighted HM (GWHM) operator. Moreover, Yu [21] introduced generalized geometric weighted HM (GGWHM) operator with IFS. Nevertheless, it seems to be counterintuitive that the GWHM and GGWHM operators do not have the characteristics of reducibility and idempotency. Therefore, Chu and Liu [22] proposed novel weighted HM AOs to uncertain linguistic information, namely, weighted generalized HM (WGHM) operator and weighted generalized geometric HM (WGWHM) operator, and proved the properties of reducibility, idempotency, monotonicity, and boundedness. Li et al. [23] further extended weighted HM AOs to SVN and put forward improved generalized weighted HM (IGWHM) and improved generalized weighted geometric HM (IGWGHM) operators. Subsequently, IGWHM and IGWGHM operators are extended to neutrosophic hesitant fuzzy set (NHFS) [24], normal neutrosophic set (NNS) [25], neutrosophic uncertain linguistic set (NULS) [26], and bipolar neutrosophic set (BNS) [27].

Since each operator has its own merit, the combination of different operators has become a hotspot in MADM problem. Motivated by the superiorities of both PA operator and HM operator, Liu [28] combined them and proposed power Heronian (PH) operator and power weighted Heronian (PWH) operator to manage IVIFS. Later, Shi et al. [29] presented power geometric Heronian mean (PGHM) operator and weighted PGHM (WPGHM) operator to IFS. Zhao et al. [30] extended PH operator, weighted PH (WPH) operator, geometric PH (GPH) operator, and weighted GPH (WGPH) operator to SVN. Jiang et al. [31] applied operators of PH, WPH, GPH, and WGPH to interval-valued dual HFS (IVDHFS). Ju et al. [32] combined PA and generalized HM operator and applied to fuse hesitant fuzzy linguistic numbers (HFLNs). Wang et al. [33] extended power generalized HM operator and its weighted form to q-rung orthopair hesitant fuzzy set (qROHFS). Moreover, rough PH operator [34] and power partitioned Heronian AOs [35, 36] have also been explored. As mentioned above, the combination of PA operator and HM operator has received the attention of many scholars. However, up to now, there is no research regarding the combination of PA operator and IGHM operator, which has more desirable properties than HM operator.

For aggregating PFS, different AOs have been explored in the recent years. Wang et al. [37] extended weighted Muirhead mean (WMM) and weighted dual MM (WDM) to picture fuzzy set and proposed PFWMM and PFWDM

operator for financial investment risk appraisal. Jana et al. [38] established some Dombi AOs under picture fuzzy environment. Liu and Zhang [39] explored operator of A-PFLWAA to MADM under PFS. Zhang et al. [40] introduced Heronian mean (HM) operators using Dombi operations to PFS. Ates and Akay [41] expanded some Bonferroni mean (BM) to picture fuzzy set to handle MADM, including PFBM, PFNWB, and PFWBM operators. Luo and Xing [42] proposed picture fuzzy interaction partitioned Heronian (PFIPH) operator for selecting hotel. Qin et al. [43] introduced the Dempster-Shafer theory to PFS and proposed power Muirhead mean (PMM) operators utilizing the proposed operational rules. Yang and Li [9] defined multiple-valued picture fuzzy linguistic set (MVPFLS) and extended generalized HM operators to it. Up to date, AOs for PFS are almost based on Algebraic operations. Besides, there are a few researches on fusing picture fuzzy numbers (PFNs) utilizing Einstein [44], Dombi [38, 40], and Hamacher [45, 46]. Hamacher operations, as an extension of Algebraic and Einstein operations, are more agile with a general parameter. Wang et al. [47] explored PA operator based on Hamacher to assess the quality of express service. However, in terms of the existing works, there is no research regarding exploring the combination of PA and IGHM operators to fuse picture fuzzy numbers (PFNs), especially using Hamacher operations.

In sum, the motivations of the paper are presented as follows:

- (1) In complex situations, PFS is more suitable for expressing more answers of DMs, such as support, neutral, against, and refusal, which is beyond the capabilities of FS and IFS. Thus, PFS is selected to represent decision-making information in the paper.
- (2) In more sophisticated cases, interrelationships among attributes are commonly existed in MADM issue, and unreasonable data may also be provided by DMs. Nevertheless, existing picture fuzzy AOs cannot reflect the correlations between attributes and alleviate the influence of abnormal data simultaneously. Thus, to solve this problem, the combination of both PA and IGHM operators is proposed in the paper.
- (3) In MADM problem, to fuse information, aggregating operator is usually based on different operations, and Hamacher operations are more general and agile. Thus, it is an interesting topic to investigate the power improved generalized Heronian mean (PIGHM) AOs utilizing Hamacher operations.

Inspired by these factors, our contributions include the following: (1) We firstly combine PA and IGHM operators and propose some novel power improved generalized Heronian mean (PIGHM) AOs. (2) Based on Hamacher operations, we define novel weighted power improved generalized Heronian mean (WPIGHM) operator and weighted geometric power improved generalized Heronian mean (WGPIGHM) operator. (3) Based on the proposed AOs, we build a new MADM model to fuse PFNs.

The framework of the paper is constructed as follows. In Section 2, the notion and Hamacher operations of PFNs as well as distance measure and cosine coefficient are introduced. In Section 3, PIGHM AOs utilizing Hamacher operations to picture fuzzy information are developed. Moreover, their predominant properties and particular instances are also analyzed. In Section 4, the MADM method based on PIGHM AOs with PFS is proposed. In Section 5, example and analysis are performed to verify the reliability of the model. In Section 6, some remarks are provided.

2. Preliminaries

Here, PFS and its Hamacher operations are introduced, and some measures and related operators are investigated as well.

2.1. PFS

Definition 1 (see [3]). A PFS H in Z is denoted as $H = \{ \langle z, \mu_H(z), \eta_H(z), \nu_H(z) \rangle, |z \in Z \}$, where $\mu_H(z) \in [0, 1]$, $\eta_H(z) \in [0, 1]$, and $\nu_H(z) \in [0, 1]$ represent the positive, neutral, and negative memberships, respectively, meeting $0 \leq \mu_H(z) + \eta_H(z) + \nu_H(z) \leq 1$, $\forall z \in Z$, and $\pi_H(z) = 1 - \mu_H(z) - \eta_H(z) - \nu_H(z)$ is called the refusal membership degree.

Particularly, if there is only one element in Z , H is defined by $H = \langle \mu_H, \eta_H, \nu_H \rangle$, which is called picture fuzzy number (PFN).

2.2. Hamacher. Hamacher operations contain Hamacher product $e \otimes f = (ef / (\gamma + (1 - \gamma)(e + f - ef)))$ and Hamacher sum $e \oplus f = (e + f - ef - (1 - \gamma)ef / (1 - (1 - \gamma)ef))$. Some operations of PFNs utilizing Hamacher are presented below.

Definition 2 (see [45]). Let $H = \langle \mu_H, \eta_H, \nu_H \rangle$ and $G = \langle \mu_G, \eta_G, \nu_G \rangle$ be two PFNs, and $\lambda > 0$. The Hamacher operations of PFNs are defined as follows:

$$\begin{aligned} (1) \quad H \oplus G &= \left\langle \frac{(\mu_H + \mu_G - \mu_H \mu_G - (1 - \gamma)\mu_H \mu_G / (1 - (1 - \gamma)\mu_H \mu_G))}{(\eta_H \eta_G / (\gamma + (1 - \gamma)(\eta_H + \eta_G - \eta_H \eta_G))), \right. \\ &\quad \left. \frac{\nu_H \nu_G / (\gamma + (1 - \gamma)(\nu_H + \nu_G - \nu_H \nu_G))}{(\mu_H \mu_G / (\gamma + (1 - \gamma)(\mu_H + \mu_G - \mu_H \mu_G)))}, \right\rangle \\ (2) \quad H \otimes G &= \left\langle \frac{(\eta_H + \eta_G - \eta_H \eta_G - (1 - \gamma)\eta_H \eta_G / (1 - (1 - \gamma)\eta_H \eta_G))}{(\nu_H + \nu_G - \nu_H \nu_G - (1 - \gamma)\nu_H \nu_G / (1 - (1 - \gamma)\nu_H \nu_G))}, \right. \\ &\quad \left. \frac{(\mu_H \mu_G / (\gamma + (1 - \gamma)(\mu_H + \mu_G - \mu_H \mu_G)))}{(\eta_H \eta_G / (\gamma + (1 - \gamma)(\eta_H + \eta_G - \eta_H \eta_G)))}, \right\rangle \\ (3) \quad \lambda H &= \left\langle \frac{((1 + (\gamma - 1)\mu_H)^\lambda - (1 - \mu_H)^\lambda) / (1 + (\gamma - 1)\mu_H)^\lambda + (\gamma - 1)(1 - \mu_H)^\lambda)}{(\gamma \eta_H^\lambda / (1 + (\gamma - 1)(1 - \eta_H)^\lambda) + (\gamma - 1)\eta_H^\lambda)}, \right. \\ &\quad \left. \frac{(\gamma \nu_H^\lambda / (1 + (\gamma - 1)(1 - \nu_H)^\lambda) + (\gamma - 1)\nu_H^\lambda)}{(\mu_H^\lambda / (1 + (\gamma - 1)(1 - \mu_H)^\lambda) + (\gamma - 1)\mu_H^\lambda)}, \right\rangle \\ (4) \quad H^\lambda &= \left\langle \frac{(\gamma \mu_H^\lambda / (1 + (\gamma - 1)(1 - \mu_H)^\lambda) + (\gamma - 1)\mu_H^\lambda)}{((1 + (\gamma - 1)\eta_H)^\lambda - (1 - \eta_H)^\lambda) / (1 + (\gamma - 1)\eta_H)^\lambda + (\gamma - 1)(1 - \eta_H)^\lambda)}, \right. \\ &\quad \left. \frac{((1 + (\gamma - 1)\nu_H)^\lambda - (1 - \nu_H)^\lambda) / (1 + (\gamma - 1)\nu_H)^\lambda + (\gamma - 1)(1 - \nu_H)^\lambda)}{(\mu_H^\lambda / (1 + (\gamma - 1)(1 - \mu_H)^\lambda) + (\gamma - 1)\mu_H^\lambda)}, \right\rangle \end{aligned}$$

2.3. Measure

Definition 3. The normalized Hamming distance measure between two PFNs $H = \langle \mu_H, \eta_H, \nu_H \rangle$ and $G = \langle \mu_G, \eta_G, \nu_G \rangle$ is denoted by

$$d(H, G) = \frac{1}{2} (|\mu_H - \mu_G| + |\eta_H - \eta_G| + |\nu_H - \nu_G|), \quad (1)$$

where $d(H, G)$ meets these conditions: $0 \leq d(H, G) \leq 1$ and $d(H, G) = d(G, H)$.

The smaller the distance between two PFNs is, the bigger the similarity between them is.

Definition 4. The similarity measure between two PFNs $H = \langle \mu_H, \eta_H, \nu_H \rangle$ and the ideal value $H^* = \langle 1, 0, 0 \rangle$ is defined by

$$s(H, H^*) = \frac{\mu_H}{\sqrt{\mu_H^2 + \eta_H^2 + \nu_H^2}}. \quad (2)$$

2.4. PA and IGHM

Definition 5. Let β_f ($f = 1, 2, \dots, h$) be multiple aggregated values, and the power average (PA) operator [10] and power geometric (PG) operator [48] are, respectively, denoted as

$$PA(\beta_1, \beta_2, \dots, \beta_h) = \frac{\sum_{f=1}^h (1 + \varphi(\beta_f)) \beta_f}{\sum_{f=1}^h (1 + \varphi(\beta_f))}, \quad (3)$$

$$PG(\beta_1, \beta_2, \dots, \beta_h) = \prod_{f=1}^h \beta_f^{\left(\frac{1 + \varphi(\beta_f)}{\sum_{f=1}^h (1 + \varphi(\beta_f))} \right)},$$

where $\varphi(\beta_f) = \sum_{g=1, g \neq f}^h \sup(\beta_f, \beta_g)$ and $\sup(\beta_f, \beta_g) = 1 - d(\beta_f, \beta_g)$ is the support between β_f and β_g , which meets the following conditions:

- (1) $\sup(\beta_f, \beta_g) \in [0, 1]$
- (2) $\sup(\beta_f, \beta_g) = \sup(\beta_g, \beta_f)$
- (3) $\sup(\beta_f, \beta_g) \geq \sup(\beta_{f'}, \beta_{g'})$

If $d(\beta_f, \beta_g) \leq d(\beta_{f'}, \beta_{g'})$, here, $d(\beta_f, \beta_g)$ is the Hamming distance presented in Definition 3.

Definition 6. Let $s, t \geq 0$ and β_f ($f = 1, 2, \dots, h$) be multiple aggregated values, and let $\omega = (\omega_1, \omega_2, \dots, \omega_h)$ be a corresponding weight vector of β_f ($f = 1, 2, \dots, h$), where $\omega_f \geq 0$ and $\sum_{f=1}^h \omega_f = 1$. Then, the improved generalized weighted Heronian mean (IGWHM) operator [23] and improved generalized geometric weighted Heronian mean (IGGWHM) operator [23] are, respectively, denoted as

$$\text{IGWHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) = \left(\frac{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \beta_f^s \beta_g^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \right)^{(1/(s+t))}, \quad (4)$$

$$\text{IGGWHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) = \frac{1}{s+t} \prod_{f=1}^h \prod_{g=f}^h (s\beta_f + t\beta_g)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)},$$

where the IGWHM operator has idempotency, monotonicity, and boundedness, and IGGWHM operator has reducibility, idempotency, monotonicity, and boundedness.

3. Power Improved Generalized Heronian Mean Operators

Based on Hamacher operations, two novel AOs, namely, the picture fuzzy Hamacher weighted power improved generalized Heronian mean (PFHWPIGHM) operator and the picture fuzzy Hamacher weighted geometric power improved generalized Heronian mean (PFHWGPIGHM)

operator, are established. Besides, their properties and particular cases are explored as well.

3.1. PFHWPIGHM Operator

Definition 7. Let $s, t \geq 0$ and $\beta_f = \langle \mu_f, \eta_f, \nu_f \rangle$ ($f = 1, 2, \dots, h$) be multiple PFNs, and the importance of PFN β_f is represented by ω_f , meeting $\omega_f \geq 0$ and $\sum_{f=1}^h \omega_f = 1$. Then, the picture fuzzy Hamacher weighted power improved generalized Heronian mean (PFHWPIGHM) operator is denoted as

$$\text{PFHWPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) = \left(\frac{\left(\bigoplus_{f=1}^h \bigoplus_{g=f}^h \omega_f \omega_g (h\varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \beta_f)^s \otimes (h\varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \beta_g)^t \right)}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \right)^{(1/(s+t))}, \quad (5)$$

where $\varepsilon_f = (1 + \varphi(\beta_f) / \sum_{e=1}^h (1 + \varphi(\beta_e)))$, $\varphi(\beta_f) = \sum_{g=1, g \neq f}^h \sup(\beta_f, \beta_g)$, and $\sup(\beta_f, \beta_g) = 1 - d(\beta_f, \beta_g)$.

Theorem 1. Let $s, t \geq 0$ and $\beta_f = \langle \mu_f, \eta_f, \nu_f \rangle$ ($f = 1, 2, \dots, h$) be multiple PFNs, and the importance of PFN β_f is

represented by ω_f , meeting $\omega_f \geq 0$ and $\sum_{f=1}^h \omega_f = 1$. Based on Hamacher operations in Definition 2, the aggregating value of PFHWPIGHM operator in Definition 7 is still a PFN and is shown as follows:

$$\text{PFHWPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) =$$

$$\left\langle \frac{\gamma \left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{M}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{N}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{M}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{N}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{M}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{N}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \right. \\ \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{P}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{O}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{P}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{O}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{P}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{O}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \\ \left. \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{R}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{Q}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{R}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{Q}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{R}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h \mathcal{Q}_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=f}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \right\rangle. \quad (6)$$

Here,

$$\begin{aligned}
M_{fg} &= \left((1 + (\gamma - 1)\mu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \mu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\mu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \mu_g)^{h\zeta_g} \right)^t \\
&\quad + (\gamma^2 - 1) \left((1 + (\gamma - 1)\mu_f)^{h\zeta_f} - (1 - \mu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\mu_g)^{h\zeta_g} - (1 - \mu_g)^{h\zeta_g} \right)^t, \\
N_{fg} &= \left((1 + (\gamma - 1)\mu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \mu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\mu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \mu_g)^{h\zeta_g} \right)^t \\
&\quad - \left((1 + (\gamma - 1)\mu_f)^{h\zeta_f} - (1 - \mu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\mu_g)^{h\zeta_g} - (1 - \mu_g)^{h\zeta_g} \right)^t, \\
O_{fg} &= \left((1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} + (\gamma^2 - 1)\eta_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} + (\gamma^2 - 1)\eta_g^{h\zeta_g} \right)^t \\
&\quad - \left((1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} - \eta_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} - \eta_g^{h\zeta_g} \right)^t, \\
P_{fg} &= \left((1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} + (\gamma^2 - 1)\eta_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} + (\gamma^2 - 1)\eta_g^{h\zeta_g} \right)^t \\
&\quad + (\gamma^2 - 1) \left((1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} - \eta_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} - \eta_g^{h\zeta_g} \right)^t, \\
Q_{fg} &= \left((1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} + (\gamma^2 - 1)\nu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} + (\gamma^2 - 1)\nu_g^{h\zeta_g} \right)^t \\
&\quad - \left((1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} - \nu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} - \nu_g^{h\zeta_g} \right)^t, \\
R_{fg} &= \left((1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} + (\gamma^2 - 1)\nu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} + (\gamma^2 - 1)\nu_g^{h\zeta_g} \right)^t \\
&\quad + (\gamma^2 - 1) \left((1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} - \nu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} - \nu_g^{h\zeta_g} \right)^t, \\
\zeta_f &= \frac{\varepsilon_f \omega_f}{\sum_{e=1}^h \varepsilon_e \omega_e}, \\
\zeta_g &= \frac{\varepsilon_g \omega_g}{\sum_{e=1}^h \varepsilon_e \omega_e}.
\end{aligned} \tag{7}$$

Proof. Based on Hamacher operations of PFNs presented in Definition 2, we get

$$\begin{aligned}
h\zeta_f\beta_f &= \left\langle \frac{(1+(\gamma-1)\mu_f)^{h\zeta_f} - (1-\mu_f)^{h\zeta_f}}{(1+(\gamma-1)\mu_f)^{h\zeta_f} + (\gamma-1)(1-\mu_f)^{h\zeta_f}}, \frac{\gamma\eta_f^{h\zeta_f}}{(1+(\gamma-1)(1-\eta_f))^{h\zeta_f} + (\gamma-1)\eta_f^{h\zeta_f}}, \right. \\
&\quad \left. \cdot \frac{\gamma\nu_f^{h\zeta_f}}{(1+(\gamma-1)(1-\nu_f))^{h\zeta_f} + (\gamma-1)\nu_f^{h\zeta_f}} \right\rangle, \\
h\zeta_g\beta_g &= \left\langle \frac{(1+(\gamma-1)\mu_g)^{h\zeta_g} - (1-\mu_g)^{h\zeta_g}}{(1+(\gamma-1)\mu_g)^{h\zeta_g} + (\gamma-1)(1-\mu_g)^{h\zeta_g}}, \frac{\gamma\eta_g^{h\zeta_g}}{(1+(\gamma-1)(1-\eta_g))^{h\zeta_g} + (\gamma-1)\eta_g^{h\zeta_g}}, \right. \\
&\quad \left. \cdot \frac{\gamma\nu_g^{h\zeta_g}}{(1+(\gamma-1)(1-\nu_g))^{h\zeta_g} + (\gamma-1)\nu_g^{h\zeta_g}} \right\rangle, \\
(h\zeta_f\beta_f)^s &= \left\langle \frac{\gamma\left((1+(\gamma-1)\mu_f)^{h\zeta_f} - (1-\mu_f)^{h\zeta_f}\right)^s}{\left((1+(\gamma-1)\mu_f)^{h\zeta_f} + (\gamma^2-1)(1-\mu_f)^{h\zeta_f}\right)^s + (\gamma-1)\left((1+(\gamma-1)\mu_f)^{h\zeta_f} - (1-\mu_f)^{h\zeta_f}\right)^s}, \right. \\
&\quad \cdot \frac{\left((1+(\gamma-1)(1-\eta_f))^{h\zeta_f} + (\gamma^2-1)\eta_f^{h\zeta_f}\right)^s - \left((1+(\gamma-1)(1-\eta_f))^{h\zeta_f} - \eta_f^{h\zeta_f}\right)^s}{\left((1+(\gamma-1)(1-\eta_f))^{h\zeta_f} + (\gamma^2-1)\eta_f^{h\zeta_f}\right)^s + (\gamma-1)\left((1+(\gamma-1)(1-\eta_f))^{h\zeta_f} - \eta_f^{h\zeta_f}\right)^s}, \\
&\quad \cdot \frac{\left((1+(\gamma-1)(1-\nu_f))^{h\zeta_f} + (\gamma^2-1)\nu_f^{h\zeta_f}\right)^s - \left((1+(\gamma-1)(1-\nu_f))^{h\zeta_f} - \nu_f^{h\zeta_f}\right)^s}{\left((1+(\gamma-1)(1-\nu_f))^{h\zeta_f} + (\gamma^2-1)\nu_f^{h\zeta_f}\right)^s + (\gamma-1)\left((1+(\gamma-1)(1-\nu_f))^{h\zeta_f} - \nu_f^{h\zeta_f}\right)^s} \Big\rangle, \\
(h\zeta_g\beta_g)^t &= \left\langle \frac{\gamma\left((1+(\gamma-1)\mu_g)^{h\zeta_g} - (1-\mu_g)^{h\zeta_g}\right)^t}{\left((1+(\gamma-1)\mu_g)^{h\zeta_g} + (\gamma^2-1)(1-\mu_g)^{h\zeta_g}\right)^t + (\gamma-1)\left((1+(\gamma-1)\mu_g)^{h\zeta_g} - (1-\mu_g)^{h\zeta_g}\right)^t}, \right. \\
&\quad \cdot \frac{\left((1+(\gamma-1)(1-\eta_g))^{h\zeta_g} + (\gamma^2-1)\eta_g^{h\zeta_g}\right)^t - \left((1+(\gamma-1)(1-\eta_g))^{h\zeta_g} - \eta_g^{h\zeta_g}\right)^t}{\left((1+(\gamma-1)(1-\eta_g))^{h\zeta_g} + (\gamma^2-1)\eta_g^{h\zeta_g}\right)^t + (\gamma-1)\left((1+(\gamma-1)(1-\eta_g))^{h\zeta_g} - \eta_g^{h\zeta_g}\right)^t}, \\
&\quad \cdot \frac{\left((1+(\gamma-1)(1-\nu_g))^{h\zeta_g} + (\gamma^2-1)\nu_g^{h\zeta_g}\right)^t - \left((1+(\gamma-1)(1-\nu_g))^{h\zeta_g} - \nu_g^{h\zeta_g}\right)^t}{\left((1+(\gamma-1)(1-\nu_g))^{h\zeta_g} + (\gamma^2-1)\nu_g^{h\zeta_g}\right)^t + (\gamma-1)\left((1+(\gamma-1)(1-\nu_g))^{h\zeta_g} - \nu_g^{h\zeta_g}\right)^t} \Big\rangle.
\end{aligned} \tag{8}$$

Let

Then,

$$\begin{aligned}
A_f &= (1 + (\gamma - 1)\mu_f)^{h\zeta_f} - (1 - \mu_f)^{h\zeta_f}, \\
B_f &= (1 + (\gamma - 1)\mu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \mu_f)^{h\zeta_f}, \\
C_f &= (1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} + (\gamma^2 - 1)\eta_f^{h\zeta_f}, \\
D_f &= (1 + (\gamma - 1)(1 - \eta_f))^{h\zeta_f} - \eta_f^{h\zeta_f}, \\
E_f &= (1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} + (\gamma^2 - 1)\nu_f^{h\zeta_f}, \\
F_f &= (1 + (\gamma - 1)(1 - \nu_f))^{h\zeta_f} - \nu_f^{h\zeta_f}, \\
A_g &= (1 + (\gamma - 1)\mu_g)^{h\zeta_g} - (1 - \mu_g)^{h\zeta_g}, \\
B_g &= (1 + (\gamma - 1)\mu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \mu_g)^{h\zeta_g}, \\
C_g &= (1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} + (\gamma^2 - 1)\eta_g^{h\zeta_g}, \\
D_g &= (1 + (\gamma - 1)(1 - \eta_g))^{h\zeta_g} - \eta_g^{h\zeta_g}, \\
E_g &= (1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} + (\gamma^2 - 1)\nu_g^{h\zeta_g}, \\
F_g &= (1 + (\gamma - 1)(1 - \nu_g))^{h\zeta_g} - \nu_g^{h\zeta_g}.
\end{aligned} \tag{9}$$

$$\begin{aligned}
(h\zeta_f\beta_f)^s \otimes (h\zeta_g\beta_g)^t &= \left\langle \frac{\gamma A_f^s A_g^t}{B_f^s B_g^t + (\gamma - 1)A_f^s A_g^t}, \frac{C_f^s C_g^t - D_f^s D_g^t}{C_f^s C_g^t + (\gamma - 1)D_f^s D_g^t}, \frac{E_f^s E_g^t - F_f^s F_g^t}{E_f^s E_g^t + (\gamma - 1)F_f^s F_g^t} \right\rangle, \\
\omega_f \omega_g (h\zeta_f\beta_f)^s \otimes (h\zeta_g\beta_g)^t &= \left\langle \frac{(B_f^s B_g^t + (\gamma^2 - 1)A_f^s A_g^t)^{\omega_f \omega_g} - (B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g}}{(B_f^s B_g^t + (\gamma^2 - 1)A_f^s A_g^t)^{\omega_f \omega_g} + (\gamma - 1)(B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g}}, \right. \\
&\quad \cdot \frac{\gamma(C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g}}{(C_f^s C_g^t + (\gamma^2 - 1)D_f^s D_g^t)^{\omega_f \omega_g} + (\gamma - 1)(C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g}}, \\
&\quad \cdot \left. \frac{\gamma(E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g}}{(E_f^s E_g^t + (\gamma^2 - 1)F_f^s F_g^t)^{\omega_f \omega_g} + (\gamma - 1)(E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g}} \right\rangle, \\
\bigoplus_{f=1}^h \bigoplus_{g=f}^h \omega_f \omega_g (h\zeta_f\beta_f)^s \otimes (h\zeta_g\beta_g)^t &= \left\langle \frac{\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t + (\gamma^2 - 1)A_f^s A_g^t)^{\omega_f \omega_g} - \prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g}}{\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t + (\gamma^2 - 1)A_f^s A_g^t)^{\omega_f \omega_g} + (\gamma - 1) \prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g}}, \right. \\
&\quad \cdot \frac{\gamma \prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g}}{\prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t + (\gamma^2 - 1)D_f^s D_g^t)^{\omega_f \omega_g} + (\gamma - 1) \prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g}}, \\
&\quad \cdot \left. \frac{\gamma \prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g}}{\prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t + (\gamma^2 - 1)F_f^s F_g^t)^{\omega_f \omega_g} + (\gamma - 1) \prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g}} \right\rangle. \tag{10}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \frac{\oplus_{f=1}^h \oplus_{g=f}^h \omega_f \omega_g (h\zeta_f \beta_f)^s \otimes (h\zeta_g \beta_g)^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \\
&= \left\langle \frac{(\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t + (\gamma^2 - 1) A_f^s A_g^t)^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)} - (\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}}{(\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t + (\gamma^2 - 1) A_f^s A_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h (B_f^s B_g^t - A_f^s A_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}}, \right. \\
& \quad \cdot \frac{\gamma (\prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}}{(\prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t + (\gamma^2 - 1) D_f^s D_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h (C_f^s C_g^t - D_f^s D_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}}, \\
& \quad \cdot \left. \frac{\gamma (\prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}}{(\prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t + (\gamma^2 - 1) F_f^s F_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h (E_f^s E_g^t - F_f^s F_g^t)^{\omega_f \omega_g})^{1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g}} \right\rangle. \tag{11}
\end{aligned}$$

Let

$$\begin{aligned}
M_{fg} &= B_f^s B_g^t + (\gamma^2 - 1) A_f^s A_g^t, \\
N_{fg} &= B_f^s B_g^t - A_f^s A_g^t, \\
O_{fg} &= C_f^s C_g^t - D_f^s D_g^t, \\
P_{fg} &= C_f^s C_g^t + (\gamma^2 - 1) D_f^s D_g^t, \\
Q_{fg} &= E_f^s E_g^t - F_f^s F_g^t, \\
R_{fg} &= E_f^s E_g^t + (\gamma^2 - 1) F_f^s F_g^t.
\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}
& \frac{\oplus_{f=1}^h \oplus_{g=f}^h \omega_f \omega_g (h\zeta_f \beta_f)^s \otimes (h\zeta_g \beta_g)^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} = \left\langle \frac{(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)} - (\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}}{(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}}, \right. \\
& \quad \cdot \frac{\gamma (\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}}{(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}}, \\
& \quad \cdot \left. \frac{\gamma (\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}}{(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)} + (\gamma - 1) (\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g})^{(1/\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g)}} \right\rangle. \tag{13}
\end{aligned}$$

Thus,

$$\begin{aligned}
 \text{PFHWPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) &= \left(\frac{\bigoplus_{f=1}^h \bigoplus_{g=f}^h \omega_f \omega_g (h \zeta_f \beta_f)^s \otimes (h \zeta_g \beta_g)^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \right)^{(1/(s+t))} \\
 &= \left\langle \frac{\gamma \left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \right. \\
 &\quad \cdot \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))} - \left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \\
 &\quad \cdot \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))} - \left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} + (\gamma^2 - 1) \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} - \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{(1/\sum_{j=1}^h \sum_{g=j}^h \omega_j \omega_g)} \right)^{(1/(s+t))}} \right\rangle. \tag{14}
 \end{aligned}$$

Therefore, Theorem 1 is true. $\square$

Theorem 2 (idempotency). Let $s, t \geq 0$ and $\beta_f = \langle \mu_f, \eta_f, \nu_f \rangle$ ($f = 1, 2, \dots, h$) be multiple PFNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_h)^T$, satisfying $\omega_f = (1/h)$ ($f = 1, 2, \dots, h$) and $\sum_{g=1}^h \omega_g = 1$. When $\beta_f = \beta$ ($f = 1, 2, \dots, h$), the PFHWPIGHM operator has the property of idempotency.

Proof. Since $\beta_f = \beta$ for each f , $\sup(\beta_f, \beta_g) = 1$ and $\varphi(\beta_f) = h - 1$.

Then, $\varepsilon_f = (1 + \varphi(\beta_f) / \sum_{e=1}^h (1 + \varphi(\beta_e))) = (1/h)$, ($h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e = h \omega_f$, and $(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e) = h \omega_g$).

Thus,

$$\begin{aligned}
 \text{PFHWPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) &= \left(\frac{\bigoplus_{f=1}^h \bigoplus_{g=f}^h \omega_f \omega_g ((h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e) \beta_f)^s \otimes (h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \beta_g)^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \right)^{(1/(s+t))} \\
 &= \left(\frac{\bigoplus_{f=1}^h \bigoplus_{g=f}^h \omega_f \omega_g (\beta)^s \otimes (\beta)^t}{\sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g} \right)^{(1/(s+t))} \\
 &= (\beta^{s+t})^{(1/(s+t))} \\
 &= \beta.
 \end{aligned} \tag{15}$$

Therefore, Theorem 2 is true.

Next, we will discuss some special instances of the PFHWPIGHM operator.

(1) When $\gamma = 1$, the PFHWPIGHM operator is degenerated to picture fuzzy Algebraic weighted power improved generalized Heronian mean (PFAWPIGHM) operator as follows:

PFHWPIGHM^{s,t}($\beta_1, \beta_2, \dots, \beta_h$)

$$\begin{aligned}
 &= \left\langle \left(1 - \left(\prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - (1 - \mu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - (1 - \mu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} \right. \\
 &\quad \left. 1 - \left(1 - \left(\prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - \eta_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - \eta_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} \right. \\
 &\quad \left. 1 - \left(1 - \left(\prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - \nu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - \nu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} \right\rangle.
 \end{aligned} \tag{16}$$

(2) When $\gamma = 2$, the PFHWPIGHM operator is degenerated to picture fuzzy Einstein weighted power

improved generalized Heronian mean (PFEWPIGHM) operator as follows:

PFHWPIGHM^{s,t}($\beta_1, \beta_2, \dots, \beta_h$)

$$\begin{aligned}
 &= \left\langle \frac{2 \left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} + 3 \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} + \left(\left(\prod_{f=1}^h \prod_{g=f}^h M_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{1/(s+t)}}, \\
 &\quad \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} + 3 \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} - \left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} + 3 \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} + \left(\left(\prod_{f=1}^h \prod_{g=f}^h P_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))}}, \\
 &\quad \frac{\left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} + 3 \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} - \left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))}}{\left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} + 3 \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))} + \left(\left(\prod_{f=1}^h \prod_{g=f}^h R_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} - \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{\omega_f \omega_g} \right)^{\left(1 / \sum_{f=1}^h \sum_{g=f}^h \omega_f \omega_g \right)} \right)^{(1/(s+t))}} \right\rangle.
 \end{aligned} \tag{17}$$

Here,

$$\begin{aligned}
M_{fg} &= \left((1 + \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3(1 - \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((1 + \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3(1 - \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad + 3 \left((1 + \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - (1 - \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((1 + \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - (1 - \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t, \\
N_{fg} &= \left((1 + \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3(1 - \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((1 + \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3(1 - \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad - \left((1 + \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - (1 - \mu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((1 + \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - (1 - \mu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t, \\
O_{fg} &= \left((2 - \eta_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\eta_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \eta_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\eta_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad - \left((2 - \eta_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \eta_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \eta_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \eta_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t, \\
P_{fg} &= \left((2 - \eta_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\eta_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \eta_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\eta_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad + 3 \left((2 - \eta_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \eta_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \eta_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \eta_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t, \\
Q_{fg} &= \left((2 - \nu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\nu_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \nu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\nu_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad - \left((2 - \nu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \nu_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \nu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \nu_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t, \\
R_{fg} &= \left((2 - \nu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\nu_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \nu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} + 3\nu_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t \\
&\quad + 3 \left((2 - \nu_f)^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \nu_f^{\left(h\varepsilon_f\omega_f/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^s \left((2 - \nu_g)^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} - \nu_g^{\left(h\varepsilon_g\omega_g/\sum_{e=1}^h\varepsilon_e\omega_e\right)} \right)^t.
\end{aligned} \tag{18}$$

(18)

3.2. PFHWGPIGHM Operator

Definition 8. Let $s, t \geq 0$ and β_f ($f = 1, 2, \dots, h$) be multiple PFNs, and the importance of PFN β_f is represented by

ω_f , meeting $\omega_f \geq 0$ and $\sum_{f=1}^h \omega_f = 1$. Then, the picture fuzzy Hamacher weighted geometric power improved generalized Heronian mean (PFHWGPIGHM) operator is denoted as

□

$$\text{PFHWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h)$$

$$= \frac{1}{s+t} \bigotimes_{f=1}^h \bigotimes_{g=f}^h \left(s\beta_f^{\left(h\varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \oplus t\beta_g^{\left(h\varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)}, \quad (19)$$

where $\varepsilon_f = (1 + \varphi(\beta_f) / \sum_{e=1}^h (1 + \varphi(\beta_e)))$, $\varphi(\beta_f) = \sum_{g=1, g \neq f}^h \sup(\beta_f, \beta_g)$, and $\sup(\beta_f, \beta_g) = 1 - d(\beta_f, \beta_g)$.

Theorem 3. Let $s, t \geq 0$ and $\beta_f = \langle \mu_f, \eta_f, \nu_f \rangle$ ($f = 1, 2, \dots, h$) be multiple PFNs, and the importance of PFN β_f is

represented by ω_f , meeting $\omega_f \geq 0$ and $\sum_{f=1}^h \omega_f = 1$. Based on Hamacher operations in Definition 2, the aggregating value of PFHWGPIGHM operator in Definition 8 is still a PFN and is shown as follows:

$$\text{PFHWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h)$$

$$= \left\langle \frac{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}, \right. \\ \left. \frac{\gamma \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}, \right. \\ \left. \frac{\gamma \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}} \right). \quad (20)$$

Here,

$$M_{fg} = \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma^2 - 1)\mu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma^2 - 1)\mu_g^{h\zeta_g} \right)^t \\ - \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} - \mu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} - \mu_g^{h\zeta_g} \right)^t, \\ N_{fg} = \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma^2 - 1)\mu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma^2 - 1)\mu_g^{h\zeta_g} \right)^t \\ + (\gamma^2 - 1) \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} - \mu_f^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} - \mu_g^{h\zeta_g} \right)^t, \\ O_{fg} = \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \eta_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \eta_g)^{h\zeta_g} \right)^t \\ + (\gamma^2 - 1) \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g} \right)^t,$$

$$\begin{aligned}
P_{fg} &= \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \eta_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \eta_g)^{h\zeta_g} \right)^t \\
&\quad - \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g} \right)^t, \\
Q_{fg} &= \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \nu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \nu_g)^{h\zeta_g} \right)^t \\
&\quad + (\gamma^2 - 1) \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g} \right)^t, \\
R_{fg} &= \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \nu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \nu_g)^{h\zeta_g} \right)^t \\
&\quad - \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f} \right)^s \left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g} \right)^t, \\
\zeta_f &= \frac{\varepsilon_f \omega_f}{\sum_{e=1}^h \varepsilon_e \omega_e}, \\
\zeta_g &= \frac{\varepsilon_g \omega_g}{\sum_{e=1}^h \varepsilon_e \omega_e}.
\end{aligned} \tag{21}$$

Proof. Based on Hamacher operations of PFNs presented in Definition 2, we get

$$\begin{aligned}
\beta_f^{h\zeta_f} &= \left\langle \frac{\gamma \mu_f^{h\zeta_f}}{\left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma - 1)\mu_f^{h\zeta_f} \right)}, \frac{(1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f}}{(1 + (\gamma - 1)\eta_f)^{h\zeta_f} + (\gamma - 1)(1 - \eta_f)^{h\zeta_f}}, \right. \\
&\quad \left. \cdot \frac{(1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f}}{(1 + (\gamma - 1)\nu_f)^{h\zeta_f} + (\gamma - 1)(1 - \nu_f)^{h\zeta_f}} \right\rangle, \\
\beta_g^{h\zeta_g} &= \left\langle \frac{\gamma \mu_g^{h\zeta_g}}{\left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma - 1)\mu_g^{h\zeta_g} \right)}, \frac{(1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g}}{(1 + (\gamma - 1)\eta_g)^{h\zeta_g} + (\gamma - 1)(1 - \eta_g)^{h\zeta_g}}, \right. \\
&\quad \left. \cdot \frac{(1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g}}{(1 + (\gamma - 1)\nu_g)^{h\zeta_g} + (\gamma - 1)(1 - \nu_g)^{h\zeta_g}} \right\rangle, \\
s\beta_f^{h\zeta_f} &= \left\langle \frac{\left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma^2 - 1)\mu_f^{h\zeta_f} \right)^s - \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} - \mu_f^{h\zeta_f} \right)^s}{\left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma^2 - 1)\mu_f^{h\zeta_f} \right)^s + (\gamma - 1) \left((1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} - \mu_f^{h\zeta_f} \right)^s}, \right. \\
&\quad \frac{\gamma \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f} \right)^s}{\left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \eta_f)^{h\zeta_f} \right)^s + (\gamma - 1) \left((1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f} \right)^s}, \\
&\quad \left. \cdot \frac{\gamma \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f} \right)^s}{\left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \nu_f)^{h\zeta_f} \right)^s + (\gamma - 1) \left((1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f} \right)^s} \right\rangle,
\end{aligned}$$

$$t\beta_g^{h\zeta_g} = \left\langle \frac{\left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma^2 - 1)\mu_g^{h\zeta_g} \right)^t - \left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} - \mu_g^{h\zeta_g} \right)^t}{\left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma^2 - 1)\mu_g^{h\zeta_g} \right)^t + (\gamma - 1)\left((1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} - \mu_g^{h\zeta_g} \right)^t}, \right. \\ \left. \frac{\gamma \left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g} \right)^t}{\left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \eta_g)^{h\zeta_g} \right)^t + (\gamma - 1)\left((1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g} \right)^t}, \right. \\ \left. \frac{\gamma \left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g} \right)^t}{\left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \nu_g)^{h\zeta_g} \right)^t + (\gamma - 1)\left((1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g} \right)^t} \right\rangle. \quad (22)$$

Let

Then,

$$\begin{aligned} A_f &= (1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} + (\gamma^2 - 1)\mu_f^{h\zeta_f}, \\ B_f &= (1 + (\gamma - 1)(1 - \mu_f))^{h\zeta_f} - \mu_f^{h\zeta_f}, \\ C_f &= (1 + (\gamma - 1)\eta_f)^{h\zeta_f} - (1 - \eta_f)^{h\zeta_f}, \\ D_f &= (1 + (\gamma - 1)\eta_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \eta_f)^{h\zeta_f}, \\ E_f &= (1 + (\gamma - 1)\nu_f)^{h\zeta_f} - (1 - \nu_f)^{h\zeta_f}, \\ F_f &= (1 + (\gamma - 1)\nu_f)^{h\zeta_f} + (\gamma^2 - 1)(1 - \nu_f)^{h\zeta_f}, \\ A_g &= (1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} + (\gamma^2 - 1)\mu_g^{h\zeta_g}, \\ B_g &= (1 + (\gamma - 1)(1 - \mu_g))^{h\zeta_g} - \mu_g^{h\zeta_g}, \\ C_g &= (1 + (\gamma - 1)\eta_g)^{h\zeta_g} - (1 - \eta_g)^{h\zeta_g}, \\ D_g &= (1 + (\gamma - 1)\eta_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \eta_g)^{h\zeta_g}, \\ E_g &= (1 + (\gamma - 1)\nu_g)^{h\zeta_g} - (1 - \nu_g)^{h\zeta_g}, \\ F_g &= (1 + (\gamma - 1)\nu_g)^{h\zeta_g} + (\gamma^2 - 1)(1 - \nu_g)^{h\zeta_g}. \end{aligned} \quad (23)$$

$$s\beta_f^{h\zeta_f} \oplus t\beta_g^{h\zeta_g} = \left\langle \frac{A_f^s A_g^t - B_f^s B_g^t}{A_f^s A_g^t + (\gamma - 1)B_f^s B_g^t}, \frac{\gamma C_f^s C_g^t}{D_f^s D_g^t + (\gamma - 1)C_f^s C_g^t}, \frac{\gamma E_f^s E_g^t}{F_f^s F_g^t + (\gamma - 1)E_f^s E_g^t} \right\rangle, \\ \cdot \left(s\beta_f^{h\zeta_f} \oplus t\beta_g^{h\zeta_g} \right)^{(2(h+1-f)/h(h+1))} \cdot \left(\omega_g^l \sum_{k=f}^h \omega_k \right)$$

$$\begin{aligned}
&= \left\langle \frac{\gamma(A_f^s A_g^t - B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{(A_f^s A_g^t + (\gamma-1)B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1)(A_f^s A_g^t - B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \right. \\
&\quad \frac{(D_f^s D_g^t + (\gamma-1)C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - (D_f^s D_g^t - C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{(D_f^s D_g^t + (\gamma-1)C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1)(D_f^s D_g^t - C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \\
&\quad \frac{(F_f^s F_g^t + (\gamma-1)E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - (F_f^s F_g^t - E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{(F_f^s F_g^t + (\gamma-1)E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1)(F_f^s F_g^t - E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \Bigg\rangle, \\
&\quad \cdot \bigotimes_{f=1}^h \bigotimes_{g=f}^h \left(s\beta_f^{h\zeta_f} \oplus t\beta_g^{h\zeta_g} \right)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} \\
&= \left\langle \frac{\gamma \prod_{f=1}^h \prod_{g=f}^h (A_f^s A_g^t - B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{\prod_{f=1}^h \prod_{g=f}^h (A_f^s A_g^t + (\gamma-1)B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1) \prod_{f=1}^h \prod_{g=f}^h (A_f^s A_g^t - B_f^s B_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \right. \\
&\quad \frac{\prod_{f=1}^h \prod_{g=f}^h (D_f^s D_g^t + (\gamma-1)C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h (D_f^s D_g^t - C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{\prod_{f=1}^h \prod_{g=f}^h (D_f^s D_g^t + (\gamma-1)C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1) \prod_{f=1}^h \prod_{g=f}^h (D_f^s D_g^t - C_f^s C_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \\
&\quad \frac{\prod_{f=1}^h \prod_{g=f}^h (F_f^s F_g^t + (\gamma-1)E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h (F_f^s F_g^t - E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}}{\prod_{f=1}^h \prod_{g=f}^h (F_f^s F_g^t + (\gamma-1)E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)} + (\gamma-1) \prod_{f=1}^h \prod_{g=f}^h (F_f^s F_g^t - E_f^s E_g^t)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g / \sum_{k=f}^h \omega_k)}} \Bigg\rangle. \\
&\hspace{20em} (24)
\end{aligned}$$

Let

Thus,

$$\begin{aligned}
M_{fg} &= A_f^s A_g^t - B_f^s B_g^t, \\
N_{fg} &= A_f^s A_g^t + (\gamma-1)B_f^s B_g^t, \\
O_{fg} &= D_f^s D_g^t + (\gamma-1)C_f^s C_g^t, \\
P_{fg} &= D_f^s D_g^t - C_f^s C_g^t, \\
Q_{fg} &= F_f^s F_g^t + (\gamma-1)E_f^s E_g^t, \\
R_{fg} &= F_f^s F_g^t - E_f^s E_g^t.
\end{aligned} \tag{25}$$

$$\begin{aligned}
\text{PFHWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) &= \frac{1}{s+t} \bigotimes_{f=1}^h \bigotimes_{g=f}^h \left(s\beta_f^{(h\varepsilon_f\omega_f/\sum_{e=1}^h \varepsilon_e\omega_e)} \oplus t\beta_g^{(h\varepsilon_g\omega_g/\sum_{e=1}^h \varepsilon_e\omega_e)} \right)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \\
&= \left\langle \frac{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}} \right. \\
&\quad \cdot \frac{\gamma \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}} \\
&\quad \cdot \left. \frac{\gamma \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} + (\gamma^2 - 1) \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))} + (\gamma - 1) \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \right)^{(1/(s+t))}} \right\rangle. \tag{26}
\end{aligned}$$

Therefore, Theorem 3 is true. $\square$

Theorem 4 (idempotency). Let $s, t \geq 0$ and $\beta_f = \langle \mu_f, \eta_f, \nu_f \rangle$ ($f = 1, 2, \dots, h$) be multiple PFNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_h)^T$, satisfying $\omega_f = (1/n)$ ($f = 1, 2, \dots, h$) and $\sum_{g=1}^h \omega_g = 1$. When $\beta_f = \beta$ ($f = 1, 2, \dots, h$), the PFHWGPIGHM operator has the property of idempotency.

Proof. Since $\beta_f = \beta$ for each f , $\sup(\beta_f, \beta_g) = 1$, and $\varphi(\beta_f) = h - 1$. Then, $\varepsilon_f = (1 + \varphi(\beta_f)/\sum_{e=1}^h (1 + \varphi(\beta_e))) = (1/h)$, and $(h\varepsilon_f\omega_f/\sum_{e=1}^h \varepsilon_e\omega_e) = h\omega_f = 1$. $(h\varepsilon_g\omega_g/\sum_{e=1}^h \varepsilon_e\omega_e) = h\omega_g = 1$. Thus,

$$\begin{aligned}
&\text{PFHWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) \\
&= \frac{1}{s+t} \bigotimes_{f=1}^h \bigotimes_{g=f}^h \left(s\beta_f^{(h\varepsilon_f\omega_f/\sum_{e=1}^h \varepsilon_e\omega_e)} \oplus t\beta_g^{(h\varepsilon_g\omega_g/\sum_{e=1}^h \varepsilon_e\omega_e)} \right)^{(2(h+1-f)/h(h+1)) \cdot (\omega_g/\sum_{k=f}^h \omega_k)} \\
&= \frac{1}{s+t} \bigotimes_{f=1}^h \bigotimes_{g=f}^h (s\beta \oplus t\beta)^{(2(h+1-f)/h(h+1)) \cdot (1/h+1-f)} \\
&= \frac{1}{s+t} ((s\beta \oplus t\beta)^{h(h+1)/2})^{(2/h(h+1))} \\
&= \beta.
\end{aligned} \tag{27}$$

Therefore, Theorem 4 is true.

Next, we will discuss some special instances of the PFHWGPIGHM operator.

- (1) When $\gamma = 1$, the PFHWGPIGHM operator is degenerated to picture fuzzy Algebraic weighted geometric power improved generalized Heronian mean (PFAWGPIGHM) operator as follows:

$$\text{PFHWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) =$$

$$\begin{aligned} & \left\langle 1 - \left(1 - \prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - \mu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - \mu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} \right. \\ & \cdot \left(1 - \prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - (1 - \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - (1 - \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} \\ & \cdot \left(1 - \prod_{f=1}^h \prod_{g=f}^h \left(1 - \left(1 - (1 - \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left(1 - (1 - \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \right)^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} \left. \right\rangle. \end{aligned} \quad (28)$$

(2) When $\gamma = 2$, the PFHWGPIGHM operator is degenerated to picture fuzzy Einstein weighted geometric power improved generalized Heronian mean (PFEWGPIGHM) operator as follows:

$$\text{PFEWGPIGHM}^{s,t}(\beta_1, \beta_2, \dots, \beta_h) =$$

$$\begin{aligned} & \left\langle \frac{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} + 3 \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} - \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} + 3 \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} + \left(\prod_{f=1}^h \prod_{g=f}^h N_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h M_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}} \right. \\ & \cdot \frac{2 \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} + 3 \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} + \left(\prod_{f=1}^h \prod_{g=f}^h O_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h P_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}} \\ & \cdot \frac{2 \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}}{\left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} + 3 \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))} + \left(\prod_{f=1}^h \prod_{g=f}^h Q_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} - \prod_{f=1}^h \prod_{g=f}^h R_{fg}^{(2(h+1-f)/h(h+1)) \cdot \left(\omega_g / \sum_{k=f}^h \omega_k \right)} \right)^{(1/(s+t))}} \left. \right\rangle. \end{aligned} \quad (29)$$

Here,

$$\begin{aligned}
M_{fg} &= \left((2 - \mu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3\mu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((2 - \mu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3\mu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad - \left((2 - \mu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - \mu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((2 - \mu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - \mu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t, \\
N_{fg} &= \left((2 - \mu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3\mu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((2 - \mu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3\mu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad + 3 \left((2 - \mu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - \mu_f^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((2 - \mu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - \mu_g^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t, \\
O_{fg} &= \left((1 + \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad + 3 \left((1 + \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t, \\
P_{fg} &= \left((1 + \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad - \left((1 + \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \eta_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \eta_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t, \\
Q_{fg} &= \left((1 + \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad + 3 \left((1 + \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t, \\
R_{fg} &= \left((1 + \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} + 3(1 - \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t \\
&\quad - \left((1 + \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \nu_f)^{\left(h \varepsilon_f \omega_f / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^s \left((1 + \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} - (1 - \nu_g)^{\left(h \varepsilon_g \omega_g / \sum_{e=1}^h \varepsilon_e \omega_e \right)} \right)^t.
\end{aligned} \tag{30}$$

□

4. MADM Model

A novel MADM model using the proposed operators to manage PFNs is illustrated as follows. Suppose that there are m alternatives represented by $L = \{L_1, L_2, \dots, L_m\}$ and h attributes represented by $A = \{A_1, A_2, \dots, A_h\}$, and the importance of each attribute is denoted by ω_f ($f = 1, 2, \dots, h$), satisfying $\sum_{f=1}^h \omega_f = 1$ and $\omega_f \geq 0$. The assessment value of alternative L_i relating to attribute A_f takes the form of PFN: $\alpha_{if} = \langle \mu(\alpha_{if}), \eta(\alpha_{if}), \nu(\alpha_{if}) \rangle$ ($i = 1, 2, \dots, m; f = 1, 2, \dots, h$), where $0 \leq \mu(\alpha_{if}), \eta(\alpha_{if}), \nu(\alpha_{if}) \leq 1$ and $0 \leq \mu(\alpha_{if}) + \eta(\alpha_{if}) + \nu(\alpha_{if}) \leq 1$. The initial matrix $D = [\alpha_{if}]_{m \times h}$ ($i = 1, 2, \dots, m; f = 1, 2, \dots, h$) is provided by experts.

The MADM model with PFN is shown in Figure 1 and the detailed steps are presented as follows.

Step 1. Obtain the normalized matrix $\tilde{D} = [\beta_{if}]_{m \times h}$.

The assessment value α_{if} with the cost attribute A_f needs to be transformed into β_{if} based on the following formula:

$$\beta_{if} = \begin{cases} \langle \mu(\alpha_{if}), \eta(\alpha_{if}), \nu(\alpha_{if}) \rangle, & \text{for benefit attribute,} \\ \langle \nu(\alpha_{if}), \eta(\alpha_{if}), \mu(\alpha_{if}) \rangle, & \text{for cost attribute.} \end{cases} \tag{31}$$

Here, $i = 1, 2, \dots, m; f = 1, 2, \dots, h$.

Step 2. Compute the supports $\sup(\beta_{if}, \beta_{ig})$.

The supports are calculated based on the following formula:

$$\sup(\beta_{if}, \beta_{ig}) = 1 - d(\beta_{if}, \beta_{ig}). \tag{32}$$

FIGURE 1: The proposed model.

Here, $i = 1, 2, \dots, m$; $f, g = 1, 2, \dots, h$, and $d(\beta_{if}, \beta_{ig})$ is the normalized Hamming distance between PFNs β_{if} and β_{ig} provided in Definition 3.

Step 3. Compute the power weights ε_{if} .

$$\varepsilon_{if} = \frac{1 + \varphi(\beta_{if})}{\sum_{e=1}^h (1 + \varphi(\beta_{ie}))}, \quad (33)$$

where $\varphi(\beta_{if}) = \sum_{\substack{g=1 \\ g \neq f}}^h \sup(\beta_{if}, \beta_{ig})$, and $i = 1, 2, \dots, m$; $f, g = 1, 2, \dots, h$.

Step 4. Compute the weights ζ_{if} of PFN β_{if} .

$$\zeta_{if} = \frac{\varepsilon_{if} \omega_f}{\sum_{e=1}^h \varepsilon_{ie} \omega_e}, \quad (34)$$

where $\zeta_{if} \geq 0$ and $\sum_{f=1}^h \zeta_{if} = 1$, and $i = 1, 2, \dots, m$; $f, g = 1, 2, \dots, h$.

Step 5. Compute the global value β_i of each alternative L_i .

The global value β_i can be obtained utilizing the PFHWPIGHM operator or the PFHWGPIGHM operator as defined in Theorems 1 and 3.

$$\beta_i = \text{PFHWPIGHM}^{st}(\beta_{i1}, \beta_{i2}, \dots, \beta_{ih}), \quad (35)$$

or

$$\beta_i = \text{PFHWGPIGHM}^{st}(\beta_{i1}, \beta_{i2}, \dots, \beta_{ih}). \quad (36)$$

Here $i = 1, 2, \dots, m$.

Step 6. Compute the similarity of alternative L_i and the ideal alternative L^* .

According to Definition 4, the similarity can be calculated based on the following formula:

$$s(L, L^*) = \frac{\mu(\beta_i)}{\sqrt{\mu^2(\beta_i) + \eta^2(\beta_i) + \nu^2(\beta_i)}} \quad (37)$$

Step 7. Rank and select alternative(s).

Based on the similarity measure calculated above, all alternatives are ranked, and the best vendor is obtained.

5. Example and Analysis

An example based on the MADM approach proposed above is applied to verify its practicability, and corresponding analysis is conducted as well.

5.1. Example. Assume that a company wants to purchase an ERP system from five vendors represented by L_i ($i = 1, 2, 3, 4, 5$), and there are four attributes represented by A_f ($f = 1, 2, 3, 4$) to evaluate different vendors, where A_1 denotes function, A_2 denotes fitness, A_3 denotes capability, and A_4 denotes reputation, whose weight vector is $\omega = (0.2, 0.1, 0.3, 0.4)$. The assessment value of each vendor L_i with attribute A_f is denoted by PFN α_{if} , and the matrix $D = [\alpha_{if}]_{5 \times 4}$ below is provided:

$$D = [\alpha_{if}]_{5 \times 4} = \begin{bmatrix} \langle 0.53, 0.33, 0.09 \rangle & \langle 0.89, 0.08, 0.03 \rangle & \langle 0.42, 0.35, 0.18 \rangle & \langle 0.08, 0.89, 0.02 \rangle \\ \langle 0.73, 0.12, 0.08 \rangle & \langle 0.13, 0.64, 0.21 \rangle & \langle 0.03, 0.82, 0.13 \rangle & \langle 0.73, 0.15, 0.08 \rangle \\ \langle 0.91, 0.03, 0.02 \rangle & \langle 0.07, 0.09, 0.05 \rangle & \langle 0.04, 0.85, 0.10 \rangle & \langle 0.68, 0.26, 0.06 \rangle \\ \langle 0.85, 0.09, 0.05 \rangle & \langle 0.74, 0.16, 0.10 \rangle & \langle 0.02, 0.89, 0.05 \rangle & \langle 0.08, 0.84, 0.06 \rangle \\ \langle 0.90, 0.05, 0.02 \rangle & \langle 0.68, 0.08, 0.21 \rangle & \langle 0.05, 0.87, 0.06 \rangle & \langle 0.13, 0.75, 0.09 \rangle \end{bmatrix}. \quad (38)$$

To choose the best vendor, the detailed steps are shown as follows:

Step 1. Obtain the normalized matrix $\tilde{D} = [\beta_{if}]_{5 \times 4}$.

Since all attributes are benefit type, the normalized matrix $\tilde{D} = [\beta_{if}]_{5 \times 4}$ is equal to the original matrix $D = [\alpha_{if}]_{5 \times 4}$.

Step 2. Compute the supports $\sup(\beta_{if}, \beta_{ig})$.

Based on the normalized Hamming distance between PFNs β_{if} and β_{ig} in Definition 3, we can get

$$\begin{aligned}
 d(\beta_{11}, \beta_{12}) &= 0.335, \\
 d(\beta_{11}, \beta_{13}) &= 0.110, \\
 d(\beta_{11}, \beta_{14}) &= 0.540, \\
 d(\beta_{12}, \beta_{13}) &= 0.445, \\
 d(\beta_{12}, \beta_{14}) &= 0.815, \\
 d(\beta_{13}, \beta_{14}) &= 0.520, \\
 d(\beta_{21}, \beta_{22}) &= 0.625, \\
 d(\beta_{21}, \beta_{23}) &= 0.725, \\
 d(\beta_{21}, \beta_{24}) &= 0.015, \\
 d(\beta_{22}, \beta_{23}) &= 0.180, \\
 d(\beta_{22}, \beta_{24}) &= 0.610, \\
 d(\beta_{23}, \beta_{24}) &= 0.710, \\
 d(\beta_{31}, \beta_{32}) &= 0.465, \\
 d(\beta_{31}, \beta_{33}) &= 0.885, \\
 d(\beta_{31}, \beta_{34}) &= 0.250, \\
 d(\beta_{32}, \beta_{33}) &= 0.420, \\
 d(\beta_{32}, \beta_{34}) &= 0.395, \\
 d(\beta_{33}, \beta_{34}) &= 0.635, \\
 d(\beta_{41}, \beta_{42}) &= 0.115, \\
 d(\beta_{41}, \beta_{43}) &= 0.815, \\
 d(\beta_{41}, \beta_{44}) &= 0.765, \\
 d(\beta_{42}, \beta_{43}) &= 0.750, \\
 d(\beta_{42}, \beta_{44}) &= 0.690, \\
 d(\beta_{43}, \beta_{44}) &= 0.060, \\
 d(\beta_{51}, \beta_{52}) &= 0.220, \\
 d(\beta_{51}, \beta_{53}) &= 0.855, \\
 d(\beta_{51}, \beta_{54}) &= 0.770, \\
 d(\beta_{52}, \beta_{53}) &= 0.785, \\
 d(\beta_{52}, \beta_{54}) &= 0.670, \\
 d(\beta_{53}, \beta_{54}) &= 0.115.
 \end{aligned} \tag{39}$$

Then,

$$\begin{aligned}
 \sup(\beta_{11}, \beta_{12}) &= \sup(\beta_{12}, \beta_{11}) = 0.665, \\
 \sup(\beta_{11}, \beta_{13}) &= \sup(\beta_{13}, \beta_{11}) = 0.890, \\
 \sup(\beta_{11}, \beta_{14}) &= \sup(\beta_{14}, \beta_{11}) = 0.460, \\
 \sup(\beta_{12}, \beta_{13}) &= \sup(\beta_{13}, \beta_{12}) = 0.555, \\
 \sup(\beta_{12}, \beta_{14}) &= \sup(\beta_{14}, \beta_{12}) = 0.185, \\
 \sup(\beta_{13}, \beta_{14}) &= \sup(\beta_{14}, \beta_{13}) = 0.480, \\
 \sup(\beta_{21}, \beta_{22}) &= \sup(\beta_{22}, \beta_{21}) = 0.375, \\
 \sup(\beta_{21}, \beta_{23}) &= \sup(\beta_{23}, \beta_{21}) = 0.275, \\
 \sup(\beta_{21}, \beta_{24}) &= \sup(\beta_{24}, \beta_{21}) = 0.985, \\
 \sup(\beta_{22}, \beta_{23}) &= \sup(\beta_{23}, \beta_{22}) = 0.820, \\
 \sup(\beta_{22}, \beta_{24}) &= \sup(\beta_{24}, \beta_{22}) = 0.390, \\
 \sup(\beta_{23}, \beta_{24}) &= \sup(\beta_{24}, \beta_{23}) = 0.290, \\
 \sup(\beta_{31}, \beta_{32}) &= \sup(\beta_{32}, \beta_{31}) = 0.535, \\
 \sup(\beta_{31}, \beta_{33}) &= \sup(\beta_{33}, \beta_{31}) = 0.115, \\
 \sup(\beta_{31}, \beta_{34}) &= \sup(\beta_{34}, \beta_{31}) = 0.750, \\
 \sup(\beta_{32}, \beta_{33}) &= \sup(\beta_{33}, \beta_{32}) = 0.580, \\
 \sup(\beta_{32}, \beta_{34}) &= \sup(\beta_{34}, \beta_{32}) = 0.605, \\
 \sup(\beta_{33}, \beta_{34}) &= \sup(\beta_{34}, \beta_{33}) = 0.365, \\
 \sup(\beta_{41}, \beta_{42}) &= \sup(\beta_{42}, \beta_{41}) = 0.885, \\
 \sup(\beta_{41}, \beta_{43}) &= \sup(\beta_{43}, \beta_{41}) = 0.185, \\
 \sup(\beta_{41}, \beta_{44}) &= \sup(\beta_{44}, \beta_{41}) = 0.235, \\
 \sup(\beta_{42}, \beta_{43}) &= \sup(\beta_{43}, \beta_{42}) = 0.250, \\
 \sup(\beta_{42}, \beta_{44}) &= \sup(\beta_{44}, \beta_{42}) = 0.310, \\
 \sup(\beta_{43}, \beta_{44}) &= \sup(\beta_{44}, \beta_{43}) = 0.940, \\
 \sup(\beta_{51}, \beta_{52}) &= \sup(\beta_{52}, \beta_{51}) = 0.780, \\
 \sup(\beta_{51}, \beta_{53}) &= \sup(\beta_{53}, \beta_{51}) = 0.145, \\
 \sup(\beta_{51}, \beta_{54}) &= \sup(\beta_{54}, \beta_{51}) = 0.230, \\
 \sup(\beta_{52}, \beta_{53}) &= \sup(\beta_{53}, \beta_{52}) = 0.215, \\
 \sup(\beta_{52}, \beta_{54}) &= \sup(\beta_{54}, \beta_{52}) = 0.330, \\
 \sup(\beta_{53}, \beta_{54}) &= \sup(\beta_{54}, \beta_{53}) = 0.885.
 \end{aligned} \tag{40}$$

Step 3. Compute the power weights ε_{if} .

As an example, $\varphi(\beta_{11})$ can be computed as follows based on the formula $\varphi(\beta_{if}) = \sum_{\substack{g=1 \\ g \neq f}}^h \sup(\beta_{if}, \beta_{ig})$:

$$\begin{aligned} \varphi(\beta_{11}) &= \sup(\beta_{11}, \beta_{12}) + \sup(\beta_{11}, \beta_{13}) + \sup(\beta_{11}, \beta_{14}) \\ &= 0.665 + 0.89 + 0.46 \\ &= 2.015. \end{aligned} \quad (41)$$

Then, we can get

$$[\varphi(\beta_{if})]_{5 \times 4} = \begin{bmatrix} 2.015 & 1.405 & 1.925 & 1.125 \\ 1.635 & 1.585 & 1.385 & 1.665 \\ 1.4 & 1.72 & 1.06 & 1.72 \\ 1.305 & 1.445 & 1.375 & 1.485 \\ 1.155 & 1.325 & 1.245 & 1.445 \end{bmatrix}. \quad (42)$$

Then, based on formula (33), we can get

$$[\varepsilon_{if}]_{5 \times 4} = \begin{bmatrix} 0.2880 & 0.2297 & 0.2794 & 0.2030 \\ 0.2566 & 0.2517 & 0.2322 & 0.2595 \\ 0.2424 & 0.2747 & 0.2081 & 0.2747 \\ 0.2399 & 0.2544 & 0.2471 & 0.2586 \\ 0.2350 & 0.2535 & 0.2448 & 0.2666 \end{bmatrix}. \quad (43)$$

Step 4. Compute the weights ζ_{if} of PFN β_{if} .

Based on formula (34), we can get

$$[\zeta_{if}]_{5 \times 4} = \begin{bmatrix} 0.2345 & 0.0935 & 0.3413 & 0.3306 \\ 0.2053 & 0.1007 & 0.2787 & 0.4153 \\ 0.1953 & 0.1107 & 0.2514 & 0.4426 \\ 0.1911 & 0.1014 & 0.2954 & 0.4121 \\ 0.1862 & 0.1004 & 0.2909 & 0.4225 \end{bmatrix}. \quad (44)$$

Step 5. Compute the global value β_i of each alternative L_i .

Let $\gamma = 2, s = t = 1$. The global value β_i can be obtained utilizing the PFHWPIGHM operator as defined in Theorem 1.

$$\begin{aligned} \beta_1 &= \langle 0.3779, 0.4754, 0.0420 \rangle, \\ \beta_2 &= \langle 0.6544, 0.2185, 0.0637 \rangle, \\ \beta_3 &= \langle 0.6804, 0.2195, 0.0374 \rangle, \\ \beta_4 &= \langle 0.2992, 0.6013, 0.0335 \rangle, \\ \beta_5 &= \langle 0.3504, 0.5146, 0.0388 \rangle. \end{aligned} \quad (45)$$

Similarly, the global value β_i can be obtained utilizing the PFHWGPIGHM operator as defined in Theorem 3.

$$\begin{aligned} \beta_1 &= \langle 0.3214, 0.5766, 0.1029 \rangle, \\ \beta_2 &= \langle 0.3678, 0.4453, 0.1110 \rangle, \\ \beta_3 &= \langle 0.3665, 0.4298, 0.0665 \rangle, \\ \beta_4 &= \langle 0.1872, 0.7498, 0.0649 \rangle, \\ \beta_5 &= \langle 0.2426, 0.6444, 0.0870 \rangle. \end{aligned} \quad (46)$$

Step 6. Compute the similarity of alternative L_i and the ideal alternative L^* .

The similarity can be calculated based on formula (37).

For the PFHWPIGHM operator,

$$\begin{aligned} s(L_1, L^*) &= 0.6208, \\ s(L_2, L^*) &= 0.9445, \\ s(L_3, L^*) &= 0.9504, \\ s(L_4, L^*) &= 0.4449, \\ s(L_5, L^*) &= 0.5617. \end{aligned} \quad (47)$$

For the PFHWGPIGHM operator,

$$\begin{aligned} s(L_1, L^*) &= 0.4811, \\ s(L_2, L^*) &= 0.6254, \\ s(L_3, L^*) &= 0.6444, \\ s(L_4, L^*) &= 0.2414, \\ s(L_5, L^*) &= 0.3496. \end{aligned} \quad (48)$$

Step 7. Rank and select alternative(s).

The final ranking using the PFHWPIGHM operator or the PFHWGPIGHM operator is always $L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$, and the best vendor is L_3 , while L_4 is the worst one.

5.2. Sensitivity Analysis. In this part, we will discuss the impact of parameters γ, s , and t to the final ranking result.

When parameters $s = t = 1$, γ is assigned to different values, and the rankings utilizing the PFHWPIGHM operator and the PFHWGPIGHM operator are presented in Tables 1 and 2, respectively.

From Tables 1 and 2, it can be clearly observed that the worst vendor is always L_4 , while the best vendor is always L_3 utilizing the PFHWPIGHM operator, and the best vendor is L_3 or L_2 utilizing the PFHWGPIGHM operator.

When parameter $\gamma = 1$, s and t are assigned to different values, and the rankings utilizing the PFHWPIGHM operator and the PFHWGPIGHM operator are presented in Tables 3 and 4, respectively.

From Tables 3 and 4, it can be clearly observed that the worst vendor is always L_4 utilizing the PFHWGPIGHM operator, while the best vendor is L_3 or L_2 utilizing the PFHWPIGHM operator with different parameters s and t .

TABLE 1: Rankings based on the PFHWPIGHM operator with different parameter γ .

γ	Rankings
1	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
2	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
3	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
5	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
10	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
15	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
20	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
50	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$

TABLE 2: Rankings based on the PFHWGPIGHM operator with different parameter γ .

γ	Rankings
1	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
2	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
3	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
5	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
10	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
15	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
20	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
50	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$

TABLE 3: Rankings based on the PFHWPIGHM operator with $\gamma = 1$.

s, t	Rankings
$s = t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0.5, t = 0$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = 1, t = 0$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = 0, t = 0.5$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 0, t = 1$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 2, t = 1$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = 5, t = 1$	$L_3 \succ L_2 \succ L_5 \succ L_4 \succ L_1$
$s = 1, t = 2$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 5$	$L_2 \succ L_3 \succ L_5 \succ L_1 \succ L_4$
$s = t = 2$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = t = 5$	$L_2 \succ L_3 \succ L_5 \succ L_4 \succ L_1$
$s = t = 10$	$L_2 \succ L_3 \succ L_5 \succ L_4 \succ L_1$

TABLE 4: Rankings based on the PFHWGPIGHM operator with $\gamma = 1$.

s, t	Rankings
$s = t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0.5, t = 0$	$L_3 \succ L_1 \succ L_5 \succ L_2 \succ L_4$
$s = 1, t = 0$	$L_1 \succ L_3 \succ L_2 \succ L_5 \succ L_4$
$s = 0, t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0, t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 2, t = 1$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 5, t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 2$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 2$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 10$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$

TABLE 5: Rankings based on the PFHWPIGHM operator with $\gamma = 2$.

s, t	Rankings
$s = t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0.5, t = 0$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 0$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0, t = 0.5$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 0, t = 1$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 2, t = 1$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = 5, t = 1$	$L_3 \succ L_2 \succ L_5 \succ L_4 \succ L_1$
$s = 1, t = 2$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 5$	$L_2 \succ L_3 \succ L_5 \succ L_1 \succ L_4$
$s = t = 2$	$L_3 \succ L_2 \succ L_5 \succ L_1 \succ L_4$
$s = t = 5$	$L_2 \succ L_3 \succ L_5 \succ L_4 \succ L_1$
$s = t = 10$	$L_2 \succ L_3 \succ L_5 \succ L_4 \succ L_1$

TABLE 6: Rankings based on the PFHWGPIGHM operator with $\gamma = 2$.

s, t	Rankings
$s = t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0.5, t = 0$	$L_3 \succ L_1 \succ L_5 \succ L_2 \succ L_4$
$s = 1, t = 0$	$L_3 \succ L_1 \succ L_2 \succ L_5 \succ L_4$
$s = 0, t = 0.5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 0, t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 2, t = 1$	$L_2 \succ L_3 \succ L_1 \succ L_5 \succ L_4$
$s = 5, t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 2$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = 1, t = 5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 2$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 5$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
$s = t = 10$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$

TABLE 7: Rankings of different operators.

Operator	Ranking
PFHWA [45], $\gamma = 2$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
PFHWG [45], $\gamma = 2$	$L_3 \succ L_1 \succ L_2 \succ L_5 \succ L_4$
PFNWB [41], $s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
PFHWPIGHM, $\gamma = 2, s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$
PFHWGPIGHM, $\gamma = 2, s = t = 1$	$L_3 \succ L_2 \succ L_1 \succ L_5 \succ L_4$

When parameter $\gamma = 2$, s and t are assigned to different values, and the rankings utilizing the PFHWPIGHM operator and the PFHWGPIGHM operator are presented in Tables 5 and 6, respectively.

From Tables 5 and 6, it can be clearly observed that the best vendor is L_3 or L_2 , while the worst vendor is always L_4 utilizing the PFHWGPIGHM operator, and the worst vendor is L_4 or L_1 utilizing the PFHWPIGHM operator with different parameters s and t .

From the sensitivity analysis above, we know that different parameters in the PFHWPIGHM operator and the PFHWGPIGHM operator may cause different ranking orders; thus, parameters γ , s , and t play an essential part in the

TABLE 8: Comparison of characteristics between different methods.

Characteristic	Wei [45]	Ates and Akay [41]	Proposed method
Consider correlations of attributes	No	Yes	Yes
Reduce the influence of biased data	No	No	Yes
Operational rules	Hamacher	Algebraic	Hamacher
Number of parameters	γ	s, t	γ, s, t

MADM problem. When $\gamma = 1$ or $\gamma = 2$, picture fuzzy Hamacher operations can be degenerated to picture fuzzy Algebraic or Einstein operations, respectively. Therefore, the picture fuzzy Hamacher operators proposed in this paper are more general. Furthermore, parameter s or t can illustrate the risk preference of DMs who can decide the value of parameter s or t in terms of actual situation. Generally, the risk seeker will assign big value to parameter s or t . Therefore, the PFHWPIGHM operator and the PFHWGPIGHM operator are more flexible.

5.3. Comparison Analysis. To further demonstrate the superiorities of the proposed MADM method using the PFHWPIGHM operator and the PFHWGPIGHM operator, we compare the method proposed in this paper with the existing method proposed by Wei [45], based on the PFHWA operator and the PFHWG operator, and the method proposed by Ates and Akay [41], based on the PFNWB operator. The rankings of different AOs are listed in Table 7.

From Table 7, it is clearly observed that the ranking orders of different AOs are the same except the PFHWG operator. Besides, the best vendor is always L_3 , and the worst one is always L_4 for all AOs presented above.

Furthermore, a comparison between our method and these methods mentioned above as regards some characteristics is presented in Table 8.

The method of Wei [45] does not consider the correlations of attributes and the influence of biased data, and only one parameter is considered. The method of Ates and Akay [41] considers the correlations of attributes but cannot reduce the influence of biased data, and it is based on Algebraic operations, which is only a special case of Hamacher operations, and only two parameters are considered. In contrast, our method not only considers the correlations of attributes but also reduces the influence of biased data. Moreover, it is based on Hamacher operational rules and three parameters γ, s , and t . These characteristics make the proposed method more general and flexible.

6. Conclusion and Future Directions

On the basis of the related studies of AOs, the PIGHM operators, which combine the PA operator and the IGHM operator, are firstly explored in this paper. Considering the superiorities of PFS and Hamacher operations, two novel AOs, the PFHWPIGHM operator and the PFHWGPIGHM operator, are proposed, and some particular cases are also discussed. Further, based on these proposed AOs, a MADM

model is constructed under PFS environment, and the detailed steps are also investigated. Eventually, an example on ERP vendor selection is conducted to demonstrate the practicality and credibility of the proposed model. Besides, we discuss the impact of different parameters γ, s , and t on the final ranking, and the comparable analysis from different aspects is also concluded.

In the recent years, to better convey the vagueness and uncertainty of cognitive information, some different fuzzy sets and their applications have been proposed and developed, such as Pythagorean fuzzy set [47], intuitionistic fuzzy set [48], Pythagorean 2-tuple linguistic set [49], probabilistic linguistic set [50], picture fuzzy labelling of graphs [51], picture fuzzy soft set [52], bipolar picture fuzzy set [53], intuitionistic multiplicative set [54], and t -spherical fuzzy set [55]. In future research, it is meaningful to explore the application of the proposed AOs and MADM model to these fuzzy sets and apply them to different fields.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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