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Research Article

Averaging Principle for Caputo Fractional Stochastic Differential Equations Driven by Fractional Brownian Motion with Delays

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In this article, we investigate a class of Caputo fractional stochastic differential equations driven by fractional Brownian motion with delays. Under some novel assumptions, the averaging principle of the system is obtained. Finally, we give an example to show that the solution of Caputo fractional stochastic differential equations driven by fractional Brownian motion with delays converges to the corresponding averaged stochastic differential equation.

1. Introduction

Fractional differential equations have been developing as an active area on medicine, electrical engineering, biochemistry, and mechanical systems [1–5]. Because the systems are often subjected to noisy fluctuations, it is important to consider randomness into models. Since the fractional Brownian motion (fBm for short) owns many excellent properties, for example, long-range dependence and self-similar, it is usually used to describe the uncertainty. Since then, stochastic calculus with respect to fBm has been paid much attention in the stochastic analysis field, and many interesting works have obtained both qualitative and quantitative properties of stochastic differential equations (SDEs for short) driven by fBm [6–8]. Furthermore, the applications of fractional stochastic differential equations (FSDEs for short) driven by fBm have been widely applied in mathematical quantum, physics, and biology [9–11].

For the deterministic systems, many varieties of methods are proposed for average systems, such as gradient-based and least squares-based iterative algorithms for Hammerstein systems using the hierarchical identification principle, two-stage least squares-based iterative estimation algorithm for CARARMA system modeling, decomposition-based fast least squares algorithm for output error systems, and gradient-based and least squares-based iterative estimation algorithms for multi-input multi-output systems [12–14]. Compared to the deterministic systems, due to influence of stochastic factors, the exact solution of FSDEs driven by fBm is difficult to realize, and the above-mentioned methods do not work. Because the averaging principle shows that the complex original systems can be ignored and one can only concentrate on the average systems instead, it is usually taken as an effective tool to reduce the amount of calculation of the original systems. Khasminskii [15] first started with the averaging method to approximate the complex system with a simpler system. In recent years, the averaging method has been developed in many ways [16–18]. For example, Pei et al. [19] investigated stochastic averaging for stochastic differential equations driven by fBm and Brownian motion. Recently, Luo et al. [20] discussed the averaging principle for FSDEs of Itô-Doob with delays driven by Brownian motion. Xu et al. [21] presented the averaging principle for stochastic differential equations with Caputo fractional derivative.

Inspired by the above works, we will discuss averaging principle of a new kind of SDEs with Caputo derivative driven by fBm and Brownian motion, which is a general case of [19, 20]. On the other hand, because Lipschitz conditions restrict the application, we will adopt weakened Lipschitz conditions to obtain the result. Moreover, in order to overcome the influence of Caputo derivative and fBm, we introduce a new averaging method to realize the stochastic averaging principle.

In this article, we will deal with averaging principle of the following Caputo FSDEs driven by fBm and Brownian motion with delays:

$$\begin{cases} D_t^{\alpha} X(t) = f(t, X(t), X_t) + g(t, X(t), X_t)\dfrac{\mathrm{d}W_t}{\mathrm{d}t} + \sigma(t, X(t), X_t)\dfrac{\mathrm{d}B_t^H}{\mathrm{d}t}, & t \in [0, T], \\ X(t) = \varphi(t), & -\tau \le t \le 0, \end{cases} \tag{1}$$

where D_t^{α} is the Caputo fractional derivative, $\alpha \in (1/2, 1)$. f: $[0, T] \times U \times \mathrm{B} \longrightarrow U$, g: $[0, T] \times U \times \mathrm{B} \longrightarrow \mathscr{L}_2^0(V, U)$, σ: $[0, T] \times U \times \mathrm{B} \longrightarrow \mathscr{L}_2^0(V, U)$. B_t^H is a V-valued Q-cylindrical fBm with the Hurst parameter $H \in (1/2, 1)$, W_t is a standard Wiener process on a real and separable Hilbert space V independent of B_t^H, and $X_t = \{X(t+\theta), \theta \in [-\tau, 0]\}$ is the B value stochastic process. The initial value $\varphi = \{\varphi(\theta)\colon -\tau \le \theta \le 0\}$ is an $\mathscr{F}_0$-measurable B-valued random variable independent of fBm B_t^H and Wiener process W_t with finite second moment.

The rest part is arranged as follows. Section 2 is devoted to some preliminary results and assumptions. In Section 3, the averaging principle is presented. An example is provided to show the result in Section 4.

2. Preliminary

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a completed probability space. For any $t \in [0, T]$, $\mathscr{F}_t$ denotes the σ field generated by $B_s^H, W_s, s \in [0, t]$, and all $\mathbb{P}$ null sets. A one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $\beta^H = \beta^H(t)$ with the covariance function:

$$R(t, s) = E\left[\beta^H(t)\beta^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{2}$$

When $H > 1/2$, $\beta^H(t)$ has the following representation:

$$\beta^H(t) = \int_0^t K(t, s)\mathrm{d}\beta(s), \tag{3}$$

where $\beta(s)$ is a standard Brownian motion, and the kernel $K(t, s)$ is given by

$$K(t, s) = c_H s^{(1/2)-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} \mathrm{d}u, \quad t \ge s, \tag{4}$$

where c_H is a nonnegative constant with respect to H.

For function $\varphi \in L^2([0, T])$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_0^T \varphi(s)\mathrm{d}\beta^H(s) = \int_0^T K_H^* \varphi(s)\mathrm{d}\beta(s), \tag{5}$$

where $K_H^*\varphi(s) = \int_s^T \varphi(t)(\partial K(t, s)/\partial t)\mathrm{d}t$.

$(V, \|\cdot\|_V)$ and $(U, \|\cdot\|_U)$ are the two real separable Hilbert spaces with their vectors norms. Let $\mathscr{L}(V, U)$ denote the collection of all linear-bounded operators from V to U equipped with the norm $\|\cdot\|$. For the sake of convenience, we shall use the same notation $\|\cdot\|$ to denote the norms in $V, U, \mathscr{L}(V, U)$. $Q \in \mathscr{L}(V, U)$ is an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{tr}Q = \sum_{n=1}^{+\infty}\lambda_n < +\infty$, where $\lambda_n (n = 1, 2, \ldots)$ are nonnegative real numbers, and $e_n (n = 1, 2, \ldots)$ denotes a complete orthonormal basis in V. We define the infinite dimensional fBm on V with covariance Q as

$$B_t^H = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{1/2} e_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t), \tag{6}$$

where $\beta_n^H(t)$ are the real, independent fBms. The process B_t^H called V-valued Q-fBm, starts from 0, has zero mean and covariance:

$$E\langle B_t^H, x\rangle\langle B_s^H, y\rangle = R(t, s)\langle Q(x), y\rangle, \quad \text{for all } x, y \in V \text{ and } t, s \in [0, T]. \tag{7}$$

In the following parts, we shall introduce Wiener integral with respect to the Q-fBm B_t^H. Let $\mathscr{L}_2^0 = \mathscr{L}_2^0(V, U)$ denote the space of all Q-Hilbert–Schmidt operators ψ: $V \longrightarrow U$ equipped with the norm

$$\|\psi\|_{\mathscr{L}_2^0} = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n}\, \psi e_n \right\|^2 < \infty, \tag{8}$$

and the inner product $\langle \varphi, \psi\rangle_{\mathscr{L}_2^0} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$ for $\varphi, \psi \in \mathscr{L}_2^0$.

Now, we give the definition of the fractional Wiener integral of the function ψ: $[0, T] \longrightarrow \mathscr{L}_2^0$ with respect to Q-fBm as follows:

$$\begin{aligned} \int_0^t \psi(s)\mathrm{d}B_s^H &= \sum_{n=1}^{\infty} \int_0^t \psi(s) Q^{1/2} e_n \mathrm{d}\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \left(K_H^*\left(\psi Q^{1/2} e_n\right)\right)(s)\mathrm{d}\beta_n(s), \end{aligned} \tag{9}$$

where β_n is the standard Brownian motion with respect to β_n^H.

We introduce $B([-\tau,0],L^2(\Omega,U))$ (B for simply) denotes the family of all $\mathscr{F}_0$-measurable bounded continuous functions ξ: $[-\tau,0]\longrightarrow L^2(\Omega,U)$ endowed with the norm $\|\xi(t)\|^2=\sup_{-\tau\le\theta\le t}E\|\xi(\theta)\|^2$. It is important of the following lemma to prove our main results, which is appeared in [22].

Lemma 1. *If ψ: $[0,T]\longrightarrow\mathscr{L}_2^0(V,U)$ satisfies*

$$\int_0^T\|\psi(t)\|_{\mathscr{L}_2^0}^2\mathrm{d}t<\infty, \tag{10}$$

then, for any $0\le s\le t\le T$,

$$E\left(\left\|\int_s^t\sigma(r)\mathrm{d}B_r^H\right\|^2\right)\le C_H(t-s)^{2H-1}\int_s^t\|\sigma(r)\|_{\mathscr{L}_2^0}^2\mathrm{d}r. \tag{11}$$

Now, we recall some notations and preliminary results about fractional calculus and some special functions.

Definition 1. For any $\alpha\in(0,1)$ and function f: $[0,T]\longrightarrow U$, the Riemann–Liouville fractional integral operator of order α is defined

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\mathrm{d}s,\quad 0\le t\le T, \tag{12}$$

where $\Gamma(\alpha)=\int_0^\infty r^{\alpha-1}e^{-r}\mathrm{d}r$.

Definition 2. The Caputo fractional derivative with order α of function $f(t)\in\mathscr{C}^n([0,T];U)$ is defined as

$$D_t^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s, & n-1<\alpha<n,\\[2ex] \dfrac{\mathrm{d}^n}{\mathrm{d}t^n}f(t), & \alpha=n.\end{cases} \tag{13}$$

In order to study the averaging principle of the system (1), we impose the following assumptions on the coefficient functions.

Assumption 1. For each $x_i\in U$, $y_i\in B$, $i=1,2$, there exists a nonnegative function $\lambda(t)$, such that

$$\begin{aligned}&\|f(t,x_1,y_1)-f(t,x_2,y_2)\|^2\vee\|g(t,x_1,y_1)-g(t,x_2,y_2)\|_{\mathscr{L}_2^0}^2\\&\quad\vee\|\sigma(t,x_1,y_1)-\sigma(t,x_2,y_2)\|_{\mathscr{L}_2^0}^2\\&\le\lambda(t)\left(\|x_1-x_2\|^2+\|y_1-y_2\|^2\right),\end{aligned} \tag{14}$$

where $\sup_{0\le t\le T}\lambda(t)<+\infty$.

Assumption 2. For each $T_1\in[0,T]$, $x\in U$ and $y\in$ B, there exist positive bounded functions $\lambda_i(t)$, $i=1,2$ and measurable functions $\overline{f}$: $U\times\mathscr{R}\longrightarrow U$, $\overline{g}\,\overline{\sigma}$: $U\times\mathscr{R}\longrightarrow\mathscr{L}_2^0(V,U)$, such that

$$\begin{aligned}&\frac{1}{T_1}\int_0^{T_1}(t-s)^{2\alpha-2}\|f(s,x,y)-\overline{f}(x,y)\|^2\mathrm{d}s\le\lambda_1(T_1)\left(\|x\|^2+\|y\|^2\right),\\&\frac{1}{T_1}\int_0^{T_1}(t-s)^{2\alpha-2}\|g(s,x,y)-\overline{g}(x,y)\|_{\mathscr{L}_2^0}^2\mathrm{d}s\le\lambda_2(T_1)\left(\|x\|^2+\|y\|^2\right),\\&\frac{1}{T_1}\int_0^{T_1}(t-s)^{2\alpha-2}\|\sigma(s,x,y)-\overline{\sigma}(x,y)\|_{\mathscr{L}_2^0}^2\mathrm{d}s\le\lambda_3(T_1)\left(\|x\|^2+\|y\|^2\right).\end{aligned} \tag{15}$$

Remark 1. In Assumption 1, if we let $\lambda(t)=C$, then it becomes the Lipschitz condition.

Remark 2. If we let $\alpha=1$, then (16) is the common stochastic delay differential equation driven by fBm, and the Assumption 2 turns to classical averaging principle.

Definition 3. A U value stochastic process $\{X(t)\}_{-\tau\le t\le T}$ is called a mild solution of (16) if $X(t)$ satisfies the following:

(1) $X(t)$ is continuous and $\mathscr{F}_t$-measurable, X_t is a B value stochastic process;

(2) For $t\in[-\tau,0]$, $X(t)=\varphi(t)$

(3) For each $0 \le t \le T$, $X(t)$ satisfies the following integral equation:

$$X(t) = \begin{cases} X(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, X(s), X_s) \mathrm{d}s + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} g(s, X(s), X_s) \mathrm{d}W_s \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} \sigma(s, X(s), X_s) \mathrm{d}B_s^H, \\ X(t) = \xi, \quad -\tau \le t \le 0. \end{cases} \tag{16}$$

Remark 3. If we let $\sigma(t, \cdot, \cdot) = 0$, then (16) becomes stochastic differential equations with Caputo fractional derivative in [21]. Moreover, the averaging principle in [21] is the special case of this study.

3. Main Results

In this section, combining the existence and uniqueness results in the second part, we investigate the averaging principle for the Caputo FSDEs. Let us consider the standard form of (16):

$$\begin{aligned} X^\varepsilon(t) = X(0) &+ \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X^\varepsilon(s), X_s^\varepsilon) \mathrm{d}s \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X^\varepsilon(s), X_s^\varepsilon) \mathrm{d}W_s \\ &+ \frac{\varepsilon^H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, X^\varepsilon(s), X_s^\varepsilon) \mathrm{d}B_s^H, \end{aligned} \tag{17}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a positive small parameter with ε_0 being a fixed number.

The following step is to introduce the original solution $X^\varepsilon(t)$ converges, as ε tends to zero, to the solution $Y^\varepsilon(t)$ of the averaged system:

$$\begin{aligned} Y^\varepsilon(t) = X(0) &+ \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(Y^\varepsilon(s), Y_s^\varepsilon) \mathrm{d}s \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{g}(Y^\varepsilon(s), Y_s^\varepsilon) \mathrm{d}W_s \\ &+ \frac{\varepsilon^H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{\sigma}(Y^\varepsilon(s), Y_s^\varepsilon) \mathrm{d}B_s^H. \end{aligned} \tag{18}$$

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then, for a given arbitrary small $\delta > 0$, there exist constants $L > 0, \varepsilon_1 \in (0, \varepsilon_0]$ and $\beta \in (0, 1]$, such that for all $\varepsilon \in (0, \varepsilon_1]$,*

$$\sup_{t \in [-\tau, L\varepsilon^{-\beta}]} E\left(\left\| X^\varepsilon(t) - Y^\varepsilon(t) \right\|^2 \right) \le \delta. \tag{19}$$

Proof. Based on the standard forms of (17) and (18), it deduces

$$\begin{aligned} X^\varepsilon(t) - Y^\varepsilon(t) = &\frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s, X^\varepsilon(s), X_s^\varepsilon) - \overline{f}(Y^\varepsilon(s), Y_s^\varepsilon) \right) \mathrm{d}s \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(g(s, X^\varepsilon(s), X_s^\varepsilon) - \overline{g}(Y^\varepsilon(s), Y_s^\varepsilon) \right) \mathrm{d}W_s \\ &+ \frac{\varepsilon^H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\sigma(s, X^\varepsilon(s), X_s^\varepsilon) - \overline{\sigma}(Y^\varepsilon(s), Y_s^\varepsilon) \right) \mathrm{d}B_s^H. \end{aligned} \tag{20}$$

For $t \in [0, T]$, by the elementary inequality, we have

$$
\begin{aligned}
E\left(\left\|X^{\varepsilon}(t)-Y^{\varepsilon}(t)\right\|^{2}\right) \leq & \frac{3\varepsilon^{2}}{\Gamma(\alpha)^{2}} E\left(\left\|\int_{0}^{t}(t-s)^{\alpha-1}\left(f\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-\overline{f}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right) \mathrm{d} s\right\|^{2}\right) \\
& +\frac{3\epsilon}{\Gamma(\alpha)^{2}} \varepsilon E\left(\left\|\int_{0}^{t}(t-s)^{\alpha-1}\left(g\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-\overline{g}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right) \mathrm{d} W_{s}\right\|^{2}\right) \\
& +\frac{3\varepsilon^{2H}}{\Gamma(\alpha)^{2}} E\left(\left\|\int_{0}^{t}(t-s)^{\alpha-1}\left(\sigma\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-\overline{\sigma}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right) \mathrm{d} B_{s}^{H}\right\|^{2}\right) =: I_{1}+I_{2}+I_{3}.
\end{aligned} \tag{21}
$$

By the elementary inequality, Cauchy–Schwarz inequality, the Assumptions 1 and 2, we get

$$
\begin{aligned}
I_{1} \leq & \frac{6\varepsilon^{2}}{\Gamma(\alpha)^{2}} E\left(\left\|\int_{0}^{t}(t-s)^{\alpha-1}\left(f\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-f\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right) \mathrm{d} s\right\|^{2}\right) \\
& +\frac{6\varepsilon^{2}}{\Gamma(\alpha)^{2}} E\left(\left\|\int_{0}^{t}(t-s)^{\alpha-1}\left(f\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)-\overline{f}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right) \mathrm{d} s\right\|^{2}\right) \\
\leq & \frac{6\varepsilon^{2} t}{\Gamma(\alpha)^{2}} E\left(\int_{0}^{t}(t-s)^{2\alpha-2}\left\|f\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-f\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right\|^{2} \mathrm{d} s\right) \\
& +\frac{6\varepsilon^{2} t}{\Gamma(\alpha)^{2}} E\left(\int_{0}^{t}(t-s)^{2\alpha-2}\left\|f\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)-\overline{f}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right\|^{2} \mathrm{d} s\right) \\
\leq & \frac{6\varepsilon^{2} t \sup_{0 \leq t \leq T} \lambda(t)}{\Gamma(\alpha)^{2}} \int_{0}^{t}(t-s)^{2\alpha-2}\left(E\left\|X^{\varepsilon}(r)-Y^{\varepsilon}(r)\right\|^{2}+E\left\|X_{s}^{\varepsilon}-Y_{s}^{\varepsilon}\right\|^{2}\right) \mathrm{d} s \\
& +\frac{6\varepsilon^{2} t^{2}}{\Gamma(\alpha)^{2}} \sup_{0 \leq s \leq t} \lambda_{1}(t)\left(\sup_{0 \leq s \leq t} E\left\|Y^{\varepsilon}(s)\right\|^{2}+\sup_{0 \leq s \leq t} E\left\|Y_{s}^{\varepsilon}\right\|^{2}\right).
\end{aligned} \tag{22}
$$

By the Itô isometry, the elementary inequality, Cauchy–Schwarz inequality, the Assumptions 1 and 2, we get

$$
\begin{aligned}
I_{2} \leq & 3\varepsilon \Gamma(\alpha)^{2} E \int_{0}^{t}(t-s)^{2\alpha-2}\left\|g\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-\overline{g}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right\|_{\mathscr{L}_{2}^{0}}^{2} \mathrm{d} s \\
\leq & \frac{6\varepsilon}{\Gamma(\alpha)^{2}} E \int_{0}^{t}(t-s)^{2\alpha-2}\left\|g\left(s, X^{\varepsilon}(s), X_{s}^{\varepsilon}\right)-g\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right\|_{\mathscr{L}_{2}^{0}}^{2} \mathrm{d} s \\
& +\frac{6\varepsilon}{\Gamma(\alpha)^{2}} E \int_{0}^{t}(t-s)^{2\alpha-2}\left\|g\left(s, Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)-\overline{g}\left(Y^{\varepsilon}(s), Y_{s}^{\varepsilon}\right)\right\|_{\mathscr{L}_{2}^{0}}^{2} \mathrm{d} s \\
\leq & \frac{6\varepsilon \sup_{0 \leq t \leq T \lambda(t)}}{\Gamma(\alpha)^{2}} \int_{0}^{t}(t-s)^{2\alpha-2}\left(E\left\|X^{\varepsilon}(s)-Y^{\varepsilon}(s)\right\|^{2}+E\left\|X_{s}^{\varepsilon}-Y_{s}^{\varepsilon}\right\|^{2}\right) \mathrm{d} s \\
& +\frac{6\varepsilon t}{\Gamma(\alpha)^{2}} \sup_{0 \leq s \leq t} \lambda_{2}(s)\left(\sup_{0 \leq s \leq t} E\left\|Y^{\varepsilon}(s)\right\|^{2}+\sup_{0 \leq s \leq t} E\left\|Y_{s}^{\varepsilon}\right\|^{2}\right).
\end{aligned} \tag{23}
$$

By Lemma 1, elementary inequality, we have

$$
\begin{aligned}
I_3 &\le \frac{3\varepsilon^{2H}}{\Gamma(\alpha)^2} E\left(\left\|\int_0^t (t-s)^{\alpha-1}\left(\sigma\left(s, X^\varepsilon(s), X_s^\varepsilon\right) - \overline{\sigma}\left(Y^\varepsilon(s), Y_s^\varepsilon\right)\right)\mathrm{d}B_s^H\right\|^2\right) \\
&\le \frac{3C_H\varepsilon^{2H}t^{2H-1}}{\Gamma(\alpha)^2} E\int_0^t (t-s)^{2\alpha-2}\left\|\sigma\left(s, X^\varepsilon(s), X_s^\varepsilon\right) - \overline{\sigma}\left(Y^\varepsilon(s), Y_s^\varepsilon\right)\right\|_{\mathscr{L}_2^0}^2 \mathrm{d}s \\
&\le \frac{6C_H\varepsilon^{2H}t^{2H-1}}{\Gamma(\alpha)^2} E\int_0^t (t-s)^{2\alpha-2}\left\|\sigma\left(s, X^\varepsilon(s), X_s^\varepsilon\right) - \sigma\left(s, Y^\varepsilon(s), Y_s^\varepsilon\right)\right\|_{\mathscr{L}_2^0}^2 \mathrm{d}s \\
&\quad + \frac{6C_H\varepsilon^{2H}t^{2H-1}}{\Gamma(\alpha)^2} E\int_0^t (t-s)^{2\alpha-2}\left\|\sigma\left(s, Y^\varepsilon(s), Y_s^\varepsilon\right) - \overline{\sigma}\left(Y^\varepsilon(s), Y_s^\varepsilon\right)\right\|_{\mathscr{L}_2^0}^2 \mathrm{d}s \\
&\le \frac{6C_H\varepsilon^{2H}t^{2H-1}\sup_{0\le t\le T}\lambda(t)}{\Gamma(\alpha)^2}\int_0^t (t-s)^{2\alpha-2}\left(E\left\|X^\varepsilon(s) - Y^\varepsilon(s)\right\|^2 + E\left\|X_s^\varepsilon - Y_s^\varepsilon\right\|^2\right)\mathrm{d}s \\
&\quad + \frac{6C_H\varepsilon^{2H}t^{2H}}{\Gamma(\alpha)^2}\sup_{0\le s\le t}\lambda_3(s)\left(\sup_{0\le s\le t} E\left\|Y^\varepsilon(s)\right\|^2 + \sup_{0\le s\le t} E\left\|Y_s^\varepsilon\right\|^2\right).
\end{aligned}
\tag{24}
$$

Submitting (22), (23), (24) to (21), we get

$$
\begin{aligned}
&E\left(\left\|X^\varepsilon(t) - Y^\varepsilon(t)\right\|^2\right) \\
&\le \frac{6\varepsilon^2 t + 6\varepsilon + 6C_H\varepsilon^{2H}t^{2H-1}}{\Gamma(\alpha)^2}\sup_{0\le t\le T}\lambda(t)\int_0^t (t-s)^{2\alpha-2}\left(E\left\|X^\varepsilon(s) - Y^\varepsilon(s)\right\|^2 + E\left\|X_s^\varepsilon - Y_s^\varepsilon\right\|^2\right)\mathrm{d}s \\
&\quad + \frac{6\varepsilon^2 t^2 + 6\varepsilon t + 6C_H\varepsilon^{2H}t^{2H}}{\Gamma(\alpha)^2}\sum_{i=1}^{3}\sup_{0\le s\le t}\lambda_i(s)\left(\sup_{0\le s\le t} E\left\|Y^\varepsilon(s)\right\|^2 + \sup_{0\le s\le t} E\left\|Y_s^\varepsilon\right\|^2\right).
\end{aligned}
\tag{25}
$$

Noting that $E(\|X^\varepsilon(t) - Y^\varepsilon(t)\|^2) = 0$ when $-\tau \le t \le 0$, it reduces

$$
\begin{aligned}
&E\left(\left\|X^\varepsilon(t) - Y^\varepsilon(t)\right\|^2\right) \\
&\le \frac{12\varepsilon^2 t + 12\varepsilon + 12C_H\varepsilon^{2H}t^{2H-1}}{\Gamma(\alpha)^2}\sup_{0\le t\le T}\lambda(t)\int_0^t (t-s)^{2\alpha-2}\left(E\left\|X^\varepsilon(s) - Y^\varepsilon(s)\right\|^2\right)\mathrm{d}s \\
&\quad + \frac{6\varepsilon^2 t^2 + 6\varepsilon t + 6C_H\varepsilon^{2H}t^{2H}}{\Gamma(\alpha)^2}\sum_{i=1}^{3}\sup_{0\le s\le t}\lambda_i(s)\left(\sup_{0\le s\le t} E\left\|Y^\varepsilon(s)\right\|^2 + \sup_{0\le s\le t} E\left\|Y_s^\varepsilon\right\|^2\right).
\end{aligned}
\tag{26}
$$

By the Gronwall–Bellman inequality ([23]), we have

$$E\left(\left\|X^{\varepsilon}(t)-Y^{\varepsilon}(t)\right\|^{2}\right)$$
$$\leq \frac{6\varepsilon^{2}t^{2}+6\varepsilon t+6C_{H}\varepsilon^{2H}t^{2H}}{\Gamma(\alpha)^{2}}\sum_{i=1}^{3}\sup_{0\leq s\leq t}\lambda_{i}(s)\left(\sup_{0\leq s\leq t}E\left\|Y^{\varepsilon}(s)\right\|^{2}+\sup_{0\leq s\leq t}E\left\|Y_{s}^{\varepsilon}\right\|^{2}\right)$$
$$\times\sum_{k=0}^{\infty}\frac{\left(12\varepsilon^{2}t^{\alpha+1}+12\varepsilon t^{\alpha}+12C_{H}\varepsilon^{2H}t^{2H+\alpha-1}\right)^{k}\sup_{0\leq t\leq T}\lambda(t)}{\Gamma(\alpha)^{k}\Gamma(k\alpha+1)}. \tag{27}$$

So,

$$\sup_{-\tau\leq s\leq t}E\left\|X^{\varepsilon}(t)-Y^{\varepsilon}(t)\right\|^{2}$$
$$\leq \frac{6\varepsilon^{2}t^{2}+6\varepsilon t+6C_{H}\varepsilon^{2H}t^{2H}}{\Gamma(\alpha)^{2}}\sum_{i=1}^{3}\sup_{0\leq s\leq t}\lambda_{i}(s)\left(\sup_{0\leq s\leq t}E\left\|Y^{\varepsilon}(s)\right\|^{2}+\sup_{0\leq s\leq t}E\left\|Y_{s}^{\varepsilon}\right\|^{2}\right)$$
$$\times\sum_{k=0}^{\infty}\frac{\left(12\varepsilon^{2}t^{\alpha+1}+12\varepsilon t^{\alpha}+12C_{H}\varepsilon^{2H}t^{2H+\alpha-1}\right)^{k}\left(\sup_{0\leq t\leq T}\lambda(t)\right)^{k}}{\Gamma(\alpha)^{k}\Gamma(k\alpha+1)}. \tag{28}$$

So, we can select $\beta\in(0,1)$ and $L>0$, such that for every $t\in[0,L\varepsilon^{-\beta}]\subseteq[0,T]$,

$$\sup_{-\tau\leq t\leq L\varepsilon^{-\beta}}E\left\|X^{\varepsilon}(t)-Y^{\varepsilon}(t)\right\|^{2}\leq C\varepsilon^{1-\beta}, \tag{29}$$

where

$$C=6L^{2}\varepsilon^{1-\beta}+6L+6L^{2H}C_{H}\varepsilon^{(2H-1)(1-\beta)}\Gamma(\alpha)^{2}$$
$$\times\sum_{i=1}^{3}\sup_{0\leq s\leq L\varepsilon^{-\beta}}\lambda_{i}(s)\left(\sup_{0\leq s\leq L\varepsilon^{-\beta}}E\left\|Y^{\varepsilon}(s)\right\|^{2}+\sup_{0\leq s\leq L\varepsilon^{-\beta}}E\left\|Y_{s}^{\varepsilon}\right\|^{2}\right)$$
$$\times\sum_{k=0}^{\infty}\frac{\left(12L^{\alpha+1}\varepsilon^{2-\alpha\beta-\beta}+12L^{\alpha}\varepsilon^{1-\alpha\beta}+12L^{2H+\alpha-1}C_{H}\varepsilon^{2H-\beta(2H+\alpha-1)}\right)^{k}\left(\sup_{0\leq t\leq T}\lambda(t)\right)^{k}}{\Gamma(\alpha)^{k}\Gamma(k\alpha+1)}. \tag{30}$$

Therefore, for any $\delta>0$, there exists $\varepsilon_{1}\in(0,\varepsilon_{0}]$, such that for any $\varepsilon\in(0,\varepsilon_{1}]$ and $t\in[0,L\varepsilon^{-\beta}]$,

$$\sup_{-\tau\leq t\leq L\varepsilon^{-\beta}}E\left\|X^{\varepsilon}(t)-Y^{\varepsilon}(t)\right\|^{2}\leq\delta. \tag{31}$$

The proof is completed. □

4. Example

Let us consider the following FSDEs with delays:

$$D_{t}^{\alpha}x(t)=\left[x(t)+x(t)(t-1)^{2}\right]+1\frac{\mathrm{d}W_{t}}{\mathrm{d}t}+1\frac{\mathrm{d}B_{t}^{H}}{\mathrm{d}t}, \tag{32}$$

where $t\in[0,1]$, $f(t,x(t),x_{t})=x(t)+x(t)(t-1)^{2}$, $g(t,x(t),x_{t})=1$, $\sigma(t,x(t),x_{t})=1$, $1/2<\alpha<1$.

It is easy to verify that f,g,σ satisfy the Assumptions 1 and 2. Define $\overline{f}(x,y)$ as follows:

$$\int_{0}^{1}\overline{f}(x,y)(1-t)^{2\alpha-2}\mathrm{d}t=\int_{0}^{1}f(t,x,y)(1-t)^{2\alpha-2}\mathrm{d}t. \tag{33}$$

We can derive that $\overline{f}(x,y)=x(1+((2\alpha-1)/(2\alpha+1)))$. Similarly, $\overline{g}(x,y)=1,\overline{\sigma}(x,y)=1$. The averaging form of (32) can be defined as

$$D_{t}^{\alpha}y(t)=y(t)\left(1+\frac{2\alpha-1}{2\alpha+1}\right)+1\frac{\mathrm{d}W_{t}}{\mathrm{d}t}+1\frac{\mathrm{d}B_{t}^{H}}{\mathrm{d}t}. \tag{34}$$

According to Theorem 1, as ε goes to zero, the solutions $x(t)$ and $y(t)$ are equivalent in the sense of mean square. So, the results can be checked.

Data Availability

The data used to support the findings of this study are freely available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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