Research Article

Ordered Variables and Their Concomitants under Extropy via COVID-19 Data Application

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Extropy, as a complementary dual of entropy, has been discussed in many works of literature, where it is declared for other measures as an extension of entropy. In this article, we obtain the extropy of generalized order statistics via its dual and give some examples from well-known distributions. Furthermore, we study the residual and past extropy for such models. On the other hand, based on Farlie–Gumbel–Morgenstern distribution, we consider the residual extropy of concomitants of m -generalized order statistics and present this measure with some additional features. In addition, we provide the upper bound and stochastic orders of it. Finally, nonparametric estimation of the residual extropy of concomitants of m -generalized order statistics is included using simulated and real data connected with COVID-19 virus.

1. Introduction

Shannon [1] introduced a well-known vintage measure of uncertainty called Shannon entropy. This information theoretic entropy manipulates in diverse fields such as financial analysis, computer science, and medical research. The extropy proposed by Lad et al. [2] is an accomplishment to notions of information based on entropy. They exhibited that entropy has a complementary dual function known as “extropy.” In the view of extropy in discrete density, the extropy measure $-\sum_{i=1}^N (1 - \theta_i) \log(1 - \theta_i)$ is neatly closer to $(-1/2) \sum_{i=1}^N \theta_i^2$ when the range of possibilities increases (as a result of larger N). Therefore, to realize extropy for a continuous density, the extropy of a nonnegative continuous random variable (r.v.) X , with probability density function (PDF) $f(x)$ is defined as

$$J(X) = \frac{-1}{2} \int_0^{\infty} f^2(x) dx. \quad (1)$$

The extropy measure has been developed for ordered variables. Qiu [3] was the first to apply extropy for order statistics and record values and present several of their properties. After that, the researchers manifested to present extension measures of extropy. Qiu and Jia [4] investigated the connotation of residual extropy of a nonnegative r.v. as

$$J^t(X) = J^R(X; t) = \frac{-1}{2\bar{F}^2(t)} \int_t^{\infty} f^2(x) dx, \quad t \geq 0. \quad (2)$$

Qiu et al. [5] presented a mixed systems lifetime via extropy and obtained some features and bounds of it. Recently, Jose and Sathar [6, 7] exploited the residual and past extropy of k -records, respectively, emerging from any continuous distribution. For extra studies on extropy, see Qiu and Jia [8], Yang et al. [9], Noughabi and Jarrahiferiz [10], Raqab and Qiu [11], and Lad et al. [12].

Krishnan et al. [13] presented the past extropy, for a fixed $t > 0$, for past lifetime of r.v. $X_t = [t - X | X \leq t]$ as follows:

$$J_t(X) = J_P(X; t) = \frac{-1}{2F^2(t)} \int_0^t f^2(x) dx. \quad (3)$$

Jahanshahi et al. [14] proposed cumulative residual entropy (CREX). For a nonnegative r.v. X with an absolutely continuous survival function $\bar{F}$, the CREX is given by

$$\zeta(X) = \frac{-1}{2} \int_0^\infty \bar{F}^2(x) dx. \quad (4)$$

In analogy with Jahanshahi et al. [14], Abdul Sathar and Dhanya [15] introduced the CREX and refer to it as survival entropy. Moreover, they conduct the dynamic survival entropy as

$$\zeta(X; t) = \frac{-1}{2\bar{F}^2(t)} \int_t^\infty \bar{F}^2(x) dx, \quad t \geq 0. \quad (5)$$

It is easy to observe that entropy and its related measures are constantly negative.

The connotation of generalized order statistics (gos) that contains all forms of ordered random observations was first proposed by Kamps [16]. Let $k \geq 1$, $n \in \mathbb{N}$, $\tilde{m} = (m_1, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$ be parameters such that $M_r = \sum_{j=r}^{n-1} m_j$, $\gamma_r = k + n - r + M_r \geq 1$ for all $1 \leq r \leq n-1$. For a subclass of gos (called m -gos), where $m_1 = m_2 = \dots = m_{n-1} = m$, the PDF of the r th m -gos, $X_{(r;n,k,m)}$, can be written as

$$f_{(r;n,k,m)}(x) = \frac{c_{r-1}}{(r-1)!} (1 - F(x))^{\gamma_r-1} f(x) g_m^{r-1}(F(x)), \quad (6)$$

where $c_{r-1} = \prod_{j=1}^r \gamma_j$, $\gamma_r = k + (n-r)(m+1)$, for $0 < z < 1$,

$$g_m(z) = \begin{cases} \frac{1 - (1-z)^{m+1}}{m+1}, & m \neq -1, \\ -\ln(1-z), & m = -1. \end{cases} \quad (7)$$

We will write $g_m(z) = (1 - (1-z)^{m+1})/(m+1)$, $0 < z < 1$ and for all m , since $\lim_{m \rightarrow -1} (1 - (1-z)^{m+1})/(m+1) = -\ln(1-z)$.

Based on descending ordered r.v.'s, Pawlas and Szynal [17] and Burkschat et al. [18] presented the dual generalized order statistics (dgos). By the same manner and parameters in m -gos, when $m_1 = m_2 = \dots = m_{n-1} = m$, the PDF of m -dgos $X_{d(r;n,k,m)}$ is defined by

$$f_{d(r;n,k,m)}(x) = \frac{c_{r-1}}{(r-1)!} \left(\frac{1}{m+1} \right)^{r-1} (F(x))^{\gamma_r-1} (1 - F^{m+1}(x))^{r-1} f(x). \quad (8)$$

The concept of concomitants of ordinary order statistic was derived by David et al. [19]. Let (X_i, Y_i) , $i = 1, 2, \dots, n$, be n pairs of independent r.v.'s drawn from some bivariate distributions with cumulative distribution function (CDF) $F(x, y)$. Let $X_{(r;n)}$ be the r th order statistic, then the r.v. Y concerned with $X_{(r;n)}$ is called the concomitant of r th order statistics and is specified by $Y_{[r;n]}$.

The Farlie–Gumbel–Morgenstern (FGM) family is an extremely supple class of bivariate family; it was primarily derived by Morgenstern [20], which is set by CDF and PDF, respectively, as follows:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))], \quad (9)$$

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)[1 + \alpha(2F_X(x) - 1)(2F_Y(y) - 1)], \quad (10)$$

where $F_X(x)$, $F_Y(y)$ and $f_X(x)$, $f_Y(y)$ are the marginal CDF's and PDF's of X and Y , respectively, $-1 \leq \alpha \leq 1$. If the dependent parameter $\alpha = 0$, then X and Y are not dependent. Beg and Ahsanullah [21] introduced the PDF of the concomitant of m -gos $Y_{[r;n,k,m]}$, $1 \leq r \leq n$, under the FGM family as follows:

$$g_{[r;n,k,m]}(y) = f_Y(y)[1 + \alpha T^*(r; n, k, m)(2F_Y(y) - 1)], \quad (11)$$

where

$$T^*(r; n, k, m) = 1 - 2 \frac{\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i + 1)}, \quad (12)$$

with parameters $n \in \mathbb{N}$, $k \geq 1$, $m \in \mathbb{R}$, such that $\gamma_r = k + (n-r)(m+1) \geq 1$, for all $1 \leq r \leq n$.

Throughout this paper, we propose the entropy of m -gos and m -dgos and study those models for the related measures of entropy. In the second part of the paper, we deal with the concomitants of m -gos of FGM family to extract the residual entropy and give some of its properties. The paper is organized as follows: Section 2 contains entropy of m -gos and m -dgos obtained from uniform distribution. Moreover, we obtain them for any distribution in terms of the obtained entropy from uniform distribution. Meanwhile, we produce some examples of some well-known distributions. Furthermore, we obtain the lower bound of the entropy of m -gos in terms of the mode. In addition, the residual and past entropy of m -gos and m -dgos is considered in Section 3. In Section 4, we derive the residual entropy of concomitants of m -gos of FGM family and discuss its relation with the stop-loss transform and Gini index. Besides, we consider this model in terms of its upper bound and produce some examples on it. Finally, in Section 5, real-life data connected with the COVID-19 virus is applied for the nonparametric estimation of residual entropy of concomitants of order statistics under the FGM family.

2. Entropy of m -Generalized Order Statistics and Its Dual

In this section, we discuss the entropy of m -gos and m -dgos for uniform distribution and for any distribution, which depends on beta function and its generalized first kind.

Based on uniform distribution $U(0, 1)$, the extropy of the r th m -gos, $X_{(r;n,k,m)}$, $1 \leq r \leq n$, can be obtained by the following theorem.

Theorem 1. Suppose the nonnegative continuous r.v. X be emerging from $U(0, 1)$ distribution. Let $U_{(r;n,k,m)}$ be its r th m -gos; then, from (1) and (6), the extropy of $U_{(r;n,k,m)}$ is given by

$$\begin{aligned} J(U_{(r;n,k,m)}) &= \frac{-(c_{r-1})^2}{2((r-1)!)^2} \left(\frac{1}{m+1}\right)^{2r-1} B\left(2r-1, \frac{2\gamma_r-1}{m+1}\right) \\ &= \frac{-(c_{r-1})^2}{2((r-1)!)^2} \frac{(2r-2)!}{\left(\prod_{i=1}^r (\gamma_1 + \gamma_i - 1)\right) \left(\prod_{i=1}^{r-1} (\gamma_r + \gamma_{1+i} - 1)\right)}. \end{aligned} \quad (13)$$

Proof. From (1) and (6), we have

$$\begin{aligned} B\left(2r-1, \frac{(2\gamma_r-1)}{(m+1)}\right) &= \frac{\Gamma(2r-1)\Gamma((2\gamma_r-1)/(m+1))}{\Gamma(2r-1 + ((2\gamma_r-1)/(m+1)))} \\ &= \frac{(2r-2)!\Gamma((2\gamma_r-1)/(m+1))}{\underbrace{(2r + ((2\gamma_r-1)/(m+1)) - 2)(2r + ((2\gamma_r-1)/(m+1)) - 3), \dots, ((2\gamma_r-1)/(m+1)) + r - 1}_{r\text{-times}}} \\ &\quad \times \frac{1}{\underbrace{(((2\gamma_r-1)/(m+1)) + r - 2)((2\gamma_r-1)/(m+1)) + r - 3, \dots, ((2\gamma_r-1)/(m+1))\Gamma((2\gamma_r-1)/(m+1))}_{(r-1)\text{-times}}}. \end{aligned} \quad (16)$$

Furthermore,

$$\begin{aligned} \left(\frac{1}{(m+1)}\right)^{2r-1} B\left(2r-1, \frac{(2\gamma_r-1)}{(m+1)}\right) &= \left(\frac{1}{(m+1)}\right)^r \left(\frac{1}{(m+1)}\right)^{r-1} B\left(2r-1, \frac{(2\gamma_r-1)}{(m+1)}\right) \\ &= \left(\frac{1}{(m+1)}\right)^r \left(\frac{1}{(m+1)}\right)^{r-1} \frac{(2r-2)!}{\underbrace{(2r + ((2\gamma_r-1)/(m+1)) - 2), \dots, ((2\gamma_r-1)/(m+1)) + r - 1}_{r\text{-times}}} \\ &\quad \times \frac{1}{\underbrace{(((2\gamma_r-1)/(m+1)) + r - 2)((2\gamma_r-1)/(m+1)) + r - 3, \dots, ((2\gamma_r-1)/(m+1))}_{(r-1)\text{-times}}} \\ &= \frac{(2r-2)!}{\left(\prod_{i=1}^r (\gamma_1 + \gamma_i - 1)\right) \left(\prod_{i=1}^{r-1} (\gamma_r + \gamma_{1+i} - 1)\right)}, \end{aligned} \quad (17)$$

and the result is taken after.

In the following theorem, we derive the extropy of m -gos, $X_{(r;n,k,m)}$, from any continuous distribution based on extropy of m -gos emerging from $U(0, 1)$. $\square$

$$\begin{aligned} J(U_{(r;n,k,m)}) &= \frac{-1}{2} \int_0^1 f_{(r;n,k,m)}^2(u) du \\ &= A_{(r;n,k,m)} \int_0^1 (1-u)^{2\gamma_r-2} (1-(1-u)^{m+1})^{2r-2} du, \end{aligned} \quad (14)$$

where $A_{(r;n,k,m)} = (-1/2)((c_{r-1})^2/((r-1)!)^2)$ $(1/(m+1))^{2r-2}$. Using the transformation $z = 1 - u$, thus

$$\begin{aligned} J(U_{(r;n,k,m)}) &= A_{(r;n,k,m)} \int_0^1 z^{2\gamma_r-2} (1-z^{m+1})^{2r-2} dz \\ &= A_{(r;n,k,m)} \frac{1}{m+1} B\left(2r-1, \frac{2\gamma_r-1}{m+1}\right), \end{aligned} \quad (15)$$

where $B(x, y/z) = z \int_0^1 t^{y-1} (1-t^z)^{x-1} dt = (\Gamma(x)\Gamma(y/z))/\Gamma(x + (y/z))$ is the beta function. Therefore, we can reduce the beta function in (15) as follows:

Theorem 2. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$. Then, from (1), (6), and (13), the extropy of the r th m -gos, $X_{(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{(r;n,k,m)}) = J(U_{(r;n,k,m)}) \mathbb{E}(f(F^{-1}(1 - V_r))), \quad (18)$$

where V_r has generalized beta of first kind distribution (i.e., $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$) and $\text{GB1}(x; a, p, q)$ has PDF:

$$f(x; a, p, q) = \frac{ax^{ap-1}(1-x^a)^{q-1}}{B(p, q)}, \quad 0 < x < 1, a, p, q > 0. \quad (19)$$

Proof. From (1) and (6), we have

$$J(X_{(r;n,k,m)}) = \frac{-1}{2} \int_0^\infty f_{(r;n,k,m)}^2(x) dx = A_{(r;n,k,m)} \int_0^\infty (1-F(x))^{2\gamma_r-2} (1-(1-F(x))^{m+1})^{2r-2} f^2(x) dx. \quad (20)$$

Putting $u = 1 - F(x)$, thus

$$\begin{aligned} J(X_{(r;n,k,m)}) &= A_{(r;n,k,m)} \int_0^1 u^{2\gamma_r-2} (1-u^{m+1})^{2r-2} f(F^{-1}(1-u)) du \\ &= A_{(r;n,k,m)} \frac{1}{m+1} B(2r-1, (2\gamma_r-1)/(m+1)) \\ &\quad \times \frac{m+1}{B(2r-1, (2\gamma_r-1)/(m+1))} \int_0^1 u^{2\gamma_r-2} (1-u^{m+1})^{2r-2} f(F^{-1}(1-u)) du \\ &= J(U_{(r;n,k,m)}) \mathbb{E}(f(F^{-1}(1-V_r))), \end{aligned} \quad (21)$$

which proves the theorem.

From the previous theorem, we show that the extropy of m -gos is the product of extropy of m -gos emerging from $U(0,1)$ distribution and expectation of the first kind generalized beta distributed r.v.

Now, we will give some special cases on Theorem 2 by the following examples. $\square$

Example 1. Suppose the nonnegative continuous r.v. X arising from exponential distribution, denoted by $\text{EXP}(\lambda)$, with CDF

$$F(x) = 1 - e^{-\lambda x}, \quad \lambda > 0, x > 0, \quad (22)$$

$$F^{-1}(x) = \frac{-1}{\lambda} \ln(1-x).$$

Therefore,

$$f(F^{-1}(1-u)) = \lambda u. \quad (23)$$

Thus,

$$\mathbb{E}(f(F^{-1}(1-V_r))) = \frac{\lambda}{B(2r-1, (2\gamma_r-1)/(m+1))} B\left(2r-1, \frac{2\gamma_r}{m+1}\right). \quad (24)$$

Thence,

$$J(X_{(r;n,k,m)}) = \frac{-\lambda(c_{r-1})^2}{2((r-1)!)^2} \left(\frac{1}{m+1}\right)^{2r-1} B\left(2r-1, \frac{2\gamma_r}{m+1}\right). \quad (25)$$

By the same manner in (17), we can reduce (25) as

$$J(X_{(r;n,k,m)}) = \frac{-\lambda(c_{r-1})^2}{2((r-1)!)^2} \frac{(2r-2)!}{(\prod_{i=1}^r (\gamma_i + \gamma_i)) (\prod_{i=1}^{r-1} (\gamma_r + \gamma_{1+i}))}. \quad (26)$$

Example 2. Suppose the nonnegative continuous r.v. X arising from Pareto distribution with CDF

$$F(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\alpha}, \quad \sigma > 0, x > \sigma, \quad (27)$$

$$F^{-1}(x) = \sigma(1-x)^{-(1/\alpha)},$$

Therefore,

$$f(F^{-1}(1-u)) = \frac{\alpha}{\sigma} u^{1+(1/\alpha)}. \quad (28)$$

Thus,

$$\mathbb{E}(f(F^{-1}(1-V_r))) = \frac{\alpha}{\sigma B(2r-1, (2\gamma_r-1)/(m+1))} B\left(2r-1, \frac{2\gamma_r + (1/\alpha)}{m+1}\right). \quad (29)$$

Thence,

$$J(X_{(r;n,k,m)}) = \frac{-\alpha(c_{r-1})^2}{2\sigma((r-1)!)^2} \left(\frac{1}{m+1}\right)^{2r-1} B\left(2r-1, \frac{2\gamma_r + (1/\alpha)}{m+1}\right). \quad (30)$$

In the next corollary, the lower bound for the extropy of m -gos will be obtained in terms of the extropy of m -gos emerging from $U(0,1)$ and the mode of the distribution.

Corollary 1. Suppose that M is the mode of the r.v. X such that $M = f(m)$ which exists; therefore,

$$J(X_{(r;n,k,m)}) \geq J(U_{(r;n,k,m)})M. \quad (31)$$

2.1. Entropy of Dual m -Generalized Order Statistics. Based on $U(0, 1)$ distribution, from (1) and (8) and Theorem 1, we can obviously see that the entropy of r th m -gos, $U_{(r;n,k,m)}$, is the same as the entropy of the r th m -dgos $U_{d(r;n,k,m)}$. On the other hand, we can obtain the entropy of r th m -dgos, $X_{d(r;n,k,m)}$, $1 \leq r \leq n$, for any distribution from the following theorem.

Theorem 3. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$ distribution. Then, from (1), (8), and (13), the entropy of the r th m -dgos, $X_{d(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{d(r;n,k,m)}) = J(U_{d(r;n,k,m)})\mathbb{E}(f(F^{-1}(V_r))), \quad (32)$$

where $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$ with PDF defined in (19).

Proof. From (1) and (8), we have

$$\begin{aligned} J(X_{d(r;n,k,m)}) &= -\frac{1}{2} \int_0^\infty f_{d(r;n,k,m)}^2(x) dx \\ &= A_{(r;n,k,m)} \int_0^\infty (F(x))^{2\gamma_r-2} (1 - (F(x))^{m+1})^{2r-2} f^2(x) dx. \end{aligned} \quad (33)$$

Putting $u = F(x)$, thus

$$\begin{aligned} J(X_{d(r;n,k,m)}) &= A_{(r;n,k,m)} \frac{1}{m+1} B\left(2r-1, \frac{(2\gamma_r-1)}{(m+1)}\right) \\ &\quad \times \frac{m+1}{B(2r-1, (2\gamma_r-1)/(m+1))} \int_0^1 u^{2\gamma_r-2} (1-u^{m+1})^{2r-2} f(F^{-1}(u)) du \\ &= J(U_{d(r;n,k,m)})\mathbb{E}(f(F^{-1}(V_r))), \end{aligned} \quad (34)$$

where the entropy of r th m -dgos $J(U_{d(r;n,k,m)}) = J(U_{(r;n,k,m)})$ obtained in (13), which proves the theorem. $\square$

3. Residual and Past Entropy of m -Generalized Order Statistics and Its Dual

Under the condition of parameters in (6), Kamps [16] obtained the CDF of m -gos, $X_{(r;n,k,m)}$, $1 \leq r \leq n$, as follows:

$$F_{(r;n,k,m)}(x) = 1 - c_{r-1} (1 - F(x))^{\gamma_r} \sum_{j=0}^{r-1} \frac{1}{j! c_{r-j-1}} \left[\frac{1}{m+1} (1 - (1 - F(x))^{m+1}) \right]^j, \quad (35)$$

where $c_{r-j-1} = \prod_{i=1}^{r-j} \gamma_i$. Furthermore, Shahbaz et al. [22] derived the CDF of both m -gos and m -dgos, $X_{(r;n,k,m)}$, $X_{d(r;n,k,m)}$, $1 \leq r \leq n$, respectively, in terms of incomplete beta function ratio as follows:

$$F_{(r;n,k,m)}(x) = I_{\beta[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right), \quad (36)$$

$$F_{d(r;n,k,m)}(x) = I_{\beta^*[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right), \quad (37)$$

where $I_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt$ is incomplete beta function ratio, $\beta[F(x)] = 1 - (1 - F(x))^{m+1}$, $\beta^*[F(x)] = F(x)^{m+1}$.

In the next theorems, we obtain the residual and past entropy of m -gos and m -dgos emerging from any distribution.

Theorem 4. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$ distribution. Then, from (2), (6), and (36), the residual entropy of r th m -gos, $X_{(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{(r;n,k,m)}) = \frac{1}{\bar{F}_{(r;n,k,m)}^2(x)} J(U_{(r;n,k,m)}) \mathbb{E}(f(F^{-1}(1 - V_r)) | V_r < 1 - F(t)), \quad (38)$$

where $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$ with PDF defined in (19) and $\bar{F}_{(r;n,k,m)}(x) = 1 - F_{(r;n,k,m)}(x)$ defined in (36).

Theorem 5. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$ distribution. Then, from (3), (6), and (36), the past extropy of the r th m -gos, $X_{(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{(r;n,k,m)}) = \frac{1}{F_{(r;n,k,m)}^2(x)} J(U_{(r;n,k,m)}) \mathbb{E}(f(F^{-1}(1 - V_r)) | V_r > 1 - F(t)), \quad (39)$$

where $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$ with PDF defined in (19) and $F_{(r;n,k,m)}(x)$ defined in (36).

Theorem 6. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$ distribution. Then, from (2), (8), and (37), the residual extropy of the r th m -dgos, $X_{d(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{d(r;n,k,m)}) = \frac{1}{\bar{F}_{d(r;n,k,m)}^2(x)} J(U_{d(r;n,k,m)}) \mathbb{E}(f(F^{-1}(V_r)) | V_r < 1 - F(t)), \quad (40)$$

where $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$ with PDF defined in (19) and $\bar{F}_{d(r;n,k,m)}(x) = 1 - F_{d(r;n,k,m)}(x)$ defined in (37).

Theorem 7. Let X be a continuous r.v. that is nonnegative with CDF $F(x)$ distribution. Then, from (3), (8), and (37), the past extropy of the r th m -dgos, $X_{d(r;n,k,m)}$, $1 \leq r \leq n$, is given by

$$J(X_{d(r;n,k,m)}) = \frac{1}{F_{d(r;n,k,m)}^2(x)} J(U_{d(r;n,k,m)}) \mathbb{E}(f(F^{-1}(V_r)) | V_r > 1 - F(t)), \quad (41)$$

where $V_r \sim \text{GB1}(x; m+1, (2\gamma_r - 1)/(m+1), 2r-1)$ with PDF defined in (19) and $F_{d(r;n,k,m)}(x)$ defined in (37).

4. Residual Extropy of Concomitants of m -Generalized Order Statistics

In this section, we will discuss the residual extropy of concomitants of m -gos under the FGM family. From equation (9), the conditional CDF of Y given $X = x$ is given by

$$F_{Y|X}(y|x) = F_Y(y) [1 + \alpha(1 - 2F_X(x))(1 - F_Y(y))]. \quad (42)$$

Under the FGM family with conditional CDF given by equation (42), Mohie El-Din et al. [23] presented the CDF of the concomitant of m -gos $Y_{[r;n,k,m]}$, $1 \leq r \leq n$, as follows:

$$G_{[r;n,k,m]}(y) = F_Y(y) [1 - \alpha T^*(r; n, k, m)(1 - F_Y(y))], \quad (43)$$

where $f_{(r;n,k,m)}(x)$ is the PDF of m -gos $X_{(r;n,k,m)}$. Therefore,

$$\bar{G}_{[r;n,k,m]}(y) = 1 - G_{[r;n,k,m]}(y) = 1 - F_Y(y) [1 - \alpha T^*(r; n, k, m)(1 - F_Y(y))] = \bar{F}_Y(y) [1 + \alpha T^*(r; n, k, m)F_Y(y)]. \quad (44)$$

From (4) and (44), the residual extropy of concomitants of m – gos is given by

$$\begin{aligned}
 \zeta_{[r;n,k,m]}(Y) &= \frac{-1}{2} \int_0^\infty (\bar{G}_{[r;n,k,m]}(y))^2 dy \\
 &= \frac{-1}{2} \int_0^\infty \left[\bar{F}_Y^2(y) + 2\alpha T^*(r; n, k, m) \bar{F}_Y^2(y) F_Y(y) + (\alpha T^*(r; n, k, m))^2 \bar{F}_Y^2(y) F_Y^2(y) \right] dy \\
 &= \zeta(Y) + \left((\alpha T^*(r; n, k, m))^2 - 4\alpha T^*(r; n, k, m) \right) \mathbb{E}(Y F_Y(y)) \\
 &\quad + 3 \left(\alpha T^*(r; n, k, m) - (\alpha T^*(r; n, k, m))^2 \right) \mathbb{E}(Y F_Y^2(y)) \\
 &\quad + 2 (\alpha T^*(r; n, k, m))^2 \mathbb{E}(Y F_Y^3(y)) + \alpha T^*(r; n, k, m) \mathbb{E}(Y).
 \end{aligned} \tag{45}$$

Furthermore, we can write (45) in terms of the moments as follows:

$$\begin{aligned}
 \zeta_{[r;n,k,m]}(Y) &= \zeta(Y) + \frac{1}{2} \left((\alpha T^*(r; n, k, m))^2 - 4\alpha T^*(r; n, k, m) \right) \mu_{2:2} \\
 &\quad + \left(\alpha T^*(r; n, k, m) - (\alpha T^*(r; n, k, m))^2 \right) \mu_{3:3} \\
 &\quad + \frac{1}{2} (\alpha T^*(r; n, k, m))^2 \mu_{4:4} + \alpha T^*(r; n, k, m) \mu,
 \end{aligned} \tag{46}$$

where $\mu_{n:n} = \int_0^\infty y f_{y_{n:n}}(y) dy$ and $y_{n:n}$ is the n th order statistic of a random sample of size n of the Y variate, $\mu = \mathbb{E}(Y)$.

4.1. Stop-Loss Transform and Gini Coefficient. In this section, we will present $\zeta_{[r;n,k,m]}(Y)$ related to stop-loss transform and Gini index.

Definition 1. The stop-loss transform $Z_F(t)$ of the non-negative r.v. Y is defined as

$$Z_F(t) = \mathbb{E}(\max\{Y - t, 0\}) = \int_t^\infty \bar{F}_Y(y) dy. \tag{47}$$

Definition 2. Suppose X and Y are independent r.v.'s and have the same distribution as X . Then, the Gini index or Gini coefficient is given by

$$G_{\text{index}} = \frac{\mathbb{E}(|X - Y|)}{\mathbb{E}(|X + Y|)} = 1 - \frac{\int_0^\infty \bar{F}_X(x) dx}{\mathbb{E}(X)}, \tag{48}$$

(see Wang [24] for more details).

Remark 1. From Jahanshahi et al. [14], based on (4) and Definitions 1 and 2, we have

$$\begin{aligned}
 \zeta(Y) &= \frac{-1}{2} [\mathbb{E}(Y) - \mathbb{E}(Z_F(Y))] \\
 &= \frac{-1}{2} [\mathbb{E}(Y) - \mathbb{E}(\bar{F}_Y(y) m_F(Y))] = \frac{\mathbb{E}(Y)}{2} [G_{\text{index}} - 1],
 \end{aligned} \tag{49}$$

where $m_F(Y) = (Z_F(Y)/\bar{F}_Y(y))$ is the mean residual life function.

Theorem 8. Let Y be a continuous r.v. that is nonnegative with CREX $\zeta(X)$ defined in (4). Then, the cumulative residual extropy of concomitants of m – gos, $Y_{(r;n,k,m)}$, can be expressed as

$$\begin{aligned}
 \zeta_{[r;n,k,m]}(Y) &= \frac{1}{2} \mathbb{E}(Z_F(Y)) + T_{[r;n,k,m]}^* \\
 &= \frac{1}{2} \mathbb{E}(\bar{F}_Y(y) m_F(Y)) + T_{[r;n,k,m]}^* \\
 &= \frac{1}{2} \mathbb{E}(Y) G_{\text{index}} + T_{[r;n,k,m]}^*,
 \end{aligned} \tag{50}$$

where

$$T_{[r;n,k,m]}^* = \left((\alpha T^*(r; n, k, m))^2 - 4\alpha T^*(r; n, k, m) \right) \mathbb{E}(Y F_Y(y)) + 3 \left(\alpha T^*(r; n, k, m) - (\alpha T^*(r; n, k, m))^2 \right) \mathbb{E}(Y F_Y^2(y)) + 2 (\alpha T^*(r; n, k, m))^2 \mathbb{E}(Y F_Y^3(y)) + \alpha T^*(r; n, k, m) \mathbb{E}(Y). \quad (51)$$

Proof. The proof directly follows from (45) and Remark 1. $\square$

4.2. *Upper Bound of $\zeta_{[r;n,k,m]}(Y)$.* In another view of the cumulative residual entropy of concomitants of m -gos, $\zeta_{[r;n,k,m]}(Y)$, we present it depending on its upper bound.

Theorem 9. *Let Y be a continuous r.v. that is nonnegative with survival function $\bar{F}_Y(y)$. Based on FGM family with*

CDF given by (9), utilizing (4), the residual extropy of the r th concomitants m -gos, $Y_{[r;n,k,m]}$, $1 \leq r \leq n$, can be expressed by

$$\zeta_{[r;n,k,m]}(Y) \leq \zeta(y) + \alpha T^*(r; n, k, m) \left[3\mathbb{E}(Y F_Y^2(y)) + \mathbb{E}(Y) - 4\mathbb{E}(Y F_Y(y)) \right]. \quad (52)$$

Proof. Using Bernoulli's inequality, from (44), we have

$$\begin{aligned} \zeta_{[r;n,k,m]}(Y) &= \frac{-1}{2} \int_0^\infty (\bar{G}_{[r;n,k,m]}(y))^2 dy \\ &= \frac{-1}{2} \int_0^\infty \bar{F}_Y^2(y) [1 + \alpha T^*(r; n, k, m) F_Y(y)]^2 dy \\ &\leq \frac{-1}{2} \int_0^\infty \bar{F}_Y^2(y) [1 + 2\alpha T^*(r; n, k, m) F_Y(y)] dy \\ &= \zeta(y) + \alpha T^*(r; n, k, m) \left[3\mathbb{E}(Y F_Y^2(y)) + \mathbb{E}(Y) - 4\mathbb{E}(Y F_Y(y)) \right], \end{aligned} \quad (53)$$

where $T^*(r; n, k, m)$ is defined in (12).

From the previous theorem, we can conclude the following remark. $\square$

Remark 2. Since $-1 < T^*(r; n, k, m) < 1$, then the residual entropy of the concomitant of r th m -gos given in (53) can be considered in following cases:

- (1) If $T^*(r; n, k, m) = 0$, then $\zeta_{[r;n,k,m]}(Y) = \zeta(y)$.
- (2) If $0 < \alpha < 1$ and $0 < T^*(r; n, k, m) < 1$ or $-1 < \alpha < 0$ and $-1 < T^*(r; n, k, m) < 0$, then

$$\zeta_{[r;n,k,m]}(Y) < \zeta(y) + \alpha T^*(r; n, k, m) \left[3\mathbb{E}(Y F_Y^2(y)) + \mathbb{E}(Y) \right]. \quad (54)$$

- (3) If $0 < \alpha < 1$ and $-1 < T^*(r; n, k, m) < 0$ or $-1 < \alpha < 0$ and $0 < T^*(r; n, k, m) < 1$, then

$$\zeta_{[r;n,k,m]}(Y) < \zeta(y) - 4\alpha T^*(r; n, k, m) \mathbb{E}(Y F_Y(y)). \quad (55)$$

Theorem 10. *Let Y be a continuous r.v. that is nonnegative with survival function $\bar{F}_Y(y)$. Then, we have the following:*

- (1) *From Remark 2, under the conditions on the parameters in (54), we get*

$$\zeta_{[r;n,k,m]}(Y) < \frac{1}{2} \xi(Y) + F_{[r;n,k,m]}^* \leq \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + F_{[r;n,k,m]}^*. \quad (56)$$

- (2) *From Remark 2, under the conditions on the parameters in (55), we get*

$$\zeta_{[r;n,k,m]}(Y) < \frac{1}{2} \xi(Y) + R_{[r;n,k,m]}^* \leq \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + R_{[r;n,k,m]}^*, \quad (57)$$

where $\xi(Y) = -\int_0^\infty \bar{F}_Y(y) \log \bar{F}_Y(y) dy$ is the cumulative residual entropy presented by Rao et al. [25]:

$$\begin{aligned} F_{[r;n,k,m]}^* &= 3\alpha T^*(r; n, k, m) \mathbb{E}(Y F_Y^2(y)) \\ &\quad + \mathbb{E}(Y) \left(\alpha T^*(r; n, k, m) - \frac{1}{2} \right), \end{aligned} \quad (58)$$

$$R_{[r;n,k,m]}^* = -4\alpha T^*(r; n, k, m) \mathbb{E}(Y F_Y(y)) - \frac{1}{2} \mathbb{E}(Y). \quad (59)$$

Proof. The proof directly follows from inequality $-\log x \geq (1-x)$, $x > 0$ and $\xi(Y) \leq (\mathbb{E}(Y^2)/2\mathbb{E}(Y))$; see Rao et al. [25].

In the next remark, we will give an application of the previous results, taking the order statistics (with $k = 1$ and $m = 0$) as a special case under the FGM family. $\square$

Remark 3. For order statistic ($m = 0$ and $k = 1$), we have $T^*(r; n, 1, 0) = -(n - 2r + 1)/(n + 1)$. Moreover, by controlling the values of n and r , we can observe the following:

- (1) $T^*(r; n, 1, 0) = 0$, if n is odd and $r = (n + 1)/2$
- (2) $T^*(r; n + 1, 1, 0) = 0$, if n is even, $r = (n/2) + 1$ and n is replaced with $n + 1$
- (3) $T^*(r; n, 1, 0) < (>) 0$, if n is even and $r < (>) (n/2)$
- (4) $T^*(r; n, 1, 0) < (>) 0$, if n is odd and $r < (>) (n + 1)/2$

Example 3. Let Y be a continuous r.v. that is nonnegative arising from $\text{EXP}(\lambda)$ with CDF defined in (22); then, utilizing (58) and (59), we have

$$\begin{aligned}\zeta_{[r;n,k,m]}(Y) &= -\frac{6 + \alpha T^*(r; n, k, m)(4 + \alpha T^*(r; n, k, m))}{24\lambda}, \\ \frac{1}{2}\xi(Y) + F_{[r;n,k,m]}^* &= \frac{17\alpha T^*(r; n, k, m)}{6\lambda} = \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + F_{[r;n,k,m]}^*, \\ \frac{1}{2}\xi(Y) + R_{[r;n,k,m]}^* &= \frac{-3\alpha T^*(r; n, k, m)}{\lambda} = \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + R_{[r;n,k,m]}^*.\end{aligned}\quad (60)$$

For order statistics (with $m = 0$ and $k = 1$), we can apply some numerical values as follows:

- (1) For $n = 20$, $r = 13$, $\alpha = 0.4$, and $\lambda = 1$, we have $T^*(r; n, 1, 0) = (5/21)$, $\zeta_{[r;n,1,0]}(Y) = -0.266251$, and $(1/2)\xi(Y) + F_{[r;n,1,0]}^* = 0.26984 = (\mathbb{E}(Y^2)/4\mathbb{E}(Y)) + F_{[r;n,1,0]}^*$
- (2) For $n = 27$, $r = 13$, $\alpha = 0.3$, and $\lambda = 1$, we have $T^*(r; n, 1, 0) = (-1/14)$, $\zeta_{[r;n,1,0]}(Y) = -0.246448$, and $(1/2)\xi(Y) + R_{[r;n,1,0]}^* = 0.06428 = (\mathbb{E}(Y^2)/4\mathbb{E}(Y)) + R_{[r;n,1,0]}^*$

which assure Theorem 10.

Example 4. Let Y be a continuous r.v. that is nonnegative arising from uniform distribution $U(0, b)$, $b > 0$; then, utilizing (58) and (59), we have

$$\begin{aligned}\zeta_{[r;n,k,m]}(Y) &= \frac{-b}{60} (10 + 5\alpha T^*(r; n, k, m) + (\alpha T^*(r; n, k, m))^2), \\ \frac{1}{2}\xi(Y) + F_{[r;n,k,m]}^* &= \frac{b}{8} (-1 + 10\alpha T^*(r; n, k, m)), \\ \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + F_{[r;n,k,m]}^* &= \frac{b}{12} (-1 + 15\alpha T^*(r; n, k, m)), \\ \frac{1}{2}\xi(Y) + R_{[r;n,k,m]}^* &= \frac{-b}{24} (3 + 32\alpha T^*(r; n, k, m)), \\ \frac{\mathbb{E}(Y^2)}{4\mathbb{E}(Y)} + R_{[r;n,k,m]}^* &= \frac{-b}{12} (1 + 16\alpha T^*(r; n, k, m)).\end{aligned}\quad (61)$$

For order statistics (with $m = 0$ and $k = 1$), we can apply some numerical values as follows:

- (1) For $n = 20$, $r = 13$, $\alpha = 0.4$, and $b = 1$, we have $T^*(r; n, 1, 0) = (5/21)$, $\zeta_{[r;n,1,0]}(Y) = -0.174754$, $(1/2)\xi(Y) + F_{[r;n,1,0]}^* = -0.005952$, and $(\mathbb{E}(Y^2)/4\mathbb{E}(Y)) + F_{[r;n,1,0]}^* = 0.0357$
- (2) For $n = 27$, $r = 13$, $\alpha = 0.3$, and $b = 1$, we have $T^*(r; n, 1, 0) = (-1/14)$, $\zeta_{[r;n,1,0]}(Y) = -0.164889$, and $(1/2)\xi(Y) + R_{[r;n,1,0]}^* = -0.0964286$

$$^* = -0.0964286, (\mathbb{E}(Y^2)/4\mathbb{E}(Y)) + R_{[r;n,1,0]}^* = -0.0547619$$

Therefore, (1.) and (2.) assure Theorem 10.

Theorem 11. Let Y be a continuous r.v. that is nonnegative with survival function $\bar{F}_Y(y)$. From Remark 2, under the conditions on the parameters in (54), for order statistics (with $m = 0$ and $k = 1$) as a special case of m -gos, we denote

$$\zeta_{[r;n,1,0]}(Y) < \zeta(Y) + \alpha T^*(r; n, 1, 0)[s_3 + \mu] = Z_{(r;n,1,0)}^*(y), \quad (62)$$

where $s_i = m_i + i \int_{m_i}^{\infty} \bar{F}_Y(y) dy$, $m_i = \inf\{y: \bar{F}_Y(y) \leq (1/i)\}$, $i = 1, 2, \dots, n$.

Proof. A method of obtaining an upper bound for the mean of the maximum of n identically distributed nonnegative r.v.'s is given by Lai and Robbins [26] and Gravey [27]. If $\mathbb{E}(Y_i^+) < \infty$, $i = 1, 2, \dots, n$, then

$$\mu_{n:n} = \mathbb{E}(Y_{n:n}) \leq m_n + n \int_{m_n}^{\infty} \bar{F}_X(x) dx = s_n. \quad (63)$$

Thus, from (54),

$$\begin{aligned} \zeta_{[r;n,1,0]}(Y) &< \zeta(y) + \alpha T^*(r; n, 1, 0)[3\mathbb{E}(YF_Y^2(y)) + \mathbb{E}(Y)] \\ &= \zeta(y) + \alpha T^*(r; n, 1, 0)[\mu_{3:3} + \mu] \\ &< \zeta(Y) + \alpha T^*(r; n, 1, 0)[s_3 + \mu] = Z_{(r;n,1,0)}^*(y). \end{aligned} \quad (64)$$

□

Example 5. Suppose Y follows EXP(1) with survival function $\bar{F}_Y(y) = e^{-y}$, $0 < y$. From (63), we have $\mu_{n:n} \leq 1 + \log n$. Therefore, using (62) and from Remark 2, under the conditions on the parameters in (54), with $n = 50$, $r = 28$, $\alpha = 0.3$, and $T^*(r; n, 1, 0) = (5/51)$, we have $\zeta_{[r;n,1,0]}(Y) = -0.25493$ and $Z_{(r;n,1,0)}^*(y) = -0.15886$, which assure the previous results.

4.3. Stochastic Orders

Definition 3. Y_2 is known to be smaller than Y_1 in the usual stochastic order, denoted by $Y_2 \leq_{st} Y_1$ if and only if $\bar{F}_{Y_2}(t) \leq \bar{F}_{Y_1}(t)$, for all $t \in \mathbb{R}$. For more details, see Shaked and Shanthikumar [28].

Theorem 12. Let Y_1 and Y_2 be two nonnegative continuous r.v.'s with CDF's $F_{Y_1}(\cdot)$ and $F_{Y_2}(\cdot)$ and finite mean $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y_2)$, respectively. If $Y_2 \leq_{st} Y_1$, then we have the following:

- (1) From Remark 2, under the conditions on the parameters in (54), we get

$$\begin{aligned} \zeta_{[r;n,k,m]}(Y_1) - \zeta_{[r;n,k,m]}(Y_2) &\leq \alpha T^*(r; n, k, m)[\mathbb{E}(Y_1) - \mathbb{E}(Y_2)]. \end{aligned} \quad (65)$$

- (2) From Remark 2, under the conditions on the parameters in (55), we get

$$\zeta_{[r;n,k,m]}(Y_1) - \zeta_{[r;n,k,m]}(Y_2) \leq \zeta(Y_1) - \zeta(Y_2). \quad (66)$$

Proof. First, from Theorem 7 of Jahanshahi et al. [14], if $Y_2 \leq_{st} Y_1$, then $\zeta(Y_2) \geq \zeta(Y_1)$. Therefore, the proof of (1) is

$$\begin{aligned} \zeta_{[r;n,k,m]}(Y_1) - \zeta_{[r;n,k,m]}(Y_2) &\leq \zeta(Y_1) - \zeta(Y_2) \\ &+ 3\alpha T^*(r; n, k, m) \left[\int_0^{\infty} t [F_{Y_1}^2(t) f_{Y_1}(t) - F_{Y_2}^2(t) f_{Y_2}(t)] dt \right] \\ &+ \alpha T^*(r; n, k, m) [\mathbb{E}(Y_1) - \mathbb{E}(Y_2)] \\ &\leq \alpha T^*(r; n, k, m) [\mathbb{E}(Y_1) - \mathbb{E}(Y_2)]. \end{aligned} \quad (67)$$

The proof of (2) is

$$\begin{aligned} \zeta_{[r;n,k,m]}(Y_1) - \zeta_{[r;n,k,m]}(Y_2) &\leq \zeta(Y_1) - \zeta(Y_2) \\ &- 4\alpha T^*(r; n, k, m) \left[\int_0^{\infty} t [F_{Y_1}(t) f_{Y_1}(t) - F_{Y_2}(t) f_{Y_2}(t)] dt \right] \\ &\leq \zeta(Y_1) - \zeta(Y_2). \end{aligned} \quad (68)$$

Now, we will give an application of the last theorem as follows. □

Example 6. Let Y_1 and Y_2 be two r.v.'s of power function distribution with CDF's $F_{Y_1}(y) = y^2$, $0 < y < 1$ and $F_{Y_2}(y) = y$, $0 < y < 1$, respectively. For order statistics (with $m = 0$ and $k = 1$). From Theorem 12, we have the following:

- (1) For $n = 20$, $r = 12$, and $\alpha = 0.6$, we have $T^*(r; n, 1, 0) = (1/7)$, $\zeta_{[r;n,1,0]}(Y_1) - \zeta_{[r;n,1,0]}(Y_2) = -0.09935$, and $\alpha T^*(r; n, k, m) [\mathbb{E}(Y_1) - \mathbb{E}(Y_2)] = 0.01428$
- (2) For $n = 20$, $r = 8$, and $\alpha = 0.8$, we have $T^*(r; n, 1, 0) = (-5/21)$, $\zeta_{[r;n,1,0]}(Y_1) - \zeta_{[r;n,1,0]}(Y_2) = -0.10121$, and $\zeta(Y_1) - \zeta(Y_2) = -0.1$

which assure the previous results.

5. Nonparametric Estimation

In this section, we obtain a nonparametric estimation of the residual entropy of concomitants of m -gos under the FGM family by the empirical data. Let $Y_1, \dots, Y_n$ be a random sample from a population with CDF F and its empirical estimator F_n . From (45), the empirical residual entropy of concomitants of m -gos is given by

$$\begin{aligned} \zeta_{[r;n,k,m]}(F_n) &= \frac{-1}{2} \int_0^{\infty} \bar{F}_n^2(y) (1 + \alpha T^*(r; n, k, m) F_n(y))^2 dy \\ &= \frac{-1}{2} \sum_{j=1}^{n-1} \int_{Y_{(j)}}^{Y_{(j+1)}} \bar{F}_n^2(y) (1 + \alpha T^*(r; n, k, m) F_n(y))^2 dy, \end{aligned} \quad (69)$$

where $\bar{F}_n(y) = 1 - F_n(y)$, $F_n(y)$ is the empirical CDF, and $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}$ are the associated order statistics of the random sample.

To estimate $\zeta_{[r;n,k,m]}$, we consider the first empirical estimator $\zeta_{1[r;n,k,m]}(F_n)$ as follows:

$$\begin{aligned} \zeta_{1[r;n,k,m]}(F_n) \\ = \frac{-1}{2} \sum_{j=1}^{n-1} W_{j+1} \left(1 - \frac{j}{n}\right)^2 \left(1 + \alpha T^*(r; n, k, m) \frac{j}{n}\right)^2, \end{aligned} \quad (70)$$

where $W_{j+1} = Y_{(j+1)} - Y_{(j)}$, $F_n(y) = (j/n)$, $j = 1, 2, \dots, n-1$, and $W_1 = Y_{(1)}$. Moreover, the second empirical estimator (kernel-smoothed estimator) $\zeta_{2[r;n,k,m]}(F_n)$ is given by

$$\begin{aligned} \zeta_{2[r;n,k,m]}(F_n) = \frac{-1}{2} \sum_{j=1}^{n-1} W_{j+1} (1 - F_n(y_j))^2 \\ \cdot \left(1 + \alpha T^*(r; n, k, m) F_n(y_j)\right)^2, \end{aligned} \quad (71)$$

where

$$F_n(y_j) = \frac{1}{n} \sum_{i=1}^n h\left(\frac{y - Y_i}{l}\right), \quad (72)$$

$h(y) = \int_{-\infty}^y K(t)dt$ and l is a bandwidth parameter; see Nadaraya [29].

In the following examples, we apply the proposed methods to explain the performance of the empirical and kernel estimators.

Example 7. Let $X_1, \dots, X_n$ be a random sample of uniform distribution $U(0, 1)$. According to Pyke [30], the sample spacing W_{j+1} follows the beta distribution $\text{Beta}(1, n)$. Hence, from (70) and (71), we have

$$\begin{aligned} \mathbb{E}(\zeta_{1[r;n,k,m]}(F_n)) &= \frac{-1}{2(n+1)} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^2 \left(1 + \alpha T^*(r; n, k, m) \frac{j}{n}\right)^2, \\ \text{Var}(\zeta_{1[r;n,k,m]}(F_n)) &= \frac{n}{4(n+1)^2(n+2)} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^4 \left(1 + \alpha T^*(r; n, k, m) \frac{j}{n}\right)^4, \\ \mathbb{E}(\zeta_{2[r;n,k,m]}(F_n)) &= \frac{-1}{2(n+1)} \sum_{j=1}^{n-1} (1 - F_n(y_j))^2 (1 + \alpha T^*(r; n, k, m) F_n(y_j))^2, \\ \text{Var}(\zeta_{2[r;n,k,m]}(F_n)) &= \frac{n}{4(n+1)^2(n+2)} \sum_{j=1}^{n-1} (1 - F_n(y_j))^4 (1 + \alpha T^*(r; n, k, m) F_n(y_j))^4. \end{aligned} \quad (73)$$

Example 8. Let $X_1, \dots, X_n$ be a random sample of $\text{Exp}(\lambda)$. According to Pyke [30], the sample spacing

W_{j+1} follows $\text{Exp}(\lambda(n-j))$. Hence, from (70) and (71), we have

$$\begin{aligned} \mathbb{E}(\zeta_{1[r;n,k,m]}(F_n)) &= \frac{-1}{2n\lambda} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \left(1 + \alpha T^*(r; n, k, m) \frac{j}{n}\right)^2, \\ \text{Var}(\zeta_{1[r;n,k,m]}(F_n)) &= \frac{1}{4n^2\lambda^2} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^2 \left(1 + \alpha T^*(r; n, k, m) \frac{j}{n}\right)^4, \\ \mathbb{E}(\zeta_{2[r;n,k,m]}(F_n)) &= \frac{-1}{2\lambda} \sum_{j=1}^{n-1} \frac{(1 - F_n(y_j))^2}{n-j} (1 + \alpha T^*(r; n, k, m) F_n(y_j))^2, \\ \text{Var}(\zeta_{2[r;n,k,m]}(F_n)) &= \frac{1}{4\lambda^2} \sum_{j=1}^{n-1} \frac{(1 - F_n(y_j))^4}{(n-j)^2} (1 + \alpha T^*(r; n, k, m) F_n(y_j))^4. \end{aligned} \quad (74)$$

Based on order statistics ($k = 1, m = 0$), Table 1 presents the mean and variance of $\zeta_{1[r;n,1,0]}$ and $\zeta_{2[r;n,1,0]}$ from $U(0, 1)$ and $\text{EXP}(1)$, respectively, by using different values of sample

size ($n = 10, 30, 50, 70, 100$). In Table 1, for fixed n and r increases, we conclude that the mean decreases and the variance increases.

TABLE 1: Mean and variance of the empirical residual entropy of concomitants of order statistics, $\alpha = 0.8$.

n	r	$U(0, 1)$ distribution				EXP(1) distribution			
		The first estimator		The second estimator		The first estimator		The second estimator	
		$\mathbb{E}(\zeta_{1[r;n,1,0]})$	$\text{Var}(\zeta_{2[r;n,1,0]})$	$\mathbb{E}(\zeta_{2[r;n,1,0]})$	$\text{Var}(\zeta_{2[r;n,1,0]})$	$\mathbb{E}(\zeta_{1[r;n,1,0]})$	$\text{Var}(\zeta_{2[r;n,1,0]})$	$\mathbb{E}(\zeta_{2[r;n,1,0]})$	$\text{Var}(\zeta_{2[r;n,1,0]})$
10	3	-0.1042	0.0019	-0.1043	0.0013	-0.1704	0.0047	-0.2009	0.0057
	5	-0.1241	0.0024	-0.1303	0.0019	-0.2132	0.0065	-0.2528	0.0084
	8	-0.1588	0.0035	-0.17507	0.0035	-0.2904	0.0108	-0.3463	0.0145
30	10	-0.1317	0.00109	-0.13301	0.000998	-0.1977	0.0019	-0.2089	0.0019
	15	-0.1512	0.001319	-0.1544	0.00125	-0.2373	0.0025	-0.2552	0.0027
	25	-0.1966	0.00193	-0.2039	0.00197	-0.3333	0.00447	-0.3663	0.0051
50	15	-0.1334	0.000702	-0.13406	0.000667	-0.1946	0.00116	-0.20008	0.001106
	25	-0.1572	0.00086	-0.15906	0.00085	-0.2423	0.0015	-0.2539	0.0016
	40	-0.19906	0.001219	-0.2026	0.00123	-0.3294	0.0026	-0.3514	0.0028
70	20	-0.1341	0.000515	-0.1343	0.00049	-0.1933	0.00082	-0.1964	0.00078
	35	-0.1598	0.00064	-0.1609	0.00063	-0.2445	0.00115	-0.2526	0.00116
	60	-0.2111	0.00096	-0.2137	0.00097	-0.3511	0.00208	-0.3685	0.0022
100	20	-0.1265	0.00034	-0.1259	0.00032	-0.1767	0.00051	-0.1776	0.00048
	50	-0.1618	0.00046	-0.1623	0.00045	-0.2461	0.000814	-0.2516	0.000819
	90	-0.2206	0.00073	-0.2226	0.00073	-0.36809	0.00158	-0.3807	0.0016

FIGURE 1: Histogram of serious cases of infected COVID-19 virus data.

5.1. Data Application. In the following, we illustrate our empirical estimators in real and simulated data for EXP(1).

Example 9. We refer to Kasilingam et al.'s [31] research to understand the spreading patterns of the COVID-19 virus; they used exponential growth modelling and identifies countries that have shown early signs of containment until 26th of March 2020. The data represent the percentage of serious cases of infections in 42 countries listed as follows: 1.56, 8.51, 2.17, 0.37, 1.09, 9.84, 4.95, 3.18, 11.37, 2.81, 6.22, 1.87, 0.00, 0.00, 9.05,

2.44, 1.38, 4.17, 3.74, 1.37, 2.33, 7.80, 2.10, 0.47, 2.54, 0.92, 0.09, 0.18, 1.72, 1.02, 0.62, 2.34, 0.50, 2.37, 3.65, 0.59, 5.76, 2.14, 0.88, 0.95, 4.17, and 2.25.

We use Kolmogorov–Smirnov ($K-S$) test to check the fitting of the data for EXP(1), which implies that the $K-S$ statistic is 0.076282 with p value 0.9674. Thus, it is admitted to fit the data by EXP(1); furthermore, see Figure 1. Based on EXP(1), Figures 2 and 3 present the real-life and simulated data, respectively. Therefore, we can conclude that by decreasing α and increasing r , the empirical estimators approach the theoretical value and vice versa.

FIGURE 2: Empirical estimators of serious cases of infected COVID-19 virus data. (a) $\alpha = 0.8, n = 40, r = 35$. (b) $\alpha = -0.8, n = 40, r = 35$. (c) $\alpha = 0.8, n = 40, r = 10$. (d) $\alpha = -0.8, n = 40, r = 10$.

FIGURE 3: Empirical estimators of simulated data. (a) $\alpha = 0.8, n = 100, r = 90$. (b) $\alpha = -0.8, n = 100, r = 90$. (c) $\alpha = 0.8, n = 100, r = 10$. (d) $\alpha = -0.8, n = 100, r = 10$.

6. Conclusion

In this communication, we introduced the extropy of m -gos and m -dgos arising from any distribution and wrote them in terms of the extropy of m -gos and m -dgos of $U(0, 1)$ distribution, respectively. Furthermore, examples of the obtained models for exponential and Pareto distributions are provided. Also, we produced the lower bound of the extropy of m -gos emerging from any continuous distribution in terms of the extropy of m -gos of $U(0, 1)$ distribution and the mode of the parent PDF. Moreover, residual and past extropy of m -gos and m -dgos are derived. Meanwhile, the residual extropy of concomitants of m -gos, $\zeta_{[r;n,k,m]}(Y)$, of FGM distribution is presented. The measure $\zeta_{[r;n,k,m]}(Y)$ is discussed in terms of stop-loss transform and Gini coefficient. An alternative view of $\zeta_{[r;n,k,m]}(Y)$ depending on its upper bound is considered. Besides, some examples and numerical results and stochastic orders of $\zeta_{[r;n,k,m]}(Y)$ are obtained. Finally, we considered the problem of estimating $\zeta_{[r;n,k,m]}(Y)$ by proposing two different empirical estimators of CDF. We concluded that the proposed estimators are affected by sample size n , r , and α and generally the first empirical estimator is more accurate than the second estimator.

Data Availability

All the data sets are provided within the main body of the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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