Research Article

Further Study on Dynamics for a Fractional-Order Competitor-Competitor-Mutualist Lotka–Volterra System

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On the basis of the previous publications, a new fractional-order prey-predator model is set up. Firstly, we discuss the existence, uniqueness, and nonnegativity for the involved fractional-order prey-predator model. Secondly, by analyzing the characteristic equation of the considered fractional-order Lotka–Volterra model and regarding the delay as bifurcation variable, we set up a new sufficient criterion to guarantee the stability behavior and the appearance of Hopf bifurcation for the addressed fractional-order Lotka–Volterra system. Thirdly, we perform the computer simulations with Matlab software to substantiate the rationalisation of the analysis conclusions. The obtained results play an important role in maintaining the balance of population in natural world.

1. Introduction

For a long time, the dynamic characteristics of interaction between predator population and prey population has been a central issue in ecology and biomathematics due to its general appearance and potential importance [1]. General speaking, the interaction between predator population and prey population includes four cases: competition, predation, mutualism, and parasitism [2]. During the past few decades, a great deal of valuable research fruits on dynamical behavior of the above four-type predator-prey models has been covered. For example, Alidousti and Ghafari [3] investigated the Hopf bifurcation and limit cycle of the fractional-order predator-prey model; Sasmal and Takeuchi [4] studied the stability behavior of all equilibria, bifurcation nature, global features, and multi-stability of a predator-prey model; Ryu and Ko [5] discussed the asymptotic peculiarity for positive solutions for

a prey-predator model; Guo et al. [6] proved the appearance of traveling waves in a prey-predator system. Zhang et al. [7] revealed the effect of the fear factor on the periodic solution of a prey-predator model. As for concrete works, we refer the readers to [8–25].

In real life, any biological or environmental coefficients will change with time. So, the parameters in predator-prey models are usually not fixed constants. They are often functions with respect to time. In particular, the influence of a periodically varying environment on the dynamics of predator-prey models plays a vital role in maintaining population balance. Furthermore, the capture of the prey from the predator throughout its past time has an important effect on the present birth rate of the predator [26, 27]. Thus, it is of importance to establish the various type predator-prey models with periodic coefficients. Based on this idea, in 2010, Lv et al. [27] established the predator-prey model involving periodic coefficients as follows:

$$\begin{cases} \dot{u}_1(t) = u_1(t)[\gamma_1(t) - \alpha_{11}(t)u_1(t - \sigma_{11}(t)) - \alpha_{12}(t)u_2(t - \sigma_{12}(t)) + \alpha_{13}(t)u_3(t - \sigma_{13}(t))], \\ \dot{u}_2(t) = u_2(t)[\gamma_2(t) - \alpha_{21}(t)u_1(t - \sigma_{21}(t)) - \alpha_{22}(t)u_2(t - \sigma_{22}(t)) + \alpha_{23}(t)u_3(t - \sigma_{23}(t))], \\ \dot{u}_3(t) = u_3(t)[\gamma_3(t) + \alpha_{31}(t)u_1(t - \sigma_{31}(t)) + \alpha_{32}(t)u_2(t - \sigma_{32}(t)) - \alpha_{33}(t)u_3(t - \sigma_{33}(t))], \end{cases} \quad (1)$$

where $u_1(t)$ and $u_2(t)$ stand for the densities of competing species at time t , $u_3(t)$ stands for the density of cooperating species at time t . And $\gamma_l, \alpha_{lk} \in C(R, [0, \infty))$ and $\sigma_{lk} \in C(R, R)$ denote ψ -periodic function ($\psi > 0$). The parameter $\sigma_{lk}(t) \geq 0$ ($l, k = 1, 2, 3$) is the delay. Applying the fixed point theory and constructing Lyapunov functions, Lv et al. [27] set up the condition to ensure the global stability of periodic solutions for model (1). For details, one can see [27].

To reveal the Hopf bifurcation nature of model (1), Xu [26] assumes that all biological and environmental coefficients remain constants and only the feedback time delays of all species u_i ($i = 1, 2, 3$) to the growth of the species themselves exist and are same [26]. Then, model (1) is rewritten as the following form:

$$\begin{cases} \dot{u}_1(t) = u_1(t)[\gamma_1 - \alpha_{11}u_1(t - \sigma) - \alpha_{12}u_2(t) + \alpha_{13}u_3(t)], \\ \dot{u}_2(t) = u_2(t)[\gamma_2 - \alpha_{21}u_1(t) - \alpha_{22}u_2(t - \sigma) + \alpha_{23}u_3(t)], \\ \dot{u}_3(t) = u_3(t)[\gamma_3 + \alpha_{31}u_1(t) + \alpha_{32}u_2(t) - \alpha_{33}u_3(t - \sigma)]. \end{cases} \quad (2)$$

With the Hopf bifurcation theory, normal form theory, and center manifold principle, Xu [26] obtained a sufficient condition to guarantee the stability behavior and the appearance for bifurcation phenomenon of model (2). Also, they have derived the concrete expression to find the nature of bifurcation periodic solution.

However, the mentioned works above are only restricted to the integer-order differential systems. Recently, the dynamics of fractional-order dynamical models has attracted great attention of many authors due to their extensive application in numerous areas, such as electromagnetic waves, medicine, mechanics, network science, biology, and finance [28–30]. Many scholars argue that fractional calculus is a powerful tool to depict real phenomena of the object world due to its owned hereditary and memory properties of various practical dynamical models [31]. In recent several decades, fractional calculus has attracted more and more attention from a large number of researchers in various fields. In particular, a great deal of interesting fruits on various dynamical natures of prey-predator systems has sprung up. For instance, Mondal et al. [32] derived the condition to ensure the stability behavior for a class of fractional-order prey-predator system; El-Saka et al. [33] analyzed the local stability and bifurcation of fractional-order predator-prey models; Li et al. [34] dealt with the dynamical property of the solutions and global asymptotic stability for a class of fractional-order prey-predator system. For more relational publications, one can see [35–37]. Here, we particularly emphasize that the study on Hopf bifurcation for fractional-order dynamical models starts relatively late. Up to now, only a few literatures have been published. For example, Alidousti [38] investigated the bifurcation

phenomenon of a fractional-order predator-prey model; Huang et al. [30] proposed a novel control technique of bifurcation for a fractional-order prey-predator model with delays; Xu et al. [39] revealed the effect of two delays on bifurcation for fractional-order neural networks. Xiao et al. [40] put up PD control way for Hopf bifurcations of fractional-order networks. As for more related literatures, one can see [41–44].

In terms of the above analysis, we think that it is meaningful for us to study the dynamics (especially Hopf bifurcation) of fractional-order prey-predator models. Based on the previous predator-prey model (2) and assuming that the feedback time delays of all species u_i ($i = 1, 2, 3$) to the growth of the species themselves and other species exists are same, then we propose the following fractional-order competitor-competitor-mutualist Lotka-Volterra system:

$$\begin{cases} \frac{d^q u_1(t)}{dt^q} = u_1(t)[\gamma_1 - \alpha_{11}u_1(t - \sigma) - \alpha_{12}u_2(t - \sigma) + \alpha_{13}u_3(t - \sigma)], \\ \frac{d^q u_2(t)}{dt^q} = u_2(t)[\gamma_2 - \alpha_{21}u_1(t - \sigma) - \alpha_{22}u_2(t - \sigma) + \alpha_{23}u_3(t - \sigma)], \\ \frac{d^q u_3(t)}{dt^q} = u_3(t)[\gamma_3 + \alpha_{31}u_1(t - \sigma) + \alpha_{32}u_2(t - \sigma) - \alpha_{33}u_3(t - \sigma)], \end{cases} \quad (3)$$

where $u_1(t)$ and $u_2(t)$ stand for the densities of competing population and $u_3(t)$ stands for the density of cooperating population, $\gamma_l, \alpha_{lk} \in CR, [0, \infty)$ ($l, k = 1, 2, 3$), the parameter $\sigma \geq 0$ is the feedback time delay of different species, and $q \in (0, 1]$ is a constant. For more concrete meaning of coefficients for system (3), see [26, 27].

The initial condition of system (3) takes the form

$$\begin{cases} u_1(\vartheta) = u_{1\vartheta}, & -\sigma \leq \vartheta \leq t_0, \\ u_2(\vartheta) = u_{2\vartheta}, & -\sigma \leq \vartheta \leq t_0, \\ u_3(\vartheta) = u_{3\vartheta}, & -\sigma \leq \vartheta \leq t_0, \end{cases} \quad (4)$$

where $t_0 > 0$ is a constant. The key object of this work focuses on existence, uniqueness, nonnegativity, stability, and bifurcation phenomenon of model (3). Different from the methodology in [26, 27], in this paper, we will mainly discuss the various dynamics by applying fractional-order differential equation theory. Due to the introduction of fractional order, various dynamical behaviors of the predator-prey model (3) are different from the integer-order case. We think that it is necessary to reveal the effect of time delay, parameters of system, and fractional order on dynamics such as the stability and Hopt bifurcation. Thus, this study has significance in theory and practice.

We plan the structure of this manuscript as follows. Section 2 gives some related knowledge about fractional-order dynamical systems. Section 3 discusses the existence,

uniqueness, nonnegativity, local stability, and Hopf bifurcation of model (3). Section 4 gives an example to support the effectiveness of the obtained key conclusions. Section 6 ends our work.

2. Basic Knowledge

In this part, we present several related definitions and lemmas on fractional-order dynamical systems that will be applied in the later proof. Let R_+ represent the set of all nonnegative real numbers.

Definition 1. (see [45]). Define Caputo fractional-order derivative as follows:

$$\mathcal{D}^q h(w) = \frac{1}{\Gamma(n-q)} \int_{w_0}^w \frac{h^{(l)}(s)}{(w-s)^{q-l+1}} ds, \quad (5)$$

where $h(w) \in ([w_0, \infty), R)$, $\Gamma(s) = \int_0^\infty w^{s-1} e^{-w} dw$, $w \geq w_0$, and $l \in Z^+$, $l-1 \leq q < l$.

The Laplace transform of Caputo fractional-order derivative is defined as follows:

$$\begin{aligned} \mathcal{L}\{\mathcal{D}^q h(t); s\} &= s^q \mathcal{H}(s) \\ &\quad - \sum_{j=0}^{m-1} s^{q-j-1} h^{(j)}(0), \quad m-1 \leq q < m \in Z^+, \end{aligned} \quad (6)$$

where $\mathcal{H}(s) = \mathcal{L}\{h(t)\}$. Especially, if $h^{(j)}(0) = 0$, $j = 1, 2, \dots, m$, then $\mathcal{L}\{\mathcal{D}^q h(t); s\} = s^q \mathcal{H}(s)$.

Definition 2 (see [46]). (v_{1*}, v_{2*}, v_{3*}) is called an equilibrium point of model (3) provided that the following system

$$\begin{cases} v_{1*}(\gamma_1 - \alpha_{11}v_{1*} - \alpha_{12}v_{2*} + \alpha_{13}v_{3*}) = 0, \\ v_{2*}(\gamma_2 - \alpha_{21}v_{1*} - \alpha_{22}v_{2*} + \alpha_{23}v_{3*}) = 0, \\ v_{3*}(\gamma_3 + \alpha_{31}v_{1*} + \alpha_{32}v_{2*} - \alpha_{33}v_{3*}) = 0, \end{cases} \quad (7)$$

holds.

Lemma 1 (see [47]). Given the following system

$$\frac{d^q v(t)}{dt^q} = h(t, v(t)), \quad (8)$$

$$v(t_0) = v_{t_0}, \quad t_0 > 0,$$

where $q \in (0, 1)$, $h: [t_0, \infty) \times \Lambda \rightarrow R_+$, $\Lambda \subset R_+$. If $h(t, v)$ satisfies the local Lipschitz condition with respect to v , then model (3) possesses a unique solution defined on $[t_0, \infty)$.

Lemma 2 (see [48]). Suppose that $q \in (0, 1)$, $h(t) \in C[\gamma, \delta]$ and $\mathcal{D}^q h(t) \in C[\gamma, \delta]$, where $\gamma, \delta \in R$. If $\mathcal{D}^q h(t) \geq 0$, $t \in [\gamma, \delta]$, then $h(t)$ is a nondecreasing function $\forall t \in [\gamma, \delta]$. If $\mathcal{D}^q h(t) \leq 0$, $t \in [\gamma, \delta]$, then $h(t)$ is a nonincreasing function $\forall t \in [\gamma, \delta]$.

Lemma 3 (see [49, 50]). For a given fractional-order model,

$$\frac{d^q v(t)}{dt^q} = g(t, v(t)), \quad v(0) = v_0, \quad (9)$$

where $q \in (0, 1]$ and $g(t, v(t)): R_+ \times R^m \rightarrow R^m$, the equilibrium point of model (9) is locally asymptotically stable if all eigenvalues ς of $(\partial g(t, v)/\partial v)$ evaluated near the equilibrium point satisfy $|\arg(\varsigma)| > ((q\pi)/2)$.

Lemma 4 (see [51]). For given m -dimensional fractional-order system,

$$\begin{cases} \frac{d^{q_1} \mathcal{V}_1(t)}{dt^{q_1}} = l_{11} \mathcal{V}_1(t - \sigma_{11}) + l_{12} \mathcal{V}_2(t - \sigma_{12}) + \dots + l_{1m} \mathcal{V}_m(t - \sigma_{1m}), \\ \frac{d^{q_2} \mathcal{V}_2(t)}{dt^{q_2}} = l_{21} \mathcal{V}_1(t - \sigma_{21}) + l_{22} \mathcal{V}_2(t - \sigma_{22}) + \dots + l_{2m} \mathcal{V}_m(t - \sigma_{2m}), \\ \vdots \\ \frac{d^{q_m} \mathcal{V}_m(t)}{dt^{q_m}} = l_{m1} \mathcal{V}_1(t - \sigma_{m1}) + l_{m2} \mathcal{V}_2(t - \sigma_{m2}) + \dots + l_{mm} \mathcal{V}_m(t - \sigma_{mm}), \end{cases} \quad (10)$$

where $0 < q_i < 1$ ($i = 1, 2, \dots, m$), the initial values $\mathcal{V}_i(t) = \varphi_i(t) \in C[-\max_{i,l} \sigma_{il}, 0]$, and $t \in [-\max_{i,l} \sigma_{il}, 0]$, $i, l = 1, 2, \dots, m$. Denote

$$\Delta(\chi) = \begin{bmatrix} \chi^{q_1} - l_{11}e^{-\chi\sigma_{11}} & -l_{12}e^{-\chi\sigma_{12}} & \dots & -l_{1m}e^{-\chi\sigma_{1m}} \\ -l_{21}e^{-\chi\sigma_{12}} & \chi^{q_2} - l_{22}e^{-\chi\sigma_{22}} & \dots & -l_{2m}e^{-\chi\sigma_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{m1}e^{-\chi\sigma_{m1}} & -l_{m2}e^{-\chi\sigma_{m2}} & \dots & \chi^{q_m} - l_{mm}e^{-\chi\sigma_{mm}} \end{bmatrix}, \quad (11)$$

and then, the zero solution of equation (10) is Lyapunov asymptotically stable if all roots of $\det(\Delta(\chi)) = 0$ possess negative real parts.

3. Main Results

3.1. Existence and Uniqueness

Theorem 1. Let $\Theta = \{(u_1, u_2, u_3) \in \mathbb{R}^3: \max\{|u_1|, |u_2|, |u_3|\} < \mathcal{Q}\}$, where $\mathcal{Q}$ is a positive constant. Then, for every $U_{t_0} = (u_{1t_0}, u_{2t_0}, u_{3t_0}) \in \Theta$ and for every $t \geq t_0$, there exists a unique solution $U(t) \in \Theta$ of system (3) with initial value U_{t_0} .

Proof. We discuss this issue for model (3) in $\Theta \times [t_0, t^*]$, where $t^* < +\infty$. Let $U = (u_1, u_2, u_3)$ and $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, and define the following mapping: $L(U) = (L_1(U), L_2(U), L_3(U))$, where

$$\begin{cases} L_1(U) = u_1(t) [\gamma_1 - \alpha_{11}u_1(t - \sigma) - \alpha_{12}u_2(t - \sigma) + \alpha_{13}u_3(t - \sigma)], \\ L_2(U) = u_2(t) [\gamma_2 - \alpha_{21}u_1(t - \sigma) - \alpha_{22}u_2(t - \sigma) + \alpha_{23}u_3(t - \sigma)], \\ L_3(U) = u_3(t) [\gamma_3 + \alpha_{31}u_1(t - \sigma) + \alpha_{32}u_2(t - \sigma) - \alpha_{33}u_3(t - \sigma)]. \end{cases} \quad (12)$$

For arbitrary $U, \tilde{U} \in \Theta, t_1, t_2 \in \mathbb{R}, t_1, t_2 \geq t_0$, one has

$$\begin{aligned} & \|L(U) - L(\tilde{U})\| \\ &= |L_1(U) - L_1(\tilde{U})| + |L_2(U) - L_2(\tilde{U})| + |L_3(U) - L_3(\tilde{U})| \\ &= |u_1(t_1) [\gamma_1 - \alpha_{11}u_1(t_1 - \sigma) - \alpha_{12}u_2(t_1 - \sigma) + \alpha_{13}u_3(t_1 - \sigma)] \\ &\quad - u_1(t_2) [\gamma_1 - \alpha_{11}u_1(t_2 - \sigma) - \alpha_{12}u_2(t_2 - \sigma) + \alpha_{13}u_3(t_2 - \sigma)]| \\ &\quad + |u_2(t_1) [\gamma_2 - \alpha_{21}u_1(t_1 - \sigma) - \alpha_{22}u_2(t_1 - \sigma) + \alpha_{23}u_3(t_1 - \sigma)] \\ &\quad - u_2(t_2) [\gamma_2 - \alpha_{21}u_1(t_2 - \sigma) - \alpha_{22}u_2(t_2 - \sigma) + \alpha_{23}u_3(t_2 - \sigma)]| \\ &\quad + |u_3(t_1) [\gamma_3 + \alpha_{31}u_1(t_1 - \sigma) + \alpha_{32}u_2(t_1 - \sigma) - \alpha_{33}u_3(t_1 - \sigma)] \\ &\quad - u_3(t_2) [\gamma_3 + \alpha_{31}u_1(t_2 - \sigma) + \alpha_{32}u_2(t_2 - \sigma) - \alpha_{33}u_3(t_2 - \sigma)]| \\ &\leq |u_1(t_1) - u_1(t_2)| |\gamma_1 - \alpha_{11}u_1(t_1 - \sigma) - \alpha_{12}u_2(t_1 - \sigma) + \alpha_{13}u_3(t_1 - \sigma)| \\ &\quad + \max\{|u_1(t_1)|, |u_1(t_2)|\} |[\gamma_1 - \alpha_{11}u_1(t_1 - \sigma) - \alpha_{12}u_2(t_1 - \sigma) + \alpha_{13}u_3(t_1 - \sigma)] \\ &\quad - [\gamma_1 - \alpha_{11}u_1(t_2 - \sigma) - \alpha_{12}u_2(t_2 - \sigma) + \alpha_{13}u_3(t_2 - \sigma)]| \\ &\quad + |u_2(t_1) - u_2(t_2)| |\gamma_2 - \alpha_{21}u_1(t_1 - \sigma) - \alpha_{22}u_2(t_1 - \sigma) + \alpha_{23}u_3(t_1 - \sigma)| \\ &\quad + \max\{|u_2(t_1)|, |u_2(t_2)|\} |[\gamma_2 - \alpha_{21}u_1(t_1 - \sigma) - \alpha_{22}u_2(t_1 - \sigma) + \alpha_{23}u_3(t_1 - \sigma)] \\ &\quad - [\gamma_2 - \alpha_{21}u_1(t_2 - \sigma) - \alpha_{22}u_2(t_2 - \sigma) + \alpha_{23}u_3(t_2 - \sigma)]| \\ &\quad + |u_3(t_1) - u_3(t_2)| |\gamma_3 + \alpha_{31}u_1(t_1 - \sigma) + \alpha_{32}u_2(t_1 - \sigma) - \alpha_{33}u_3(t_1 - \sigma)| \\ &\quad + \max\{|u_3(t_1)|, |u_3(t_2)|\} |[\gamma_3 + \alpha_{31}u_1(t_1 - \sigma) + \alpha_{32}u_2(t_1 - \sigma) - \alpha_{33}u_3(t_1 - \sigma)] \\ &\quad - [\gamma_3 + \alpha_{31}u_1(t_2 - \sigma) + \alpha_{32}u_2(t_2 - \sigma) - \alpha_{33}u_3(t_2 - \sigma)]| \\ &\leq |u_1(t_1) - u_1(t_2)| (\gamma_1 + \alpha_{11}\mathcal{Q} + \alpha_{12}\mathcal{Q} + \alpha_{13}\mathcal{Q}) \\ &\quad + \mathcal{Q}\alpha_{11}|u_1(t_1 - \sigma) - u_1(t_2 - \sigma)| + \mathcal{Q}\alpha_{12}|u_2(t_1 - \sigma) - u_2(t_2 - \sigma)| \\ &\quad + \mathcal{Q}\alpha_{13}|u_3(t_1 - \sigma) - u_3(t_2 - \sigma)| \\ &\quad + |u_2(t_1) - u_2(t_2)| (\gamma_2 + \alpha_{21}\mathcal{Q} + \alpha_{22}\mathcal{Q} + \alpha_{23}\mathcal{Q}) \\ &\quad + \mathcal{Q}\alpha_{21}|u_1(t_1 - \sigma) - u_1(t_2 - \sigma)| + \mathcal{Q}\alpha_{22}|u_2(t_1 - \sigma) - u_2(t_2 - \sigma)| \\ &\quad + \mathcal{Q}\alpha_{23}|u_3(t_1 - \sigma) - u_3(t_2 - \sigma)| \\ &\quad + |u_3(t_1) - u_3(t_2)| (\gamma_3 + \alpha_{31}\mathcal{Q} + \alpha_{32}\mathcal{Q} + \alpha_{33}\mathcal{Q}) \\ &\quad + \mathcal{Q}\alpha_{31}|u_1(t_1 - \sigma) - u_1(t_2 - \sigma)| + \mathcal{Q}\alpha_{32}|u_2(t_1 - \sigma) - u_2(t_2 - \sigma)| \\ &\quad + \mathcal{Q}\alpha_{33}|u_3(t_1 - \sigma) - u_3(t_2 - \sigma)| \\ &\leq L_1|u_1(t_1) - u_1(t_2)| + L_2|u_2(t_1) - u_2(t_2)| + L_3|u_3(t_1) - u_3(t_2)|, \end{aligned} \quad (13)$$

where

$$\begin{cases} L_1 = \gamma_1 + \mathcal{Q}(2\alpha_{11} + \alpha_{21} + \alpha_{31} + \alpha_{12} + \alpha_{13}), \\ L_2 = \gamma_2 + \mathcal{Q}(2\alpha_{21} + \alpha_{22} + \alpha_{32} + \alpha_{22} + \alpha_{23}), \\ L_3 = \gamma_3 + \mathcal{Q}(2\alpha_{31} + \alpha_{23} + \alpha_{33} + \alpha_{32} + \alpha_{33}). \end{cases} \quad (14)$$

It follows from (13) that

$$\|L(U) - L(\tilde{U})\| \leq LU - \tilde{U}, \quad (15)$$

where

$$L = \max\{L_1, L_2, L_3\}. \quad (16)$$

In view of Lemma 1, one can conclude that Theorem 1 is true. We finish the proof. $\square$

3.2. Nonnegativity. Denote $\Theta_+ = \{(u_1, u_2, u_3) \in \Theta: u_1, u_2, u_3 \in R_+\}$.

Theorem 2. Each solution of model (3) that begins with Θ_+ remains nonnegative.

Proof. Our object is to prove that $u_i(t) \geq 0 (i = 1, 2, 3) \forall t \geq t_0$. Let $U(t_0) = (u_1(t_0), u_1(t_0), u_1(t_0)) \in \Theta_+$ be the initial value of system (3). Assume that $\exists$ is a constant and $t_0 \leq t < \rho$ such that

$$\begin{cases} u_i(t) = 0, & t_0 \leq t < \rho, \\ u_i(\rho) = 0, \\ u_i(\rho^+) < 0, \end{cases} \quad (17)$$

where $i = 1, 2, 3$. In view of system (3), one has

$$\begin{cases} \frac{d^e u_1(t)}{dt^e} \Big|_{u_i=0} = 0, \\ \frac{d^e u_2(t)}{dt^e} \Big|_{u_i=0} = 0, \\ \frac{d^e u_3(t)}{dt^e} \Big|_{u_i=0} = 0, \end{cases} \quad (18)$$

where $i = 1, 2, 3$. By Lemma 2, one knows that $u_i(\rho^+) = 0 (i = 1, 2, 3)$, which contradicts $u_i(\rho^+) < 0 (i = 1, 2, 3)$ in (17). Thus, one can conclude that $u_i(t) \geq 0 \forall t \geq t_0$. The proof is complete. $\square$

3.3. Stability and Hopf Bifurcation. In the section, we shall focus on the local stability and the appearance of Hopf bifurcation of system (3). Consider the biological implication of system (3); we only seek the sufficient condition to guarantee the local stability of the positive equilibrium and the emergence of Hopf bifurcations of system (3).

Clearly, system (3) has a unique positive equilibrium $U_0(u_1^*, u_2^*, u_3^*)$ provided that the following condition

$$(\mathcal{M}_1) \text{sign}\{\mathcal{A}\} = \text{sign}\{\mathcal{A}_1\} = \text{sign}\{\mathcal{A}_2\} = \text{sign}\{\mathcal{A}_3\}, \quad (19)$$

is satisfied, where

$$\begin{cases} \mathcal{A} = \det \begin{bmatrix} \alpha_{11}\alpha_{12}-\alpha_{13} & \alpha_{21}\alpha_{22}-\alpha_{23} & \alpha_{31}\alpha_{31}-\alpha_{33} \end{bmatrix} \neq 0, \\ \mathcal{A}_1 = \det \begin{bmatrix} \gamma_1 & \alpha_{12} & -\alpha_{13} \\ \gamma_2 & \alpha_{22} & -\alpha_{23} \\ -\gamma_3 & \alpha_{31} & -\alpha_{33} \end{bmatrix}, \\ \mathcal{A}_2 = \det \begin{bmatrix} \alpha_{11} & \gamma_1 & -\alpha_{13} \\ \alpha_{21} & \gamma_2 & -\alpha_{23} \\ \alpha_{31} & -\gamma_3 & -\alpha_{33} \end{bmatrix}, \\ \mathcal{A}_3 = \det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \gamma_1 \\ \alpha_{21} & \alpha_{22} & \gamma_2 \\ \alpha_{31} & \alpha_{31} & -\gamma_3 \end{bmatrix}, \end{cases} \quad (20)$$

$$u_1^* = \frac{\mathcal{A}_1}{\mathcal{A}}, \quad (20)$$

$$u_2^* = \frac{\mathcal{A}_2}{\mathcal{A}},$$

$$u_3^* = \frac{\mathcal{A}_3}{\mathcal{A}}.$$

The linear system of equation (3) near $U_0(u_1^*, u_2^*, u_3^*)$ takes

$$\begin{cases} \frac{d^e u_1(t)}{dt^e} = a_{11}u_1(t - \sigma) + a_{12}u_2(t - \sigma) + a_{13}u_3(t - \sigma), \\ \frac{d^e u_2(t)}{dt^e} = a_{21}u_1(t - \sigma) + a_{22}u_2(t - \sigma) + a_{23}u_3(t - \sigma), \\ \frac{d^e u_3(t)}{dt^e} = a_{31}u_1(t - \sigma) + a_{32}u_2(t - \sigma) + a_{33}u_3(t - \sigma), \end{cases} \quad (21)$$

where

$$\begin{cases} a_{11} = -u_{1*}\alpha_{11}, a_{12} = -u_{1*}\alpha_{12}, a_{13} = u_{1*}\alpha_{13}, \\ a_{21} = -u_{2*}\alpha_{21}, a_{22} = -u_{2*}\alpha_{22}, a_{23} = u_{2*}\alpha_{23}, \\ a_{31} = u_{3*}\alpha_{31}, a_{32} = u_{3*}\alpha_{32}, a_{33} = -u_{3*}\alpha_{33}. \end{cases} \quad (22)$$

The characteristic equation of equation (21) is computed as follows:

$$\det \begin{bmatrix} s^0 - a_{11}e^{-s\sigma} & -a_{12}e^{-s\sigma} & -a_{13}e^{-s\sigma} \\ -a_{21}e^{-s\sigma} & s^0 - a_{22}e^{-s\sigma} & -a_{23}e^{-s\sigma} \\ -a_{31}e^{-s\sigma} & -a_{32}e^{-s\sigma} & s^0 - a_{33}e^{-s\sigma} \end{bmatrix} = 0, \quad (23)$$

which leads to

$$s^{3\varrho} + b_1 s^{2\varrho} e^{-s\sigma} + b_2 s^{\varrho} e^{-2s\sigma} + b_3 e^{-3s\sigma} = 0, \quad (24)$$

where

$$\begin{cases} b_1 = -(a_{11} + a_{22} + a_{33}), \\ b_2 = a_{11}a_{33} + a_{22}a_{33} + a_{11}a_{22} - a_{13}a_{31} \\ \quad - a_{12}a_{21} - a_{23}a_{32}, \\ b_3 = a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} \\ \quad - a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}. \end{cases} \quad (25)$$

In terms of (24), we have

$$s^{3\varrho} e^{2s\sigma} + b_1 s^{2\varrho} e^{s\sigma} + b_2 s^{\varrho} + b_3 e^{-s\sigma} = 0. \quad (26)$$

Let $s = i\varphi = \varphi(\cos(\pi/2) + i \sin(\pi/2))$ be the root of (26); then, one has

$$\begin{cases} \mathcal{G}_1(\varrho) \cos 2\varphi\sigma + \mathcal{G}_2(\varrho) \sin 2\varphi\sigma + \mathcal{G}_3(\varrho) \cos \varphi\sigma + \mathcal{G}_4(\varrho) \sin \varphi\sigma = 0, \\ \mathcal{S}_1(\varrho) \cos 2\varphi\sigma + \mathcal{S}_2(\varrho) \sin 2\varphi\sigma + \mathcal{S}_3(\varrho) \cos \varphi\sigma + \mathcal{S}_4(\varrho) \sin \varphi\sigma = 0, \end{cases} \quad (27)$$

where

$$\begin{cases} \mathcal{G}_1(\varrho) = \varphi^{3\varrho} \cos \frac{3\varrho\pi}{2} + b_2 \varphi^{\varrho} \cos \frac{\varrho\pi}{2}, \\ \mathcal{G}_2(\varrho) = b_2 \varphi^{\varrho} \sin \frac{\varrho\pi}{2} - \varphi^{3\varrho} \sin \frac{3\varrho\pi}{2}, \\ \mathcal{G}_3(\varrho) = b_1 \varphi^{2\varrho} \cos \varrho\pi + b_3, \\ \mathcal{G}_4(\varrho) = -b_1 \varphi^{2\varrho} \sin \varrho\pi, \\ \mathcal{S}_1(\varrho) = \varphi^{3\varrho} \sin \frac{3\varrho\pi}{2} + b_2 \varphi^{\varrho} \sin \frac{\varrho\pi}{2}, \\ \mathcal{S}_2(\varrho) = \varphi^{3\varrho} \cos \frac{3\varrho\pi}{2} - b_2 \varphi^{\varrho} \cos \frac{\varrho\pi}{2}, \\ \mathcal{S}_3(\varrho) = b_1 \varphi^{2\varrho} \sin \varrho\pi, \\ \mathcal{S}_4(\varrho) = b_1 \varphi^{2\varrho} \cos \varrho\pi - b_3. \end{cases} \quad (28)$$

In (28), let

$$\begin{cases} c_1 = \cos \frac{3\varrho\pi}{2}, \\ c_2 = b_2 \cos \frac{\varrho\pi}{2}, \\ c_3 = b_2 \sin \frac{\varrho\pi}{2}, \\ c_4 = -\sin \frac{3\varrho\pi}{2}, \\ c_5 = b_1 \cos \varrho\pi, \\ c_6 = b_3, \\ c_7 = -b_1 \sin \varrho\pi, \\ c_8 = \sin \frac{3\varrho\pi}{2}, \\ c_9 = b_2 \sin \frac{\varrho\pi}{2}, \\ c_{10} = \cos \frac{3\varrho\pi}{2}, \\ c_{11} = -b_2 \cos \frac{\varrho\pi}{2}, \\ c_{12} = b_1 \sin \varrho\pi, \\ c_{13} = b_1 \cos \varrho\pi, \\ c_{14} = -b_3, \end{cases} \quad (29)$$

and then, (28) takes the form:

$$\begin{cases} \mathcal{G}_1(\varrho) = c_1 \varphi^{3\varrho} + c_2 \varphi^{\varrho}, \\ \mathcal{G}_2(\varrho) = c_3 \varphi^{\varrho} + c_4 \varphi^{3\varrho}, \\ \mathcal{G}_3(\varrho) = c_5 \varphi^{2\varrho} + c_6, \\ \mathcal{G}_4(\varrho) = c_7 \varphi^{2\varrho}, \\ \mathcal{S}_1(\varrho) = c_8 \varphi^{3\varrho} + c_9 \varphi^{\varrho}, \\ \mathcal{S}_2(\varrho) = c_{10} \varphi^{3\varrho} + c_{11} \varphi^{\varrho}, \\ \mathcal{S}_3(\varrho) = c_{12} \varphi^{2\varrho}, \\ \mathcal{S}_4(\varrho) = c_{13} \varphi^{2\varrho} + c_{14}. \end{cases} \quad (30)$$

For (27), we consider two cases.

Case 1. If $\sin \varphi\sigma = \sqrt{1 - \cos^2 \varphi\sigma}$, then the first equation of (27) becomes

$$\begin{aligned} & \mathcal{G}_1(\varrho)(2 \cos^2 \varphi\sigma - 1) + 2\mathcal{G}_2(\varrho)\cos \varphi\sigma\sqrt{1 - \cos^2 \varphi\sigma} \\ & + \mathcal{G}_3(\varrho)\cos \varphi\sigma + \mathcal{G}_4(\varrho)\sqrt{1 - \cos^2 \varphi\sigma} = 0. \end{aligned} \quad (31)$$

Hence,

$$\begin{aligned} & \left[2\mathcal{G}_2(\varrho)\cos \varphi\sigma\sqrt{1 - \cos^2 \varphi\sigma} + \mathcal{G}_4(\varrho)\sqrt{1 - \cos^2 \varphi\sigma} \right]^2 \\ & = \left[\mathcal{G}_1(\varrho)(2 \cos^2 \varphi\sigma - 1) + \mathcal{G}_3(\varrho)\cos \varphi\sigma \right]^2, \end{aligned} \quad (32)$$

which leads to

$$\vartheta_1 \cos^4 \varphi\sigma + \vartheta_2 \cos^3 \varphi\sigma + \vartheta_3 \cos^2 \varphi\sigma + \vartheta_4 \cos \varphi\sigma + \vartheta_5 = 0, \quad (33)$$

where

$$\begin{cases} \vartheta_1 = 4\mathcal{G}_1^2(\varrho) + 4\mathcal{G}_2^2(\varrho), \\ \vartheta_2 = 4\mathcal{G}_1(\varrho)\mathcal{G}_3(\varrho) + 4\mathcal{G}_2(\varrho)\mathcal{G}_4(\varrho), \\ \vartheta_3 = \mathcal{G}_3^2(\varrho) + \mathcal{G}_4^2(\varrho) - 4\mathcal{G}_1^2(\varrho) - 4\mathcal{G}_2^2(\varrho), \\ \vartheta_4 = -[2\mathcal{G}_1(\varrho)\mathcal{G}_3(\varrho) + 4\mathcal{G}_2(\varrho)\mathcal{G}_4(\varrho)], \\ \vartheta_5 = \mathcal{G}_1^2(\varrho) - \mathcal{G}_4^2(\varrho). \end{cases} \quad (34)$$

Set $\cos \varphi\sigma = r$, and let

$$z(r) = r^4 + \frac{\vartheta_2}{\vartheta_1}r^3 + \frac{\vartheta_3}{\vartheta_1}r^2 + \frac{\vartheta_4}{\vartheta_1}r + \frac{\vartheta_5}{\vartheta_1}. \quad (35)$$

Hence,

$$\frac{dz(r)}{dr} = 4r^3 + \frac{3\vartheta_2}{\vartheta_1}r^2 + \frac{2\vartheta_3}{\vartheta_1}r + \frac{\vartheta_4}{\vartheta_1}. \quad (36)$$

Denote

$$r^3 + \frac{3\vartheta_2}{\vartheta_1}r^2 + \frac{2\vartheta_3}{\vartheta_1}r + \frac{\vartheta_4}{\vartheta_1} = 0. \quad (37)$$

Set $p = r + (\vartheta_2/4\vartheta_1)$, and then, (37) takes the form

$$p^3 + f_1p + f_2 = 0, \quad (38)$$

where

$$\begin{aligned} f_1 &= \frac{\vartheta_3}{2\vartheta_1} - \frac{3\vartheta_2^2}{16\vartheta_1^2}, \\ f_2 &= \frac{\vartheta_2^3}{32\vartheta_1^3} - \frac{\vartheta_2\vartheta_3}{8\vartheta_1^2} + \frac{\vartheta_4}{4\vartheta_1}. \end{aligned} \quad (39)$$

Let

$$\begin{aligned} q_1 &= \left(\frac{f_2}{2}\right)^2 + \left(\frac{f_1}{3}\right)^3, \\ q_2 &= \frac{-1 + i\sqrt{3}}{2}. \end{aligned} \quad (40)$$

It follows from (38) that

$$\begin{cases} p_1 = \sqrt[3]{\frac{f_2}{2} + \sqrt{q_1}} + \sqrt[3]{\frac{f_2}{2} - \sqrt{q_1}}, \\ p_2 = \sqrt[3]{\frac{f_2}{2} + \sqrt{q_1}q_2} + \sqrt[3]{\frac{f_2}{2} - \sqrt{q_1}q_2}, \\ p_3 = \sqrt[3]{\frac{f_2}{2} + \sqrt{q_1}q_2^2} + \sqrt[3]{\frac{f_2}{2} - \sqrt{q_1}q_2^2}. \end{cases} \quad (41)$$

According to the discussion above, one easily obtains the expression of $\cos \varphi\sigma$. Then, one easily gets the expression of $\sin \varphi\sigma$. Here, we suppose that

$$\begin{aligned} \cos \varphi\sigma &= g_1(\varphi), \\ \sin \varphi\sigma &= g_2(\varphi). \end{aligned} \quad (42)$$

Then,

$$g_1^2(\varphi) + g_2^2(\varphi) = 1. \quad (43)$$

With the aid of Matlab 7.0, one can get the root (say φ) of equation (43). Then, one obtains

$$\sigma^{1k} = \frac{1}{\varphi} [\arccos g_1(\varphi) + 2k\pi], \quad k = 0, 1, 2, \dots \quad (44)$$

Case 2. If $\sin \varphi\sigma = -\sqrt{1 - \cos^2 \varphi\sigma}$, applying the same way to this case, we can obtain

$$\begin{aligned} \cos \varphi\sigma &= h_1(\varphi), \\ \sin \varphi\sigma &= h_2(\varphi). \end{aligned} \quad (45)$$

Then,

$$h_1^2(\varphi) + h_2^2(\varphi) = 1. \quad (46)$$

By Matlab 7.0, one can obtain the root (say φ) of equation (46). So,

$$\sigma^{2k} = \frac{1}{\varphi} [\arccos h_1(\varphi) + 2k\pi], \quad k = 0, 1, 2, \dots \quad (47)$$

Set

$$\sigma_0 = \min\{\sigma^{1k}, \sigma^{2k}\}, \quad k = 0, 1, 2, \dots, \quad (48)$$

In the sequel, we check the transversality condition of the appearance of Hopf bifurcation. We give the hypothesis as follows: $(\mathcal{M}_2)\mathcal{F}_{11}\mathcal{F}_{21} + \mathcal{F}_{12}\mathcal{F}_{22} > 0$, where

$$\begin{aligned}
& \left[\begin{aligned}
& \mathcal{E}_{11} = 3\varrho\varphi_0^{3\varrho-1} \left[\cos \frac{(3\varrho-1)\pi}{2} \cos 2\varphi_0\sigma_0 - \sin \frac{(3\varrho-1)\pi}{2} \sin 2\varphi_0\sigma_0 \right] \\
& + 2\varphi_0^{3\varrho}\sigma_0 \left[\cos \frac{3\varrho\pi}{2} \cos 2\varphi_0\sigma_0 - \sin \frac{3\varrho\pi}{2} \sin 2\varphi_0\sigma_0 \right] \\
& + 2b_1\varrho\varphi_0^{2\varrho-1} \left[\cos \frac{(2\varrho-1)\pi}{2} \cos \varphi_0\sigma_0 - \sin \frac{(2\varrho-1)\pi}{2} \sin \varphi_0\sigma_0 \right] \\
& + b_1\sigma_0\varphi_0^{2\varrho} [\cos \varrho\pi \cos \varphi_0\sigma_0 - \sin \varrho\pi \sin \varphi_0\sigma_0] \\
& + b_2\varrho\varphi_0^{\varrho-1} \cos \frac{(\varrho-1)\pi}{2} - b_3\sigma_0 \cos \varphi_0\sigma_0, \\
& \mathcal{E}_{12} = 3\varrho\varphi_0^{3\varrho-1} \left[\cos \frac{(3\varrho-1)\pi}{2} \sin 2\varphi_0\sigma_0 + \sin \frac{(3\varrho-1)\pi}{2} \cos 2\varphi_0\sigma_0 \right] \\
& + 2\varphi_0^{3\varrho}\sigma_0 \left[\cos \frac{3\varrho\pi}{2} \sin 2\varphi_0\sigma_0 + \sin \frac{3\varrho\pi}{2} \cos 2\varphi_0\sigma_0 \right] \\
& + 2b_1\varrho\varphi_0^{2\varrho-1} \left[\cos \frac{(2\varrho-1)\pi}{2} \sin \varphi_0\sigma_0 + \sin \frac{(2\varrho-1)\pi}{2} \cos \varphi_0\sigma_0 \right] \\
& + b_1\sigma_0\varphi_0^{2\varrho} [\cos \varrho\pi \sin \varphi_0\sigma_0 + \sin \varrho\pi \cos \varphi_0\sigma_0] \\
& + b_2\varrho\varphi_0^{\varrho-1} \sin \frac{(\varrho-1)\pi}{2} + b_3\sigma_0 \sin \varphi_0\sigma_0, \\
& \mathcal{E}_{21} = b_3\varphi_0 \sin \varphi_0\sigma_0 - 2\varphi_0^{3\varrho+1} \left[\cos \frac{(3\varrho+1)\pi}{2} \cos 2\varphi_0\sigma_0 - \sin \frac{(3\varrho+1)\pi}{2} \sin 2\varphi_0\sigma_0 \right] \\
& - 2 \left(b_2\varrho_0^{2\sigma} \sin \sigma\pi + b_1\varrho_0^\sigma \sin \frac{\sigma\pi}{2} \right) \varrho_0 \cos 2\varrho_0\eta_0, \\
& - b_1\varphi_0^{2\varrho+1} \left[\cos \frac{(2\varrho+1)\pi}{2} \cos 2\varphi_0\sigma_0 - \sin \frac{(2\varrho+1)\pi}{2} \sin 2\varphi_0\sigma_0 \right], \\
& \mathcal{E}_{22} = b_3\varphi_0 \cos \varphi_0\sigma_0 - 2\varphi_0^{3\varrho+1} \left[\cos \frac{(3\varrho+1)\pi}{2} \sin 2\varphi_0\sigma_0 + \sin \frac{(3\varrho+1)\pi}{2} \cos 2\varphi_0\sigma_0 \right] \\
& - b_1\varphi_0^{2\varrho+1} \left[\cos \frac{(2\varrho+1)\pi}{2} \sin 2\varphi_0\sigma_0 + \sin \frac{(2\varrho+1)\pi}{2} \cos 2\varphi_0\sigma_0 \right].
\end{aligned} \right. \tag{49}
\end{aligned}$$

Lemma 5. Suppose that $s(\sigma) = \xi(\sigma) + i\eta(\sigma)$ is a root of equation (26) near $\sigma = \sigma_0$ such that $\xi(\sigma_0) = 0, \eta(\sigma_0) = \varphi_0$; then, one has $\text{Re}[ds/d\sigma]|_{\sigma=\sigma_0, \varphi=\varphi_0} > 0$.

Proof. In terms of equation (26), one knows that

$$\begin{aligned} & 3\varrho s^{3\varrho-1} e^{2s\sigma} \frac{ds}{d\sigma} + 2s^{3\varrho} e^{2s\sigma} \left(\frac{ds}{d\sigma} \sigma + s \right) \\ & + 2b_1 \varrho s^{2\varrho-1} e^{s\sigma} \frac{ds}{d\sigma} + b_1 e^{s\sigma} s^{2\varrho} \left(\frac{ds}{d\sigma} \sigma + s \right) + b_2 \varrho s^{\varrho-1} \frac{ds}{d\sigma} - b_3 e^{-s\sigma} \left(\frac{ds}{d\sigma} \sigma + s \right) = 0, \end{aligned} \quad (50)$$

which leads to

$$\begin{aligned} & \left[3\varrho s^{3\varrho-1} e^{2s\sigma} + 2s^{3\varrho} e^{2s\sigma} \sigma + 2b_1 \varrho s^{2\varrho-1} e^{s\sigma} \right. \\ & \left. + b_1 e^{s\sigma} s^{2\varrho} \sigma + b_2 \varrho s^{\varrho-1} - b_3 e^{-s\sigma} \sigma \right] \frac{ds}{d\sigma} \\ & = b_3 s e^{-s\sigma} - 2s^{3\varrho+1} e^{2s\sigma} - b_1 s^{2\varrho+1} e^{s\sigma}. \end{aligned} \quad (51)$$

Then,

$$\left[\frac{ds}{d\sigma} \right]^{-1} = \frac{\mathcal{F}_1(s)}{\mathcal{F}_2(s)} - \frac{\sigma}{s}, \quad (52)$$

where

$$\begin{cases} \mathcal{F}_1(s) = 3\varrho s^{3\varrho-1} e^{2s\sigma} + 2s^{3\varrho} e^{2s\sigma} \sigma + 2b_1 \varrho s^{2\varrho-1} e^{s\sigma} \\ \quad + b_1 e^{s\sigma} s^{2\varrho} \sigma + b_2 \varrho s^{\varrho-1} - b_3 e^{-s\sigma} \sigma, \\ \mathcal{F}_2(s) = b_3 s e^{-s\sigma} - 2s^{3\varrho+1} e^{2s\sigma} - b_1 s^{2\varrho+1} e^{s\sigma}. \end{cases} \quad (53)$$

So,

$$\text{Re} \left\{ \frac{ds}{d\sigma} \right\} \Big|_{\sigma=\sigma_0, \varphi=\varphi_0} = \text{Re} \left\{ \frac{\mathcal{Z}_1(s)}{\mathcal{Z}_2(s)} \right\} \Big|_{\sigma=\sigma_0, \varphi=\varphi_0} = \frac{\mathcal{Z}_{11} \mathcal{Z}_{21} + \mathcal{Z}_{12} \mathcal{Z}_{22}}{\mathcal{Z}_{21}^2 + \mathcal{Z}_{22}^2}. \quad (54)$$

By $(\mathcal{M}_2)$, one obtains

$$\text{Re} \left\{ \left[\frac{ds}{d\sigma} \right]^{-1} \right\} \Big|_{\sigma=\sigma_0, \varphi=\varphi_0} > 0. \quad (55)$$

This finishes the proof of Lemma 5.

Suppose that

$$\begin{aligned} & (\mathcal{M}_3) b_1 > 0, \\ & b_1 b_2 > b_3, \\ & b_3 > 0. \end{aligned} \quad (56)$$

□

Lemma 6. Provided that $\sigma = 0$ and $(\mathcal{M}_3)$ is fulfilled, then system (3) is locally asymptotically stable.

Proof. If $\sigma = 0$, then (24) takes the form

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0. \quad (57)$$

By $(\mathcal{M}_3)$, one knows that all roots λ_i of (57) satisfy $|\arg(\lambda_i)| > ((\varrho\pi)/2)$ ($i = 1, 2, 3, 4$). Thus, one knows that Lemma 6 is right. The proof finishes.

Based on the investigation above, we have the following conclusion. □

Theorem 3. Provided that $(\mathcal{M}_1)$ – $(\mathcal{M}_3)$ hold true, then $U_0(u_{1*}, u_{1*}, u_{3*})$ of system (3) is locally asymptotically stable if $\sigma \in [0, \sigma_0)$, and a Hopf bifurcation exists near $U_0(u_{1*}, u_{1*}, u_{3*})$ if $\sigma = \sigma_0$.

4. Computer Simulations

In this paper, we apply implicit Euler's scheme which is introduced in [52] to carry our numerical simulations. Given the following fractional-order predator-prey model

$$\begin{cases} \frac{d^\varrho u_1(t)}{dt^\varrho} = u_1(t) [8 - 3u_1(t - \sigma) - 2u_2(t - \sigma) + 0.3u_3(t - \sigma)], \\ \frac{d^\varrho u_2(t)}{dt^\varrho} = u_2(t) [7 - 2u_1(t - \sigma) - 2u_2(t - \sigma) + 0.3u_3(t - \sigma)], \\ \frac{d^\varrho u_3(t)}{dt^\varrho} = u_3(t) [6 + 0.2u_1(t - \sigma) + 0.2u_2(t - \sigma) - 3u_3(t - \sigma)], \end{cases} \quad (58)$$

we can easily know that model (58) possesses the equilibrium point $U_0(1.0000, 2.8384, 2.2559)$. Let $\varrho = 0.79$. Then, $r_1 = 8, \gamma_2 = 7, \gamma_3 = 6, \alpha_{11} = 3, \alpha_{12} = 2, \alpha_{13} = 0.3, \alpha_{21} = 2, \alpha_{22} = 2, \alpha_{23} = 0.3, \alpha_{31} = 0.2, \alpha_{32} = 0.2$, and $\alpha_{33} = 3$. Hence, $a_{11} = -3, a_{12} = -2, a_{13} = 0.3, a_{21} = -5.6788, a_{22} = -5.6788, a_{23} = 0.6768, a_{31} = 0.4512, a_{32} = 0.4512, a_{33} = -6.7677, \varphi_0 = 0.3127$, and $\sigma_0 = 0.192$. Then, $b_1 = 15.4465, b_2 = 113.7636, b_3 = 151.8836, \mathcal{Z}_{11} = 0.3742, \mathcal{Z}_{12} = 0.6425, \mathcal{Z}_{21} = 0.5004$, and $\mathcal{Z}_{22} = 0.7157$. Thus, the hypothesis $(\mathcal{M}_1)$ – $(\mathcal{M}_3)$ of

FIGURE 1: State variable $u_1(t)$ versus time t of system (58) when $\sigma = 0.18 < \sigma_0 = 0.192$.

FIGURE 2: . State variable $u_2(t)$ versus time t of system (58) when $\sigma = 0.18 < \sigma_0 = 0.192$.

Theorem 3 hold true. If $\sigma \in [0, 0.192)$, the positive equilibrium point $U_0(1.0000, 2.8384, 2.2559)$ of system (58) is locally asymptotically stable. For this case, we choose $\sigma = 0.18 < \sigma_0 = 0.192$. The simulation plots are displayed in Figures 1–3. Figures 1–3 reveal that if σ is less than $\sigma_0 = 0.192$, then the densities of competing species u_1 and u_2 of system (58) will be tardily close to 1.0000, 2.8384, respectively, and the density of

cooperating species u_3 of system (58) will be tardily close to 2.2559. When $\sigma \in [0.192, +\infty)$, then system (58) loses its stability and Hopf bifurcation behavior emerges. For this case, we choose $\sigma = 0.33$. The numerical simulation diagrams are presented in Figures 4–6. Figures 4–6 confirm that when σ is greater than $\sigma_0 = 0.192$, then the densities of competing species u_1 and u_2 and the density of cooperating species u_3 will remain

FIGURE 3: State variable $u_3(t)$ versus time t of system (58) when $\sigma = 0.18 < \sigma_0 = 0.192$.

FIGURE 4: State variable $u_1(t)$ versus time t when $\sigma = 0.33 > \sigma_0 = 0.192$.

periodic motion around the positive equilibrium point $U_0(1.0000, 2.8384, 2.2559)$, namely, Hopf bifurcation phenomenon takes place around $U_0(1.0000, 2.8384, 2.2559)$. In order to illustrate the bifurcation phenomenon intuitively, we

plot the bifurcation diagrams (see Figures 7–9). From Figures 7–9, we can easily see that the bifurcation value is 0.192. In addition, we give the relationship table between φ_0 and σ_0 in Table 1.

FIGURE 5: State variable $u_2(t)$ versus time t when $\sigma = 0.33 > \sigma_0 = 0.192$.

FIGURE 6: State variable $u_3(t)$ versus time t when $\sigma = 0.33 > \sigma_0 = 0.192$.

FIGURE 7: Bifurcation diagram for system (58): σ versus u_1 .

FIGURE 8: Bifurcation diagram for system (58): σ versus u_2 . FIGURE 9: Bifurcation diagram for system (58): σ versus u_3 .TABLE 1: The magnitude relation of φ_0 and σ_0 for system (58).

φ_0	σ_0
0.3127	0.1920
0.3499	0.2173
0.4957	0.3218
0.5200	0.3400
0.6880	0.4732
0.7832	0.5543
1.0108	0.7651
1.0483	0.8022
1.1556	0.9123

5. Conclusions

Based on the previous works on predator-prey models and noticing the effect of feedback delay between the predators and preys, we have established a new fractional-order

competitor-competitor-mutualist Lotka–Volterra model. We have found the conditions to ensure the existence, uniqueness, and nonnegativity of the involved fractional-order prey-predator model. Applying the Laplace transform, stability theorem, and Hopf bifurcation theory of fractional-order dynamical systems, we set up a novel sufficient criterion to guarantee the local stability and the emergence of Hopf bifurcation of the involved fractional-order predator-prey system. The study shows that the time delay has a vital effect on controlling the stability. To check the rationality of theoretical predictions, we design the program to implement simulation experiments. The established analytical conclusions have crucial guiding significance in preserving the coexistence of biological populations. In this paper, we assume that the feedback time delay of different species is same. Of course, we can also deal with the different delay cases. We will focus on this topic in near future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no competing interest.

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