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Retraction

Retracted: The MGHSS for Solving Continuous Sylvester Equation $AX + XB = C$

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Y.-Y. Feng, Q.-B. Wu, and X.-N. Jing, "The MGHSS for Solving Continuous Sylvester Equation $AX + XB = C$," *Complexity*, vol. 2021, Article ID 6615728, 8 pages, 2021.

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Research Article

The MGHSS for Solving Continuous Sylvester Equation $AX + XB = = C$

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This paper proposes the modified generalization of the HSS (MGHSS) to solve a large and sparse continuous Sylvester equation, improving the efficiency and robustness. The analysis shows that the MGHSS converges to the unique solution of $AX + XB = C$ unconditionally. We also propose an inexact variant of the MGHSS (IMGHSS) and prove its convergence under certain conditions. Numerical experiments verify the efficiency of the proposed methods.

1. Introduction

This paper focuses on solving the continuous Sylvester equation defined as

$$AX + XB = C. \tag{1}$$

Firstly, we assume A, B, and C are large and sparse matrices, and $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, and $C \in \mathbb{C}^{m\times n}$, respectively; then, equation (1) is a large sparse equation. Then, we assume that both A and B are positive semidefinite, and at least one of them is positive definite and at least one of A and B is non-Hermitian.

Under the above assumptions, it is sufficient to prove that equation (1) has a unique solution [1]. When $B = A^*$ and C is Hermitian, the continuous Sylvester equation (1) is a special case of the continuous Lyapunov equation [2]. A^* indicates the conjugate transpose of A. The continuous Sylvester equation (1) has numerous applications in many fields, such as in control and system theory [3], signal processing [4] and image restoring [5], and stability of linear systems [6]. Many authors considered such a linear matrix equation problem and concentrated on accelerating the HSS iteration on the continuous Sylvester equation (1) [7–10], which was first proposed in [11].

By using Kronecker Product, equation (1) is rewritten as the following linear equation:

$$\mathscr{A}x = c, \tag{2}$$

where $\mathscr{A} = I_n \otimes A + B^T \otimes I_m$, and x and c are the columns of X and C, respectively. I_m represents the identity matrix whose order is m, and $\otimes$ is the Kronecker Product. However, when the size of the linear equation (2) is large, it is ill-conditioned to solve it directly.

Before the appearance of the HSS, direct algorithms were usually used to solve the continuous Sylvester equations, such as Hessenberg–Schur and Bartels–Stewart methods [1, 12]. However, they were only applicable to small-sized continuous Sylvester equations. For large and sparse continuous Sylvester equations, iteration methods were used, such as the gradient-based algorithm [13–18]. Such an iteration method has been studied in recent years, taking advantage of the low-rank and sparsity of right-hand C in equation (1).

In 2011, Bai proposed the HSS to solve large sparse continuous Sylvester equation [11]. Since then, many HSS-based iteration methods [19–23] have been widely studied and achieved certain results in solving the continuous Sylvester equation. In the same direction of the research, this paper presents a modified GHSS method to solve the continuous Sylvester equations. Besides, there are numerical research studies which focus on solving complex Sylvester matrix equation with large size, based on the HSS method for solving (1) which is proposed in [11]. Modified HSS (MHSS) iteration method [24] and preconditioned MHSS (PMHSS) method [9] for solving complex Sylvester matrix equation were presented, and then, the generalized MHSS (GMHSS) method [10] is also based on the MHSS iteration method by parameterizing it. In recent years, some neural network methods for time-varying complex Sylvester equation were proposed [25, 26]. Many methods are updated to solve various types of Sylvester equation. In this paper, we focus on solving continuous Sylvester equation with non-Hermitian and positive definite/semidefinite matrices.

Firstly, the Hermitian and skew-Hermitian splitting method in equation (1) is used [27]. Let the Hermitian part of V be $H(V) = 1/2(V + V^*)$ and the skew-Hermitian part of V be $S(V) = 1/2(V - V^*)$.

HSS [2]: the following equations are computed with an initial matrix $x^{(0)}$ until $\{x^{(k)}\}$, where $k = 0, 1, 2\ldots$ is converged:

$$\begin{cases} (\alpha I + H)x^{(k+1/2)} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+1/2)} + b. \end{cases} \tag{3}$$

where $\alpha > 0$ is a constant.

HSS for solving continuous Sylvester equations [11]: $X^{(k+1)} \in \mathbb{C}^{m\times n}$ is computed with an initial matrix $x^{(0)}$ through the following equations until $\{X^{k+1}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + H(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I + H(B)) \\ = (\alpha I - S(A))X^{(k)} + X^{(k)}(\beta I - S(B)) + C, \\ (\alpha I + S(A))X^{(k+1)} + X^{(k+1)}(\beta I + S(B)) \\ = (\alpha I - H(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I - H(B)) + C, \end{cases} \tag{4}$$

where $\alpha > 0$ and $\beta > 0$ are constants. The iteration matrix for solving continuous Sylvester equation is $M(\gamma) = (\gamma I + \mathbf{S})^{-1}(\gamma I - \mathbf{H})(\gamma I + \mathbf{H})^{-1}(\gamma I - \mathbf{S})$ and $\gamma = \alpha + \beta$, and it converges to the exact solution: $\mathbf{H} = I \otimes H(A) + H(B)^T \otimes I$ and $\mathbf{S} = I \otimes S(A) + S(B)^T \otimes I$.

GHSS [28]: similar to HSS, the following equations are computed until $\{x^{(k)}\}$ is converged:

$$\begin{cases} (\alpha I + G)x^{(k+1/2)} = (\alpha I - K - S)x^{(k)} + b, \\ (\alpha I + S + K)x^{(k+1)} = (\alpha I - G)x^{(k+1/2)} + b, \end{cases} \tag{5}$$

where $\alpha > 0$ is a constant.

GHSS for solving continuous Sylvester equations [8]: in GHSS, to solve continuous Sylvester equations, $X^{(k+1)} \in \mathbb{C}^{m\times n}$, where $k = 0, 1, 2\ldots$ is computed through the following scheme until $\{X^{k+1}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + G(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I + G(B)) \\ = (\alpha I - S(A) - K(A))X^{(k)} + X^{(k)}(\beta I - S(B) - K(B)) + C, \\ (\alpha I + S(A) + K(A))X^{(k+1)} + X^{(k+1)}(\beta I + S(B) + K(B)) \\ = (\alpha I - G(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I - G(B)) + C, \end{cases} \tag{6}$$

where $\alpha > 0$ and $\beta > 0$ are constants. The iteration matrix is $\mathcal{M}(\gamma) = (\gamma I + \mathbf{S} + \mathbf{K})^{-1}(\gamma I - \mathbf{G})(\gamma I + \mathbf{G})^{-1}(\gamma I - \mathbf{S} - \mathbf{K})$ and $\mathbf{G} = I \otimes G(A) + G(B)^T \otimes I$, $\mathbf{K} = I \otimes K(A) + K(B)^T \otimes I$, and $\mathbf{S} = I \otimes S(A) + S(B)^T \otimes I$. The convergence factor is given by the spectral radius $\rho(\mathcal{M}(\gamma))$ of the matrix $\mathcal{M}(\gamma)$, bounded by $\rho(\mathcal{M}(\gamma)) \le \sigma(\gamma) := \max_{\lambda_i \in \lambda}(G(A)) \max_{\mu_i \in \lambda(G(B))} |\gamma - (\lambda_i + \mu_i)/\gamma + (\lambda_i + \mu_i)|$, and the optimal parameter $\gamma^* = \sqrt{\lambda_{\min}\lambda_{\max}}$, where $\lambda_{\min} = \lambda_{\min}^{(G(A))} + \lambda_{\min}^{(G(B))}$ and $\lambda_{\max} = \lambda_{\max^{(G(A))}} + \lambda_{\max^{(G(B))}}$.

As an extension of those iterative schemes, this paper proposes the modified GHSS to solve the continuous Sylvester equations and proves its convergence. Section 2 presents the detailed MGHSS and analyzes its convergence for the continuous Sylvester equation. IMGHSS iteration is described in Section 3. In Section 4, we take two examples into experiments from previous experiments in other HSS-based iteration methods. The results show that the proposed MGHSS is more effective in both the iteration and runtime. Section 5 concludes this paper with several thoughts.

In the remainder of this paper, especially in the proof of the convergence property of MGHSS, we rewrite the continuous Sylvester equation (1) as the linear-vector form firstly. When the vector sequence $\{y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n^2}$ converges the vector $y \in \mathbb{C}^{n^2}$, it is equivalent as the corresponding matrix sequence $\{Y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n\times n}$ converges to the corresponding matrix $Y \in \mathbb{C}^{n\times n}$, where the corresponding vector y is the concatenated columns of the correspondent matrix Y.

2. The Modified GHSS Iteration Method

This paper proposes a modified GHSS, and a direct solver can be used to solve each step of the inner iteration.

Firstly, the GHSS splits A and B into generalized Hermitian and skew-Hermitian parts [8]:

$$A = G(A) + (S(A) + K(A)), B = G(B) + (S(B) + K(B)), \tag{7}$$

where $S(A)$ and $S(B)$ are the skew-Hermitian part of A and B, respectively, and $H(A) = G(A) + K(A)$ and $H(B) = G(B) + K(B)$ are the Hermitian part of A and B, respectively.

With matrix splitting and GHSS [8], $S(A)$ and $S(B)$ are decomposed into two skew-Hermitian matrices: $S(A) = R(A) + T(A)$ and $S(B) = R(B) + T(B)$. Then, A and B are rewritten:

$$
\begin{aligned}
A &= (\alpha I + G(A) + R(A)) - (\alpha I - T(A) - K(A)) \\
&= (\alpha I + T(A) + K(A)) - (\alpha I - G(A) - R(A)), \\
B &= (\beta I + G(B) + R(B)) - (\beta I - T(B) - K(B)) \\
&= (\beta I + T(B) + K(B)) - (\beta I - G(B) - R(B)).
\end{aligned} \tag{8}
$$

Accordingly, equation (1) is rewritten in the matrix equation as follows:

$$
\begin{cases}
(\alpha I + G(A) + R(A))X + X(\beta I + G(B) + R(B)) \\
= (\alpha I - T(A) - K(A))X + X(\beta I - T(B) - K(B)) + C, \\
(\alpha I + T(A) + K(A))X + X(\beta I + T(B) + K(B)) \\
= (\alpha I - G(A) - R(A))X + X(\beta I - G(B) - R(B)) + C.
\end{cases} \tag{9}
$$

It is known that no common eigenvalue exists between $\alpha I + G(A) + R(A)$ and $-(\beta I + G(B) + R(A))$ and between $(\alpha I + T(A) + K(A))$ and $-(\beta I + T(B) + K(B))$. Thus, there exist unique solutions in both two fixed-point matrix equations. Based on this, the MHSS iteration is conducted to solve equation (1).

MGHSS: $X^{(k+1)} \in \mathbb{C}^{m\times n}$ is computed with an initial matrix $x^{(0)}$ through equation (10). The process stops when $\{X^{k+1}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ satisfies the stopping criterion:

$$
\begin{cases}
(\alpha I + G(A) + R(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I + G(B) + R(B)) \\
= (\alpha I - T(A) - K(A))X^{(k)} + X^{(k)}(\beta I - T(B) - K(B)) + C, \\
(\alpha I + T(A) + K(A))X^{(k+1)} + X^{(k+1)}(\beta I + T(B) + K(B)) \\
= (\alpha I - G(A) - R(A))X^{(k+1/2)} + X^{(k+1/2)}(\beta I - G(B) - R(B)) + C.
\end{cases} \tag{10}
$$

Lemma 1 (see [11]). *Let M_i and N_i denote two split of matrix A, where $A = M_i - N_i\,(i = 1, 2)$. Then, a two-step iteration sequence $\{X^k\}$ is defined as follows:*

$$
\begin{cases}
M_1 X^{(k+1/2)} B = N_1 X^{(k)} B + C, \\
M_2 X^{(k+1)} B = N_2 X^{(k+1/2)} B + C,
\end{cases} \tag{11}
$$

where $A, B, C \in \mathbb{C}^{m\times n}$ and $k = 1, 2 \ldots$, $x^{(0)}$ is an initial matrix. Then,

$$
X^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 X^{(k)} + M_2^{-1}\left(I + N_2^{-1} M_1\right)\mathrm{CB}^{-1}. \tag{12}
$$

This is rewritten as vector form as follows:

$$
\begin{aligned}
x^{(k+1)} &= I \otimes \left(M_2^{-1} N_2 M_1^{-1} N_1\right) x^{(k)} \\
&\quad + \left(B^{-T} \otimes M_2^{-1}\left(I + N_2 M_1^{-1}\right)\mathrm{vec}(C)\right.
\end{aligned} \tag{13}
$$

Furthermore, when the spectral radius $\rho(I \otimes (M_2^{-1} N_2 M_1^{-1} N_1)) < 1$, $\{X^k\}$ converges to $X^* \in \mathbb{C}^{m\times n}$ for all $X^{(0)} \in \mathbb{C}^{m\times n}$.

Lemma 2 (see [29, 30]). *Let $A = H + S$, where $H = 1/2(A + A^*)$. When H is positive semidefinite and $\alpha \geqslant 0$,*

$$
\left\|(\alpha I - A)(\alpha I + A)^{-1}\right\| \leq 1. \tag{14}
$$

When H is positive definite and $\alpha > 0$,

$$
\left\|(\alpha I - A)(\alpha I + A)^{-1}\right\| < 1. \tag{15}
$$

Theorem 1. *Let $\mathscr{A} = (\mathbf{G} + \mathbf{K}) + (\mathbf{R} + \mathbf{T})$, where*

$$
\begin{aligned}
\mathbf{G} &= I_n \otimes G(A) + G(B)^T \otimes I_m, \\
\mathbf{K} &= I_n \otimes K(A) + K(B)^T \otimes I_m, \\
\mathbf{R} &= I_n \otimes R(A) + R(B)^T \otimes I_m. \\
\mathbf{T} &= I_n \otimes T(A) + T(B)^T \otimes I_m, \\
\gamma &= \alpha + \beta,
\end{aligned} \tag{16}
$$

where R and T are skew-Hermitian matrices and G and K are symmetric positive semi-definite. The unique solution of (1) obtained by the MGHSS converges to the unique exact solution $X^* \in \mathbb{C}^{m\times n}$ when either G and K are symmetric positive definite.

Proof. By using the Kronecker product, we can rewrite (10) as follows:

$$
\begin{cases}
(\gamma I + \mathbf{G} + \mathbf{R})\mathrm{vec}\left(X^{(k+1/2)}\right) = (\gamma I - \mathbf{T} - \mathbf{K})\mathrm{vec}\left(X^{(k)}\right) + \mathrm{vec}(C), \\
(\gamma I + \mathbf{T} + \mathbf{K})\mathrm{vec}\left(X^{(k+1)}\right) = (\gamma I - \mathbf{G} - \mathbf{R})\mathrm{vec}\left(X^{(k+1/2)}\right) + \mathrm{vec}(C).
\end{cases} \tag{17}
$$

Then, it can reformulated as follows by eliminating $X^{(k+1/2)}$:

$$
\mathrm{vec}\left(X^{(k+1)}\right) = \mathscr{M}_\gamma \mathrm{vec}\left(X^{(k)}\right) + \mathscr{G}_\gamma \mathrm{vec}(C), \tag{18}
$$

where

$$
\begin{cases}
\mathscr{M}_\gamma = (\gamma I + \mathbf{T} + \mathbf{K})^{-1}(\gamma I - \mathbf{G} - \mathbf{R})(\gamma I + \mathbf{G} + \mathbf{R})^{-1}(\gamma I - \mathbf{T} - \mathbf{K}), \\
\mathscr{G}_\gamma = \dfrac{1}{2\alpha}(\gamma I + \mathbf{T} + \mathbf{K})^{-1}(\gamma I + \mathbf{G} + \mathbf{R})^{-1}.
\end{cases} \tag{19}
$$

By a similar transformation of the components of the iteration matrix $\mathscr{M}_\gamma$, we obtain

$$\widehat{\mathscr{M}}_\gamma = (\gamma I - \mathbf{G} - \mathbf{R})(\gamma I + \mathbf{G} + \mathbf{R})^{-1}(\gamma I - \mathbf{T} - \mathbf{K})(\gamma I + \mathbf{T} + \mathbf{K})^{-1}. \tag{20}$$

Denote $\mathscr{A}_1 = \mathbf{G} + \mathbf{R}$ and $\mathscr{A}_2 = \mathbf{T} + \mathbf{K}$, then $\mathscr{A}_1$ and $\mathscr{A}_2$ are positive semidefinite, and clearly,

$$\widehat{\mathscr{M}}_\gamma = (\gamma I - \mathscr{A}_1)(\gamma I + \mathscr{A}_1)^{-1}(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}. \tag{21}$$

From Lemma 2, we have

$$\begin{aligned} \left\|(\gamma I - \mathscr{A}_1)(\gamma I + \mathscr{A}_1)^{-1}\right\| &\leq 1, \\ \left\|(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}\right\| &\leq 1. \end{aligned} \tag{22}$$

Respectively, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are positive definite, the above inequalities are strict. Thus, when either G and K is symmetric positive definite, we have

$$\begin{aligned} \rho\left(\mathscr{M}_\gamma\right) = \rho\left(\widehat{\mathscr{M}}_\gamma\right) &\leq \left\|(\gamma I - \mathscr{A}_1)(\gamma I + \mathscr{A}_1)^{-1}\right\| \\ &\cdot \left\|(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}\right\| < 1, \end{aligned} \tag{23}$$

for any $\gamma > 0$, completing the proof. Accordingly, the MGHSS unconditionally converges to the exact solution of equation (1).

From the results in Chapter 4 in [31], denote the inner product $(x, y) = x^T y$. We know under the above definitions of $\mathscr{A}$, $\mathbf{G}$, $\mathbf{K}$, $\mathbf{R}$, $\mathbf{T}$, $\mathscr{A}_1$, and $\mathscr{A}_2$, the following inequalities are satisfied:

$$(\mathscr{A}_i u, u) \geq m(u, u) \text{ and } (\mathscr{A}_i u, u) \leq M(u, u), \quad (i = 1, 2). \tag{24}$$

The upper bound on the spectral radius of the iteration matrix $\mathscr{M}_\gamma$ is minimized with the parameter γ, which is defined as follows:

$$\gamma_{\text{opt}} = \frac{1}{\sqrt{mM}}. \tag{25}$$

It indicates that finding the optimal parameter γ is challenging but necessary because it relies on the spectral information of the selected iteration matrix.

In the following section, the improved MGHSS, the inexact MGHSS (IMGHSS) is introduced.

3. The Inexact MGHSS Method

Unlike MGHSS (Section 3) that solves the two fixed-point equations by direct algorithms, IMGHSS, presented in this section, iteratively solves the two subsystems. Similar to the IHSS [32, 33] for solving linear systems and IHSS for solving Sylvester equation [11], the process of the IMGHSS iteration scheme for solving a continuous Sylvester equation is as follows. Here, we denote $\|\cdot\|_F$ as Frobenius norm.

IMGHSS: $X^{(0)} \in \mathbb{C}^{m\times n}$ is an initial matrix. In the IMGHSS algorithm, the solution of equation (1) is derived as the following:

$$k = 0, \tag{26}$$

while (stopping condition = = false)

$$R^{(k)} = C - AX^{(k)} - X^{(k)}B \tag{27}$$

is approximated to $(\alpha I + G(A) + R(A))Z^{(k)} + Z^{(k)}(\beta I + G(B) + R(A)) = R^{(k)}$ that is solved by the residual $P^{(k)} = R^{(k)} - (\alpha I + G(A) + R(A))Z^{(k)} - (\beta I + G(B) + R(A))$ of the iteration satisfies $\|P^{(k)}\|_F \leq \eta_k \|R^{(k)}\|_F$:

$$X^{(k+1/2)} = X^{(k)} + Z^{(k)}, \tag{28}$$

$$R^{(k+1/2)} = C - AX^{(k+1/2)} - X^{(k+1/2)}B, \tag{29}$$

is approximated to $(\alpha I + T(A) + K(A))Z^{(k+1/2)} + Z^{(k+1/2)}(\beta I + T(B) + K(B))$ that is solved by the residual $Q^{(k+1/2)} = R^{(k+1/2)} - (\alpha I + T(A) + K(A))Z^{(k+1/2)} + Z^{(k+1/2)}(\beta I + T(B) + K(B))$ of the iteration which satisfies $\|Q^{(k+1/2)}\|_F \leq \varepsilon_k \|R^{(k+1/2)}\|_F$:

$$X^{(k+1)} = X^{(k+1/2)} + Z^{(k+1/2)}, \tag{30}$$

$$k = k + 1, \tag{31}$$

end.

In the scheme of IMGHSS, ε_k and η_k are prescribed tolerances that control the accuracy of the inner iterations. In implements, the values of ε_k and η_k do not necessarily decrease to zero when k increases so long as we choose suitable values of it, and we can also ensure the convergence of the IMGHSS.

In [11], the convergence of the two-step iteration was explored. We analyze the convergence of IMGHSS as follows.

Theorem 2. *Let $\{X^k\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ denote an iteration sequence produced by IMGHSS and $X^* \in \mathbb{C}^{m\times n}$ denote the exact solution of equation (1). Then, under the assumption that the conditions of Theorem 1 are met, it holds*

$$\left\|X^{k+1} - X^*\right\|_S \leq \left(\delta(\gamma) + \theta\rho\tau_k\delta_2\right)\left(1 + \theta\eta_k\delta_2^{-1}\right)\left\|X^{(k)} - X^*\right\|_S, \quad k = 1, 2\cdots, \tag{32}$$

where the norm $\|\cdot\|_S$ is defined as follows:

$$\|Y\|_S = \|(\alpha I + T(A) + K(A))Y + Y(\beta I + T(B) + K(B))\|_F, \tag{33}$$

for any matrix $Y \in \mathbb{C}^{m\times n}$, and the constant ρ and θ are given by $\rho = \|(\gamma I + \mathbf{T} + \mathbf{K})(\gamma I + \mathbf{G} + \mathbf{R})^{-1}\|_2$, $\theta = \|A(\gamma I + \mathbf{T} + \mathbf{K})^{-1}\|_2$, and $\delta(\gamma) = \delta_1\delta_2$, where $\delta_1 = \|(\gamma I - \mathscr{A}_1)(\gamma I + \mathscr{A}_1)^{-1}\|$ and $\delta_2 = \|(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}\|$.

In particular, when $(\delta(\gamma) + \theta\rho\varepsilon_{\max}\delta_2)(1 + \theta\eta_{\max}\delta_2^{-1}) < 1$, the iteration sequence $\{X^k\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ converges to $X^* \in \mathbb{C}^{m\times n}$, where $\eta_{\max} = \max_k\{\eta_k\}$ and $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$.

Proof. The IMGHSS can be rewritten in the matrix-vector form with the notations in Theorem 1 and the Kronecker product as follows:

$$\begin{cases} (\alpha I + \mathbf{G} + \mathbf{R})z^{(k)} = r^{(k)}, x^{(k+1/2)} = x^{(k)} + z^{(k)}, \\ (\alpha I + \mathbf{T} + \mathbf{K})z^{(k+1/2)} = r^{(k+1/2)}, x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}, \end{cases} \tag{34}$$

where $r^{(k)} = c - \mathscr{A}x^{(k)}$ and $r^{(k+1/2)} = c - \mathscr{A}x^{(k+1/2)}$. $z^{(k)}$ is such that the residual $p^{(k)} = r^{(k)} - (\alpha I + \mathbf{G} + \mathbf{R})z^{(k)}$ satisfies $\|p^{(k)}\|_2 \le \eta_k\|r^{(k)}\|_2$, and $z^{(k+1/2)}$ is such that the residual $q^{(k+1/2)} = r^{(k+1/2)} - (\alpha I + \mathbf{T} + \mathbf{K})z^{(k+1/2)}$ satisfies $\|q^{(k+1/2)}\|_2 \le \varepsilon_k \,\|\, \|r^{(k+1/2)}\|_2$.

Equation (34) is the IMGHSS for solving (2), with $\mathscr{A} = (\mathbf{G} + \mathbf{K}) + (\mathbf{R} + \mathbf{T})$. Then, based on Theorem 2 in [11], we have

$$\left\|x^{k+1} - x^*\right\|_S \le \left(\delta(\gamma) + \theta\rho\varepsilon_k\delta_2\right)\left(1 + \theta\eta_k\delta_2^{-1}\right)\left\|x^{(k)} - x^*\right\|_S, \quad k = 1, 2\cdots, \tag{35}$$

where the constants are given by

$$\begin{aligned} \rho &= \left\|(\gamma I + \mathbf{T} + \mathbf{K})(\gamma I + \mathbf{G} + \mathbf{R})^{-1}\right\|_2, \\ \theta &= \left\|\mathscr{A}(\gamma I + \mathbf{T} + \mathbf{K})^{-1}\right\|_2, \end{aligned} \tag{36}$$

and $\delta(\gamma) = \|(\gamma I - \mathscr{A}_1)(\gamma I + \mathscr{A}_1)^{-1}\| \ \|(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}\|$ and $\delta_2 = \|(\gamma I - \mathscr{A}_2)(\gamma I + \mathscr{A}_2)^{-1}\|$.

For a vector $y \in \mathbb{C}^{m\times n}$ that consists of the concatenated columns of Y, $\|y\|_S = \|(\alpha I + T(A) + K(A))Y + Y(\beta I + T(B) + K(B))\|_F$.

Thus, the following is obtained:

$$\left\|X^{k+1} - X^*\right\|_S \le \left(\delta(\gamma) + \theta\rho\varepsilon_k\delta_2\right)\left(1 + \theta\eta_k\delta_2^{-1}\right)\left\|X^{(k)} - X^*\right\|_S, \quad k = 1, 2\cdots. \tag{37}$$

The proof is completed.

According to Theorem 2, it is important to choose a suitable value of the tolerance ε_k and η_k to control the IMGHSS's convergence. Still, analyzing the optimal tolerances is challenging.

4. Numerical Analysis

The feasibility and efficiency of the MGHSS are verified in several examples in this section. The proposed method was compared with other methods in terms of the number of iteration steps (n_{is}) and the computational time (t [sec]). The numerical analysis was conducted in Matlab on Intel dual-core CPU (2.5 GHz) and 8 GB RAM. Zero matrix was used as an initial guess, and the termination condition is defined as

$$\frac{\left\|C - AX^{(k)} - X^{(k)}B\right\|_F}{\|C\|_F} \le 10^{-6}. \tag{38}$$

Example 1. The continuous Sylvester equation (1) with $m = n$ is considered, and the matrices are as follows:

$$M = \begin{pmatrix} 2.3 & -1 & & & \\ -1 & 2.3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2.3 & -1 \\ & & & -1 & 2.3 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0.5 & & & \\ -0.5 & 0 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.5 & 0 & 0.5 \\ & & & -0.5 & 0 \end{pmatrix}, \tag{39}$$

$A = B = M + 2rN + 100/(n+1)^2 I$, where I represents the identity matrix.

For the practical iteration parameters of all those iteration methods, we take $\alpha = \beta$. Also, all the subproblems are exactly solved by the Bartels–Stewart method [3] in each step of the HSS, GHSS, and MGHSS. Tables 1 and 2 compare HSS and MGHSS in solving the continuous Sylvester equation (1) in terms of n_{is} and t[sec]. The optimal values of $\alpha_{\exp}$ and $\beta_{\exp}$ ($\alpha_{\exp} = \beta_{\exp}$) were analyzed in Tables 3 and 4, respectively, for HSS/MGHSS and GHSS/MGHSS.

According to Table 3 and Table 4, as the rank n of equation (2) is incremented, $\alpha_{\exp}$ and $\beta_{\exp}$ of the HSS, GHSS, and MGHSS are all decreased. In Figure 1, the logarithm versus iteration of the HSS, GHSS, and MGHSS methods ($n = 128$) are shown in (a) and (b) when $r = 0.1$ and $r = 1.0$, respectively. It shows the efficiency of the MGHSS method.

Example 2. The continuous Sylvester equation (1) with $m = n$ and the matrices

$A = \mathrm{diag}(1, 2, \ldots, n) + rL^T$ and $B = 2^{-p}I + \mathrm{diag}(1, 2, \ldots, n) + rL^T + 2^{-p}L$, where L is the strictly lower triangular matrix with ones in the lower triangle part and p is a problem parameter specified in practical computations.

Table 5 shows that the MGHSS outperforms the GHSS and HSS in solving the continuous Sylvester equation. In Table 6, the continuous Sylvester equation in Example 2 are solved by the IMGHSS and MGHSS iteration methods and the results show that IMGHSS is much better than the MGHSS. Here, we set $\varepsilon_k = \eta_k = 0, 01, k = 0, 1, 2, \ldots$, and use the ADI method as the inner iteration scheme.

Table 1: Comparison of HSS and MGHSS in terms of n_{is} and t.

	HSS						MGHSS					
r	0.01		0.1		1.0		0.01		0.1		1.0	
	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t
$n = 8$	12	0.002	13	0.007	11	0.007	11	0.001	11	0.001	9	0.004
$n = 16$	20	0.012	23	0.013	15	0.017	19	0.003	20	0.004	16	0.009
$n = 32$	44	0.076	43	0.080	32	0.055	43	0.014	41	0.074	23	0.028
$n = 64$	86	1.556	73	0.543	44	0.332	72	0.622	68	0.494	29	0.077
$n = 128$	170	10.114	119	13.956	69	1.547	78	4.469	79	4.870	51	2.435
$n = 256$	335	66.758	175	34.734	83	14.378	78	27.889	79	29.082	74	11.475

Table 2: Comparison of GHSS and MGHSS in terms of n_{is} and t.

	GHSS						MGHSS					
r	0.01		0.1		1.0		0.01		0.1		1.0	
	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t	n_{is}	t
$n = 8$	11	0.003	12	0.002	10	0.004	11	0.001	11	0.001	9	0.004
$n = 16$	19	0.012	15	0.05	17	0.048	19	0.003	20	0.004	16	0.009
$n = 32$	16	0.121	17	0.175	29	0.065	43	0.014	41	0.074	23	0.028
$n = 64$	27	0.864	31	0.842	41	0.834	72	0.622	68	0.494	29	0.077
$n = 128$	44	2.754	45	6.347	55	19.437	78	4.469	79	4.870	51	2.435
$n = 256$	68	30.584	67	31.462	167	125.643	78	27.889	79	29.082	74	11.475

Table 3: The analysis of the optimal values $\alpha_{\exp}$ and $\beta_{\exp}$ for HSS and MGHSS.

r	HSS			MGHSS		
	0.01	0.1	1.0	0.01	0.1	1.0
$n = 8$	2.75	2.55	2.45	2.65	2.48	2.61
$n = 16$	1.26	1.26	1.48	1.25	1.3	2.53
$n = 32$	0.64	0.65	1.15	0.62	0.63	1.78
$n = 64$	0.32	0.32	0.93	0.27	0.3	1.71
$n = 128$	0.16	0.2	0.75	0.06	0.05	0.81
$n = 256$	0.83	0.14	0.65	0.01	0.04	0.71

Table 4: The analysis of the optimal values $\alpha_{\exp}$ and $\beta_{\exp}$ for GHSS and MGHSS.

r	GHSS			MGHSS		
	0.01	0.1	1.0	0.01	0.1	1.0
$n = 8$	2.05	2.1	2.35	2.65	2.48	2.61
$n = 16$	0.26	0.36	1.28	1.25	1.3	2.53
$n = 32$	0.34	0.35	1.05	0.62	0.63	1.78
$n = 64$	0.32	0.36	0.83	0.27	0.3	1.71
$n = 128$	0.26	0.3	0.75	0.06	0.05	0.81
$n = 256$	0.3	0.34	0.55	0.01	0.04	0.71

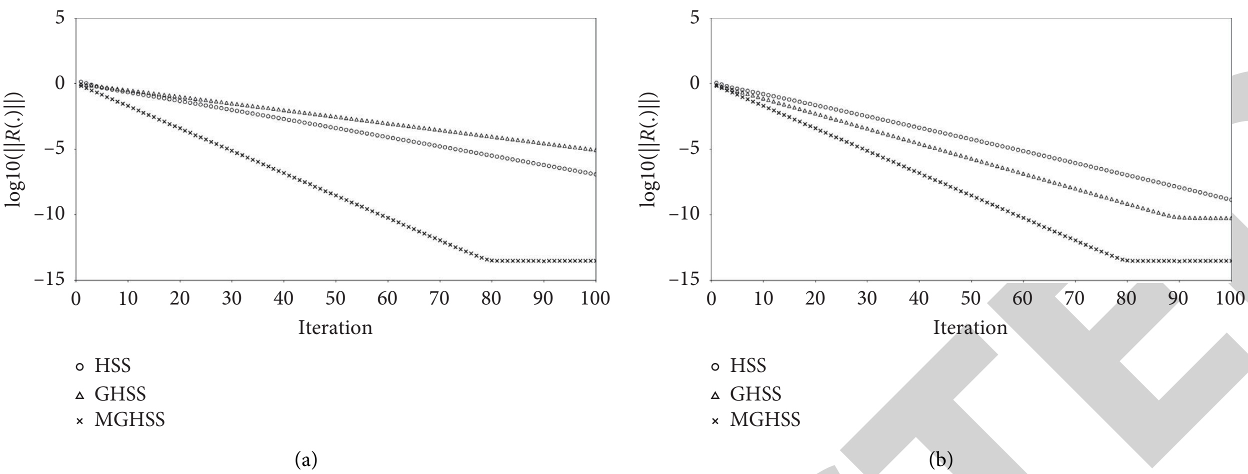

Figure 1: The logarithm $\|R(X^{(k)})\|$ versus iteration of the HSS, GHSS, and MGHSS methods.

Table 5: Comparisons of HSS, GHSS, and MGHSS in terms of n_{is} and t.

	HSS		GHSS		MGHSS	
	n_{is}	t	n_{is}	t	n_{is}	t
$n = 8$	17	0.014	17	0.022	17	0.013
$n = 16$	27	0.035	21	0.087	25	0.027
$n = 32$	35	0.608	29	0.453	34	0.048
$n = 64$	50	3.244	34	3.231	47	0.303
$n = 128$	71	19.539	43	10.543	64	3.87
$n = 256$	100	30.76	50	20.432	78	9.095

Table 6: Comparison of IMGHSS and MGHSS in terms of n_{is} and t.

	IMGHSS		MGHSS	
	n_{is}	t	n_{is}	t
$n = 8$	8	0.011	17	0.013
$n = 16$	19	0.021	25	0.027
$n = 32$	24	0.029	34	0.048
$n = 64$	27	0.256	47	0.303
$n = 128$	33	1.123	64	3.87

5. Conclusions

HSS-based methods have been widely used to solve the continuous Sylvester equations. In this paper, a modified generalization of the HSS method (MGHSS) is proposed. A preconditioner can also be taken for all of the generalizations of the HSS, although many researchers concentrated on the studies of the relations between parameters and the convergence property of each. Furthermore, we establish the IMGHSS as an efficient solver. The convergence of the MGHSS and IMGHSS were analyzed. Also, the efficiency and robustness of the proposed method were verified in several examples.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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