Averaging Principle for Caputo Fractional Stochastic Differential Equations Driven by Fractional Brownian Motion with Delays

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In this article, we investigate a class of Caputo fractional stochastic differential equations driven by fractional Brownian motion with delays. Under some novel assumptions, the averaging principle of the system is obtained. Finally, we give an example to show that the solution of Caputo fractional stochastic differential equations driven by fractional Brownian motion with delays converges to the corresponding averaged stochastic differential equation.

1. Introduction

Fractional differential equations have been developing as an active area on medicine, electrical engineering, biochemistry, and mechanical systems [1–5]. Because the systems are often subjected to noisy fluctuations, it is important to consider randomness into models. Since the fractional Brownian motion (fBm for short) owns many excellent properties, for example, long-range dependence and self-similar, it is usually used to describe the uncertainty. Since then, stochastic calculus with respect to fBm has been paid much attention in the stochastic analysis field, and many interesting works have obtained both qualitative and quantitative properties of stochastic differential equations (SDEs for short) driven by fBm [6–8]. Furthermore, the applications of fractional stochastic differential equations (FSDEs for short) driven by fBm have been widely applied in mathematical quantum, physics, and biology [9–11].

For the deterministic systems, many varieties of methods are proposed for average systems, such as gradient-based and least squares-based iterative algorithms for Hammerstein systems using the hierarchical identification principle, two-stage least squares-based iterative estimation algorithm for CARARMA system modeling, decomposition-based fast least squares algorithm for output error systems, and gradient-based and least squares-based iterative estimation algorithms for multi-input multi-output systems [12–14]. Compared to the deterministic systems, due to influence of stochastic factors, the exact solution of FSDEs driven by fBm is difficult to realize, and the above-mentioned methods do not work. Because the averaging principle shows that the complex original systems can be ignored and one can only concentrate on the average systems instead, it is usually taken as an effective tool to reduce the amount of calculation of the original systems. Khasminskii [15] first started with the averaging method to approximate the complex system with a simpler system. In recent years, the averaging method has been developed in many ways [16–18]. For example, Pei et al. [19] investigated stochastic averaging for stochastic differential equations driven by fBm and Brownian motion. Recently, Luo et al. [20] discussed the averaging principle for FSDEs of Itô-Doob with delays driven by Brownian motion. Xu et al. [21] presented the averaging principle for stochastic differential equations with Caputo fractional derivative.

Inspired by the above works, we will discuss averaging principle of a new kind of SDEs with Caputo derivative driven by fBm and Brownian motion, which is a general case of [19, 20]. On the other hand, because Lipschitz conditions restrict the application, we will adopt weakened Lipschitz conditions to obtain the result. Moreover, in order to overcome the influence of Caputo derivative and fBm, we introduce a new averaging method to realize the stochastic averaging principle.
In this article, we will deal with averaging principle of the following Caputo FSDEs driven by fBm and Brownian motion with delays:

\[
\begin{aligned}
\frac{d}{dt}X(t) &= f(t, X(t), X_\alpha) + g(t, X(t), X_\alpha)
+ \frac{dW_t}{dt} + \sigma(t, X(t), X_\alpha) \frac{dB^H_t}{dt}, \\
X(t) &= \varphi(t),
\end{aligned}
\]

where \(D^\alpha_t\) is the Caputo fractional derivative, \(\alpha \in (1/2, 1)\). \(f: [0, T] \times U \times X \rightarrow U, g: [0, T] \times U \times X \rightarrow \mathcal{L}_2^0(V, U), \sigma: [0, T] \times U \times X \rightarrow \mathcal{L}_2^0(V, U)\). \(B^H_t\) is a \(V\)-valued Q-cylindrical fBm with the Hurst parameter \(H \in (1/2, 1)\), \(W_t\) is a standard Wiener process on a real and separable Hilbert space \(V\) independent of \(B^H_t\), and \(X_\alpha = \{X(t + \theta), \theta \in [-\tau, 0]\}\) is the \(B\) value stochastic process. The initial value \(\varphi = [\varphi(\theta): -\tau \leq \theta \leq 0]\) is an \(\mathcal{F}_0\)-measurable \(B\)-valued random variable independent of fBm \(B_t^H\) and Wiener process \(W_t\) with finite second moment.

The rest part is arranged as follows. Section 2 is devoted to some preliminary results and assumptions. In Section 3, the averaging principle is presented. An example is provided to show the result in Section 4.

2. Preliminary

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a completed probability space. For any \(t \in [0, T]\), \(\mathcal{F}_t\) denotes the \(\sigma\) field generated by \(B^H_t, W_t, s \in [0, t]\), and all \(\mathbb{P}\) null sets. A one-dimensional fractional Brownian motion with Hurst parameter \(H \in (0, 1)\) is a centered Gaussian process \(\beta^H_t = \beta^H(t)\) with the covariance function:

\[
R(t, s) = E[\beta^H(t) \beta^H(s)] = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right).
\]

For \(H > 1/2\), \(\beta^H_t(t)\) has the following representation:

\[
\beta^H_t(t) = \int_0^t K(t, s) d\beta(s),
\]

where \(K(t, s) = c_H s^{(1/2) - H} \int_s^t (u - s)^{H - 3/2} u^{1/2 - H} du, \quad t \geq s\), \(c_H\) is a nonnegative constant with respect to \(H\).

For function \(\varphi \in L^2([0, T])\), the fractional Wiener integral of \(\varphi\) with respect to \(\beta^H\) is defined by

\[
\int_0^T \varphi(s) d\beta^H(s) = \int_0^T K^*_H \varphi(s) d\beta(s),
\]

where \(K^*_H\) is an operator defined by \(K^*_H \varphi(s) = \int_0^s \varphi(t) (\partial K(t, s)/\partial t) dt\).

In the following parts, we shall introduce Wiener integral with respect to the Q-fBm \(B^H_t\). Let \(\mathcal{L}^0_2 = \mathcal{L}^0_2(V, U)\) denote the space of all \(Q\)-Hilbert–Schmidt operators \(Q: V \rightarrow U\) equipped with the norm

\[
\|Q\|_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \|\lambda_n Q e_n\|_V^2 < \infty,
\]

for all \(n \in \mathbb{N}\), and the inner product \(\langle \varphi, \psi \rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle\) for \(\varphi, \psi \in \mathcal{L}^0_2\).

Now, we give the definition of the fractional Wiener integral of the function \(\psi: [0, T] \rightarrow \mathcal{L}^0_2\) with respect to Q-fBm as follows:

\[
\int_0^T \psi(s) d\beta^H(s) = \sum_{n=1}^{\infty} \int_0^T \psi(s) Q^{1/2} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^T \left( K^*_H(\psi Q^{1/2} e_n) \right) (s) d\beta_n(s),
\]

for all \(x, y \in V\) and \(t, s \in [0, T]\).
where $\beta_n$ is the standard Brownian motion with respect to $\beta^H_n$.

We introduce $B([-\tau,0],L^2(\Omega, U))$ ($B$ for simply) denotes the family of all $\mathcal{F}_0$-measurable bounded continuous functions $\xi$: $[-\tau,0] \rightarrow L^2(\Omega, U)$ endowed with the norm $\|\xi(t)\|^2 = \sup_{\tau \leq s \leq T} E\|\xi(s)\|^2$. It is important of the following lemma to prove our main results, which is appeared in [22].

**Lemma 1.** If $\psi$: $[0,T] \rightarrow \mathcal{L}^0_2(V, U)$ satisfies
\begin{equation}
\int_0^T \|\psi(t)\|^2_{\mathcal{L}_2^3} dt < \infty,
\end{equation}
then, for any $0 \leq s \leq t \leq T$,
\begin{equation}
E\left(\int_s^t \sigma(r)dB_r^H\right)^2 \leq C_H (t-s)^{2H-1} \int_s^t \|\sigma(r)\|^2_{\mathcal{L}^0_2} dr.
\end{equation}

Now, we recall some notations and preliminary results about fractional calculus and some special functions.

**Definition 1.** For any $\alpha \in (0,1)$ and function $f$: $[0,T] \rightarrow U$, the Riemann–Liouville fractional integral operator of order $\alpha$ is defined
\begin{equation}
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad 0 \leq t \leq T,
\end{equation}
where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

**Definition 2.** The Caputo fractional derivative with order $\alpha$ of function $f(t) \in \mathcal{L}^m([0,T]; U)$ is defined as
\begin{equation}
D^\alpha f(t) = \begin{cases}
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, & n-1 < \alpha < n, \\
\frac{d^n}{dt^n} f(t), & \alpha = n.
\end{cases}
\end{equation}

In order to study the averaging principle of the system (1), we impose the following assumptions on the coefficient functions.

**Assumption 1.** For each $x_i \in U, y_i \in B, i = 1, 2$, there exists a nonnegative function $\lambda(t)$, such that
\begin{align}
\|f(t,x_1,y_1) - f(t,x_2,y_2)\| & \leq \lambda(t) (\|x_1 - x_2\| + \|y_1 - y_2\|), \\
\|\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)\| & \leq \lambda(t) (\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{align}
where $\sup_{0 \leq t \leq T} \lambda(t) < +\infty$.

**Assumption 2.** For each $T_i \in [0,T], x \in U$ and $y \in B$, there exist positive bounded functions $\lambda_i(t), i = 1, 2$ and measurable functions $\mathcal{F}: U \times \mathcal{R} \rightarrow U, \mathcal{G}: U \times \mathcal{R} \rightarrow \mathcal{L}^0_2(V, U)$, such that
\begin{align}
\frac{1}{T_1} \int_0^{T_1} (t-s)^{2\alpha-2} \|f(s,x,y) - \mathcal{F}(x,y)\|^2 dt \leq \lambda_1(T_1) (\|x\|^2 + \|y\|^2), \\
\frac{1}{T_1} \int_0^{T_1} (t-s)^{2\alpha-2} \|g(s,x,y) - \mathcal{G}(x,y)\|^2 dt \leq \lambda_2(T_1) (\|x\|^2 + \|y\|^2), \\
\frac{1}{T_1} \int_0^{T_1} (t-s)^{2\alpha-2} \|\sigma(s,x,y) - \mathcal{S}(x,y)\|^2 dt \leq \lambda_3(T_1) (\|x\|^2 + \|y\|^2).
\end{align}

**Remark 1.** In Assumption 1, if we let $\lambda(t) = C$, then it becomes the Lipschitz condition.

**Remark 2.** If we let $\alpha = 1$, then (16) is the common stochastic delay differential equation driven by fBm, and the Assumption 2 turns to classical averaging principle.

**Definition 3.** A $U$ value stochastic process $\{X(t)\}_{0 \leq t \leq T}$ is called a mild solution of (16) if $X(t)$ satisfies the following:
\begin{enumerate}
\item[(1)] $X(t)$ is continuous and $\mathcal{F}_t$-measurable, $X_t$ is a $B$ value stochastic process;
\item[(2)] For $t \in [-\tau,0]$, $X(t) = \varphi(t)$.
\end{enumerate}
(3) For each $0 \leq t \leq T$, $X(t)$ satisfies the following integral equation:

$$
X(t) = \begin{cases} 
X(0) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} f(s, X(s), X_s) \, ds 
+ \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} g(s, X(s), X_s) \, dW_s 
+ \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \sigma(s, X(s), X_s) \, dB_s^H, 
\end{cases}
$$

$$
X(t) = \xi, \quad -\tau \leq t \leq 0.
$$

Remark 3. If we let $\sigma(t, \cdot, \cdot) = 0$, then (16) becomes stochastic differential equations with Caputo fractional derivative in [21]. Moreover, the averaging principle in [21] is the special case of this study.

3. Main Results

In this section, combining the existence and uniqueness results in the second part, we investigate the averaging principle for the Caputo FSDEs. Let us consider the standard form of (16):

$$
X^e(t) = X(0) + \frac{\varepsilon}{\Gamma(a)} \int_0^t (t-s)^{a-1} f(s, X^e(s), X^e_s) \, ds 
+ \frac{\sqrt{\varepsilon}}{\Gamma(a)} \int_0^t (t-s)^{a-1} g(s, X^e(s), X^e_s) \, dW_s 
+ \frac{\varepsilon H}{\Gamma(a)} \int_0^t (t-s)^{a-1} \sigma(s, X^e(s), X^e_s) \, dB_s^H,
$$

where $\varepsilon \in (0, \varepsilon_0]$ is a positive small parameter with $\varepsilon_0$ being a fixed number.

The following step is to introduce the original solution $X^e(t)$ converges, as $\varepsilon$ tends to zero, to the solution $Y^e(t)$ of the averaged system:

$$
Y^e(t) = X(0) + \frac{\varepsilon}{\Gamma(a)} \int_0^t (t-s)^{a-1} \mathcal{J}(Y^e(s), Y^e_s) \, ds 
+ \frac{\sqrt{\varepsilon}}{\Gamma(a)} \int_0^t (t-s)^{a-1} \mathcal{G}(Y^e(s), Y^e_s) \, dW_s 
+ \frac{\varepsilon H}{\Gamma(a)} \int_0^t (t-s)^{a-1} \mathcal{H}(Y^e(s), Y^e_s) \, dB_s^H.
$$

Theorem 1. Suppose that Assumptions 1 and 2 hold. Then, for a given arbitrary small $\delta > 0$, there exist constants $L > 0, \varepsilon_1 \in (0, \varepsilon_0]$ and $\beta \in (0, 1]$, such that for all $\varepsilon \in (0, \varepsilon_1]$,

$$
\sup_{t \in [-\tau, T]} E \left( \| X^e(t) - Y^e(t) \|^2 \right) \leq \delta.
$$

Proof. Based on the standard forms of (17) and (18), it deduces

$$
X^e(t) - Y^e(t) = \frac{\varepsilon}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left( f(s, X^e(s), X^e_s) - \mathcal{J}(Y^e(s), Y^e_s) \right) \, ds 
+ \frac{\sqrt{\varepsilon}}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left( g(s, X^e(s), X^e_s) - \mathcal{G}(Y^e(s), Y^e_s) \right) \, dW_s 
+ \frac{\varepsilon H}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left( \sigma(s, X^e(s), X^e_s) - \mathcal{H}(Y^e(s), Y^e_s) \right) \, dB_s^H.
$$

For $t \in [0, T]$, by the elementary inequality, we have
$E\left(\left\|X^ε(t) - Y^ε(t)\right\|^2\right) \leq \frac{3ε^2}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{α-1} \left(f\left(s, X^ε(s), X^ε_s\right) - f\left(s, Y^ε(s), Y^ε_s\right)\right)ds\right)^2$

\[ + \frac{3ε^2}{\Gamma(α)^2} eE\left(\int_0^t (t-s)^{α-1} \left(g\left(s, X^ε(s), X^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right)dW_s\right)^2 \]

\[ + \frac{3ε^2\lambda}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{α-1} \left(σ\left(s, X^ε(s), X^ε_s\right) - σ\left(s, Y^ε(s), Y^ε_s\right)\right)dB^H_t\right)^2 \right) = I_1 + I_2 + I_3. \]  

By the elementary inequality, Cauchy–Schwarz inequality, the Assumptions 1 and 2, we get

\[ I_1 \leq \frac{6ε^2}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{α-1} \left|f\left(s, X^ε(s), X^ε_s\right) - f\left(s, Y^ε(s), Y^ε_s\right)\right|ds\right)^2 \]

\[ + \frac{6ε^2}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{α-1} \left|g\left(s, X^ε(s), X^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right|ds\right)^2 \]

\[ \leq \frac{6ε^2 t}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{2α-2} \left|f\left(s, X^ε(s), X^ε_s\right) - f\left(s, Y^ε(s), Y^ε_s\right)\right|^2 ds\right) \]

\[ + \frac{6ε^2 t}{\Gamma(α)^2} E\left(\int_0^t (t-s)^{2α-2} \left|g\left(s, Y^ε(s), Y^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right|^2 ds\right) \]

\[ \leq \frac{6ε^2 t}{\Gamma(α)^2} \sup_{0 \leq t \leq T} τ_1(t) \int_0^t (t-s)^{2α-2} \left( E\|X^ε(r) - Y^ε(r)\|^2 + E\|X^ε_s - Y^ε_s\|^2 \right)^2 ds \]

\[ + \frac{6ε^2 t}{\Gamma(α)^2} \sup_{0 \leq t \leq T} λ_1(t) \left( \sup_{0 \leq s \leq t} E\|Y^ε(s)\|^2 + \sup_{0 \leq s \leq t} E\|Y^ε_s\|^2 \right). \]

By the Itô isometry, the elementary inequality, Cauchy–Schwarz inequality, the Assumptions 1 and 2, we get

\[ I_2 \leq 3εt(α)^2 E\int_0^t (t-s)^{2α-2} \left|g\left(s, X^ε(s), X^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right|^2 ds \]

\[ \leq \frac{6ε}{\Gamma(α)^2} E\int_0^t (t-s)^{2α-2} \left|g\left(s, X^ε(s), X^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right|^2 ds \]

\[ + \frac{6ε}{\Gamma(α)^2} E\int_0^t (t-s)^{2α-2} \left|g\left(s, Y^ε(s), Y^ε_s\right) - g\left(s, Y^ε(s), Y^ε_s\right)\right|^2 ds \]

\[ \leq \frac{6ε}{\Gamma(α)^2} \sup_{0 \leq t \leq T} τ_1(t) \int_0^t (t-s)^{2α-2} \left( E\|X^ε(s) - Y^ε(s)\|^2 + E\|X^ε_s - Y^ε_s\|^2 \right)^2 ds \]

\[ + \frac{6εt}{\Gamma(α)^2} \sup_{0 \leq t \leq T} λ_2(t) \left( \sup_{0 \leq s \leq t} E\|Y^ε(s)\|^2 + \sup_{0 \leq s \leq t} E\|Y^ε_s\|^2 \right). \]
By Lemma 1, elementary inequality, we have

\[
I_3 \leq \frac{2\varepsilon^2}{\Gamma(\alpha)} E \left( \int_0^t (t-s)^{\alpha-1} \left( \sigma(s, X^c(s), X^*_s) - \sigma(Y^c(s), Y^*_s) \right) dB_t^E \right)^2
\]

\[
\leq \frac{3C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} E \left( \int_0^t (t-s)^{2\alpha-2} \left\| \sigma(s, X^c(s), X^*_s) - \sigma(Y^c(s), Y^*_s) \right\|_{L_2}^2 \right) ds
\]

\[
\leq \frac{6C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} E \left( \int_0^t (t-s)^{2\alpha-2} \left\| \sigma(s, Y^c(s), Y^*_s) - \sigma(s, Y^c(s), Y^*_s) \right\|_{L_2}^2 \right) ds
\]

\[
\leq \frac{6C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sup_{0 \leq s \leq T} \lambda(t) \int_0^t (t-s)^{2\alpha-2} \left( E\|X^c(s) - Y^c(s)\|^2 + E\|X^*_s - Y^*_s\|^2 \right) ds
\]

\[
\leq \frac{6C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sum_{i=1}^3 \lambda_i(s) \left( \sup_{0 \leq s \leq t} E\|Y^c(s)\|^2 + \sup_{0 \leq s \leq t} E\|Y^*_s\|^2 \right).
\]

(24)

Submitting (22), (23), (24) to (21), we get

\[
E\left( \|X^c(t) - Y^c(t)\|^2 \right)
\]

\[
\leq \frac{6 \varepsilon^2 t + 6 \varepsilon t + 6C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sup_{0 \leq s \leq T} \lambda(t) \int_0^t (t-s)^{2\alpha-2} \left( E\|X^c(s) - Y^c(s)\|^2 + E\|X^*_s - Y^*_s\|^2 \right) ds
\]

\[
\leq \frac{6 \varepsilon^2 t + 6 \varepsilon t + 6C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sum_{i=1}^3 \lambda_i(s) \left( \sup_{0 \leq s \leq t} E\|Y^c(s)\|^2 + \sup_{0 \leq s \leq t} E\|Y^*_s\|^2 \right)
\]

(25)

Noting that \( E(\|X^c(t) - Y^c(t)\|^2) = 0 \) when \( -\tau \leq t \leq 0 \), it reduces

\[
E\left( \|X^c(t) - Y^c(t)\|^2 \right)
\]

\[
\leq \frac{12 \varepsilon^2 t + 12 \varepsilon t + 12C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sup_{0 \leq s \leq T} \lambda(t) \int_0^t (t-s)^{2\alpha-2} \left( E\|X^c(s) - Y^c(s)\|^2 \right) ds
\]

\[
\leq \frac{12 \varepsilon^2 t + 12 \varepsilon t + 12C_H \varepsilon^2 t^{2H-1}}{\Gamma(\alpha)^2} \sum_{i=1}^3 \lambda_i(s) \left( \sup_{0 \leq s \leq t} E\|Y^c(s)\|^2 + \sup_{0 \leq s \leq t} E\|Y^*_s\|^2 \right)
\]

(26)

By the Gronwall–Bellman inequality ([23]), we have
\[ E \left( \|X^\varepsilon(t) - Y^\varepsilon(t)\|^2 \right) \]
\[ \leq \frac{6\varepsilon t^2 + 6\varepsilon t + 6C_H\varepsilon^{2H+2\varepsilon}}{\Gamma(\alpha)^2} \sum_{i=1}^{3} \sup_{0 \leq s \leq t} \lambda_i(s) \left( \sup_{0 \leq s \leq t} E \|Y^\varepsilon(s)\|^2 + \sup_{0 \leq s \leq t} E \|Y^\varepsilon(s)\|^2 \right) \]
\[ \times \sum_{k=0}^{\infty} \left( 12\varepsilon t^{\alpha+1} + 12\varepsilon t^{\alpha} + 12C_H\varepsilon^{2H+\alpha-1} \right)^k \sup_{0 \leq s \leq T} \lambda(t)^k \Gamma(\alpha)^k \Gamma(\alpha + 1) \]  
(27) 

So,

\[ \sup_{-\tau \leq s \leq t} E \|X^\varepsilon(t) - Y^\varepsilon(t)\|^2 \]
\[ \leq \frac{6\varepsilon t^2 + 6\varepsilon t + 6C_H\varepsilon^{2H+2\varepsilon}}{\Gamma(\alpha)^2} \sum_{i=1}^{3} \sup_{0 \leq s \leq t} \lambda_i(s) \left( \sup_{0 \leq s \leq t} E \|Y^\varepsilon(s)\|^2 + \sup_{0 \leq s \leq t} E \|Y^\varepsilon(s)\|^2 \right) \]
\[ \times \sum_{k=0}^{\infty} \left( 12\varepsilon t^{\alpha+1} + 12\varepsilon t^{\alpha} + 12C_H\varepsilon^{2H+\alpha-1} \right)^k \sup_{0 \leq s \leq T} \lambda(t)^k \Gamma(\alpha)^k \Gamma(\alpha + 1) \]  
(28) 

Therefore, for any \( \delta > 0 \), there exists \( \varepsilon_1 \in (0, \varepsilon_0] \), such that for any \( \varepsilon \in (0, \varepsilon_1] \) and \( t \in [0, L\varepsilon^{-\beta}] \),

\[ \sup_{-\tau \leq s \leq t \leq L\varepsilon^{-\beta}} E \|X^\varepsilon(t) - Y^\varepsilon(t)\|^2 \leq \delta. \]  
(31) 

The proof is completed. □

4. Example

Let us consider the following FSDEs with delays:

\[ D^\alpha_x x(t) = \left[ x(t) + x(t)(t - 1)^{\alpha} \right] + \frac{1}{d} W_t + \frac{1}{d} B^H_t, \]  
(32)

and

\[ D^\alpha_y y(t) = y(t) \left( 1 + \frac{2\alpha - 1}{2\alpha + 1} \right) + \frac{1}{d} W_t + \frac{1}{d} B^H_t. \]  
(33)
According to Theorem 1, as $\varepsilon$ goes to zero, the solutions $x(t)$ and $y(t)$ are equivalent in the sense of mean square. So, the results can be checked.

**Data Availability**

The data used to support the findings of this study are freely available.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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