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Research Article

A New Lifetime Distribution: Properties, Copulas, Applications, and Different Classical Estimation Methods

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A new continuous version of the inverse flexible Weibull model is proposed and studied. Some of its properties such as quantile function, moments and generating functions, incomplete moments, mean deviation, Lorenz and Bonferroni curves, the mean residual life function, the mean inactivity time, and the strong mean inactivity time are derived. The failure rate of the new model can be "increasing-constant," "bathtub-constant," "bathtub," "constant," "**J**-HRF," "upside down bathtub," "increasing," "upside down-increasing-constant," and "upside down." Different copulas are used for deriving many bivariate and multivariate type extensions. Different non-Bayesian well-known estimation methods under uncensored scheme are considered and discussed such as the maximum likelihood estimation, Anderson Darling estimation, ordinary least square estimation, Cramér-von-Mises estimation, weighted least square estimation, and right tail Anderson Darling estimation methods. Simulation studies are performed for comparing these estimation methods. Finally, two real datasets are analyzed to illustrate the importance of the new model.

1. Introduction

The Weibull model [1] is a very useful distribution in modeling real data exhibiting monotonic hazard rate function (HRF). But it cannot be used in modeling and studying data which have nonmonotonic HRF such as the "bathtub shape (**U**-HRF)." For avoiding this drawback, Bebbington et al. [2] have defined a new two-parameter distribution which is an extension of the Weibull distribution referred to as a flexible Weibull (FW) extension distribution; it has a failure function that can be "decreasing," "increasing," or "bathtub-shaped." Analogously, El-Goharyet et al. [3] derived the two-parameter inverse flexible Weibull (IFW) model which is the reciprocal of a random variable (RV) which has FW model. Several mathematical properties of this distribution such as the mode, moments, and moment generating function (MGF) have been discussed. El-Gohary et al. [3] proved that the hazard rate function (RRF) of the IFW model can be "upside down constant," the cumulative distribution function (CDF) of IFW distribution is given by

$$G_{\alpha,\theta}(z) = \exp\left[-\exp\left(\frac{\alpha}{z} - \theta z\right)\right]|_{(z>0)}, \tag{1}$$

where the two parameters $\alpha > 0$ and $\theta > 0$ control the shape of the distribution. The corresponding probability density function (PDF) is

$$g_{\alpha,\theta}(z) = \left(\theta + \frac{\alpha}{z^2}\right)\exp\left(\frac{\alpha}{z} - \theta z\right)\exp\left[-\exp\left(\frac{\alpha}{z} - \theta z\right)\right]|_{(z>0)}. \tag{2}$$

Many authors introduced new families of continuous distributions that extend well-known existing ones and, at the same time, provide great flexibility in modeling data in practice, for example, the Marshall-Olkin-generated family [4], the Kumaraswamy-G [5], the Burr X generator of distributions [6], the Topp-Leone odd log-logistic family [7], the transmuted Topp-Leone family [8], the Burr XII system of densities [9], and the odd log-logistic Topp-Leone family [10], among others. Recently, Elgarhy et al. [11] represented a new flexible class of continuous distributions with an extra positive parameter called the type II Topp-Leone generated

(TIITL-G) family of distributions. Due to [11], the CDF of the TIITL-G family is given by

$$F_{\lambda,\underline{\psi}}(z) = 1 - \left[1 - G_{\underline{\psi}^2(z)]^{\lambda}}\right. \quad (3)$$

The PDF is defined by

$$f_{\lambda,\underline{\psi}}(z) = 2\lambda g_{\underline{\psi}}(z) G_{\underline{\psi}}(z)\left[1 - G_{\underline{\psi}^2(z)]^{\lambda-1}}\right., \quad (4)$$

where $\lambda > 0$ is a shape parameter, $g_{\underline{\psi}}(z) = \mathrm{d}G_{\underline{\psi}}(z)/\mathrm{d}z$ is the baseline PDF, and $G_{\underline{\psi}}(z)$ is a baseline CDF.

The remainder of the paper is organized as follows: in Section 2, we define the CDF, PDF, and HRF of TIITLIFW model and provide a simple expansion of the PDF. Simple type copula is derived in Section 3. Various mathematical properties are discussed in Section 4. Non-Bayesian estimation methods under uncensored schemes are given in Section 5. Section 6 presents a comparison under the non-Bayesian estimation methods using uncensored schemes via a simulation study. Section 7 presents a comparison under uncensorship with some competitive models. Concluding remarks are contained in Section 8.

2. The New Model

Using (1) in (3), the CDF of the TIITLIFW distribution can be written as

$$F_{\underline{V}}(z)|_{\underline{V}=(\alpha,\theta,\lambda)} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z} - \theta z\right)\right]\right\}^{\lambda}. \quad (5)$$

The corresponding PDF is given by

$$f_{\underline{V}}(z) = 2\lambda\left(\theta + \frac{\alpha}{z^2}\right)\frac{\exp[-2\exp((\alpha/z) - \theta z)]\exp((\alpha/z) - \theta z)}{\{1 - \exp[-2\exp((\alpha/z) - \theta z)]\}^{1-\lambda}}. \quad (6)$$

The HRF can be expressed as

$$h_{\underline{V}}(z) = \frac{2\lambda\exp[-2\exp((\alpha/z) - \theta z)]}{1 - \exp[-2\exp((\alpha/z) - \theta z)]}\left(\theta + \frac{\alpha}{z^2}\right)\exp\left(\frac{\alpha}{z} - \theta z\right). \quad (7)$$

Figure 1 shows some plots of the PDF of the TIITLIFW distribution for some different values of the parameters. Figure 2 shows some plots of the HRF of the TIITLIFW distribution for some different parameter values. Based on Figure 1, we conclude that the new PDF can have many right skewed heavy tail shapes. Based on Figure 2, it is observed that the new HRF can be "increasing-constant," "bathtub-constant," "bathtub," "constant," "J-HRF," "upside down bathtub," "increasing," "upside down-increasing-constant," and "upside down"

The PDF of the TIITLIFW distribution can be written as

$$f_{\underline{V}}(z) = \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left(\theta z^{-k} + \alpha z^{-k-2}\right)\exp[-(j+1)\theta z]. \quad (8)$$

Proof. Supposing $|w_1/w_2| < 1, w_3 > 0$ is a real noninteger, we have the power series expansion

$$\left(1 - \frac{w_1}{w_2}\right)^{w_3} = \sum_{i=0}^{\infty}\left(-\frac{w_1}{w_2}\right)^i \frac{\Gamma(1+w_3)}{i!\Gamma(1+w_3-i)}, \quad (9)$$

using the power series (9) in equation (8), and the fact that $0 < \exp[-2\exp((\alpha/z) - \theta z)] < 1$, we get

$$f_{\underline{V}}(z) = \sum_{i=0}^{\infty}(-1)^i \frac{2\lambda\Gamma(\lambda)}{i!\Gamma(\lambda-i)}\left(\theta + \frac{\alpha}{z^2}\right)\exp\left(\frac{\alpha}{z} - \theta z\right) \cdot \exp\left[-2(i+1)\exp\left(\frac{\alpha}{z} - \theta z\right)\right], \quad (10)$$

expanding $\exp[-2(i+1)\exp((\alpha/z) - \theta z)]$ using Taylor series

$$\exp\left[-2(i+1)\exp\left(\frac{\alpha}{z} - \theta z\right)\right] = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}[2(i+1)]^j \exp\left[j\left(\frac{\alpha}{z} - \theta z\right)\right], \quad (11)$$

again, using a series expansion of $\exp[(j+1)(\alpha/z)]$, and after some algebras, the PDF can be written as

$$f_{\underline{V}}(z) = \sum_{i,j,k=0}^{\infty}\nabla_{i,j,k}\left(\theta z^{-k} + \alpha z^{-k-2}\right)\exp[-(j+1)\theta z], \quad (12)$$

where

$$\nabla_{i,j,k} = \frac{2^{j+1}\alpha^k(-1)^{i+j}(i+1)^j(j+1)^k\Gamma(\lambda+1)}{i!j!k!\Gamma(\lambda-i)}. \quad (13)$$

3. Copula

3.1. Bivariate TIITLIFW (BTIITLIFW) via Morgenstern Family. The CDF of the Morgenstern family of two random RVs (X_1, X_2) can be derived as

$$\mathbb{H}_{\rho}(x_1, x_2)|_{(|\rho|\le 1)} = F_1(x_1)F_2(x_2)[1 + \rho\overline{F}_1(x_1)\overline{F}_2(x_2)], \quad (14)$$

where

$$\overline{F}_1(x_1) = 1 - F_1(x_1), \quad \overline{F}_2(x_2) = 1 - F_2(x_2), \quad (15)$$

setting

$$F_{\underline{V_1}}(x)|_{\underline{V_1}=(\alpha_1,\theta_1,\lambda_1)} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha_1}{x_1} - \theta_1 x_1\right)\right]\right\}^{\lambda_1},$$

$$F_{\underline{V_2}}(x)|_{\underline{V_2}=(\alpha_2,\theta_2,\lambda_2)} = 1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha_2}{x_2} - \theta_2 x_2\right)\right]\right\}^{\lambda_2}, \quad (16)$$

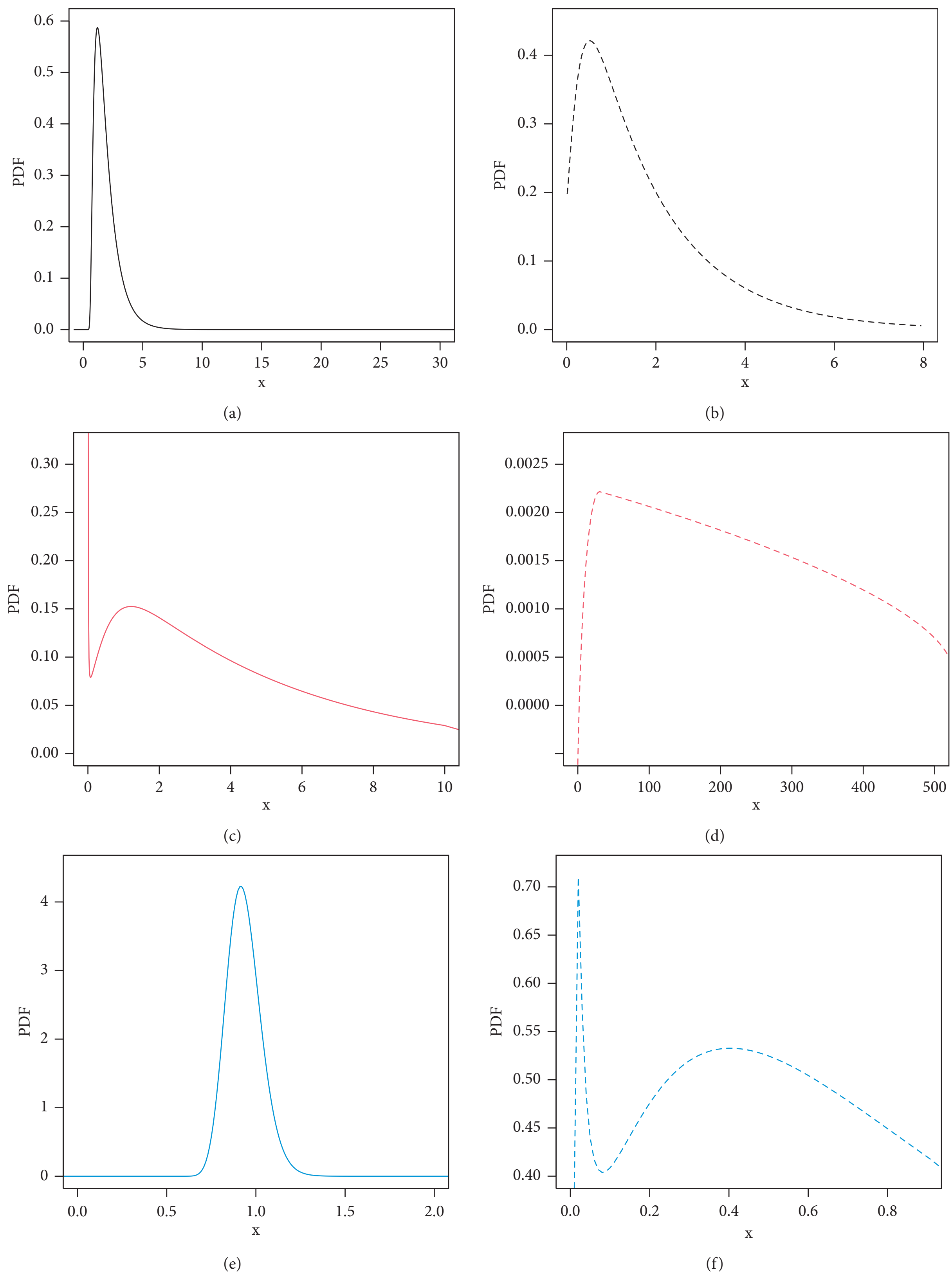


FIGURE 1: PDF plots of the TIITLIFW model for some different values of the parameters. (a) $\alpha = 1, \theta = 1, \lambda = 1$. (b) $\alpha = 0.0001, \theta = 4, \lambda = 0.15$. (c) $\alpha = 0.0001, \theta = 2, \lambda = 0.1$. (d) $\alpha = 5, \theta = 0.05, \lambda = 0.05$. (e) $\alpha = 2, \theta = 2, \lambda = 10$. (f) $\alpha = 0.1, \theta = 5, \lambda = 0.15$.

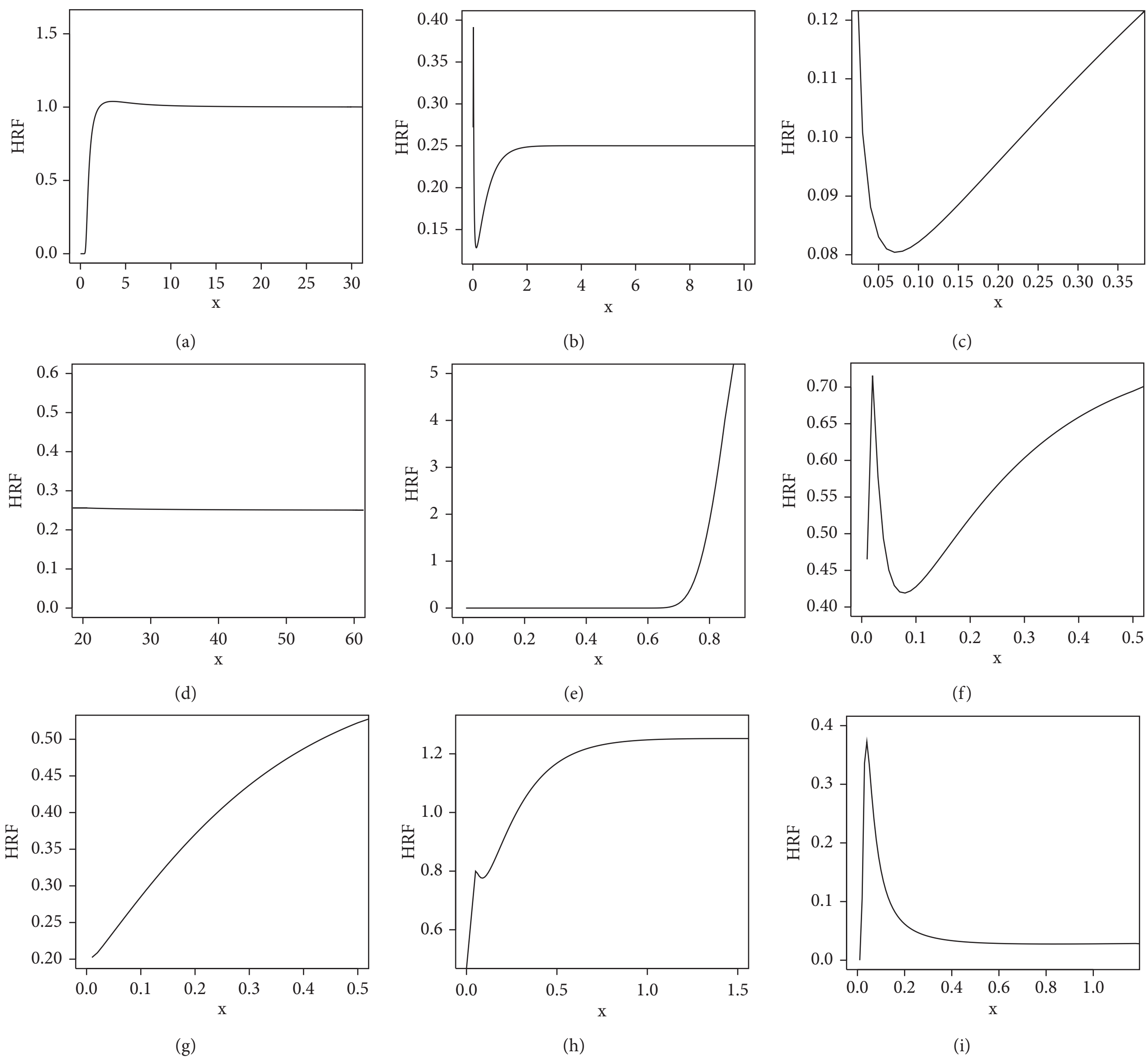


Figure 2: HRFs of the TIITLIFW model for some different values of the parameters. (a) $\alpha = 1, \theta = 1, \lambda = 1$. (b) $\alpha = 0.1, \theta = 2.5, \lambda = 0.1$. (c) $\alpha = 0.001, \theta = 2, \lambda = 0.1$. (d) $\alpha = 5, \theta = 0.5, \lambda = 0.5$. (e) $\alpha = 2, \theta = 2, \lambda = 10$. (f) $\alpha = 0.1, \theta = 5, \lambda = 0.15$. (g) $\alpha = 0.0001, \theta = 4, \lambda = 0.15$. (h) $\alpha = 0.25, \theta = 5, \lambda = 0.25$. (i) $\alpha = 0.25, \theta = 0.25, \lambda = 0.25$.

then we have

$$\begin{aligned}\mathbb{H}_{\rho,\underline{V_1},\underline{V_2}}(x_1,x_2) &= \left[1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{x_1}-\theta_1 x_1\right)\right]\right\}^{\lambda_1}\right]\times\left[1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{x_2}-\theta_2 x_2\right)\right]\right\}^{\lambda_2}\right]\\ &\quad\times\left(1+\rho\left\{\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{x_1}-\theta_1 x_1\right)\right]\right\}^{\lambda_1}\times\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{x_2}-\theta_2 x_2\right)\right]\right\}^{\lambda_2}\right\}\right).\end{aligned} \tag{17}$$

3.2. Via Clayton Copula. The BTIITLIFW type extension: the weighted version of the Clayton copula can be expressed as

$$\mathbb{H}(u,v)=\left[u^{-\rho}+v^{-(\varphi_1+\varphi_2)}-1\right]^{-1/\rho}|_{\rho\geq 0}. \quad (18)$$

Let us assume that $X\sim$ TIITLIFW $(\underline{V_1})$ and $Y\sim$ TIITLIFW $(\underline{V_2})$. Then, setting

$$u=u_{\underline{V_1}}(x)=1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{x}-\theta_1 x\right)\right]\right\}^{\lambda_1},$$
$$v=u_{\underline{V_2}}(y)=1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{y}-\theta_2 y\right)\right]\right\}^{\lambda_2}, \quad (19)$$

the associated CDF of the BTIITLIFW type distribution can be written as

$$\mathbb{H}(x,y)=\left\{\left[1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{x}-\theta_1 x\right)\right]\right\}^{\lambda_1}\right]^{-\rho}+\left[1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{y}-\theta_2 y\right)\right]\right\}^{\lambda_2}\right]^{-\rho}-1\right\}^{-1/\rho}. \quad (20)$$

A straightforward M-dimensional extension from the above will be

$$\mathbb{H}(x_1,x_2,\ldots,x_M)=\left\{\sum_{i=1}^{M}\left[1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_i}{x_i}-\theta_i x_i\right)\right]\right\}^{\lambda_i}\right]^{-\rho}-M+1\right\}^{-1/\rho}. \quad (21)$$

3.3. Via Modified Farlie-Gumbel-Morgenstern (FGM) Copula. The joint CDF of the modified FGM copula is given as $\mathbb{H}_{\rho}(u,w)=uw[1+\rho A(u)V(w)]|_{\rho\in(-1,1)}$ or $\mathbb{H}_{\rho}(u,w)=uw+\rho\dot{A}(u)\dot{V}(w)|_{\rho\in(-1,1)}$, where $\dot{A}(u)=uA(u)$, and $\dot{V}(w)=wV(w)$, where $A(u)$ and $V(w)$ are being two continuous functions on the interval $(0,1)$ where $0=A(0)=A(1)=V(0)=V(1)$. Let

$$c_{1,u}=\inf\left\{\frac{\partial}{\partial u}\dot{A}(u)|\mathbf{k}_1 t(u)\right\}<0,$$
$$c_{1,u}=\sup\left\{\frac{\partial}{\partial u}\dot{A}(u)|\mathbf{k}_1 t(u)\right\}<0,$$
$$d_{2,w}=\inf\left\{\frac{\partial}{\partial w}\dot{V}(w)|\mathbf{k}_2 t(w)\right\}>0,$$
$$d_{2,w}=\sup\left\{\frac{\partial}{\partial w}\dot{V}(w)|\mathbf{k}_2 t(w)\right\}>0. \quad (22)$$

Then, $1\leq\min(c_{1.u}c_{1,u},d_{2,w}d_{2,w})\leq\infty$, where

$$\frac{\partial}{\partial u}A(u)=A(u)+u\frac{\partial}{\partial u}A(u),$$
$$\mathbf{k}_1(u)=\left\{u\colon u\in(0,1)|\frac{\partial\dot{A}(u)}{\partial u}t\ n\text{exists}\right\},$$
$$\mathbf{k}_2(w)=\left\{w\colon w\in(0,1)|\frac{\partial\dot{V}(w)}{\partial w}t\ n\text{exists}\right\}. \quad (23)$$

Type I:
Recalling the following functional form for both $A(u)$ and $V(w)$. Then, the BTIITLIFW-FGM (Type-I) can be derived from

$$\mathbb{H}_{\rho}(u,w)=uw+\rho\dot{A}(u)\dot{V}(w)|_{\rho\in(-1,1)}, \quad (24)$$

where

$$\dot{A}(u)=u\left[1-F_{\underline{V_1}}(u)\right],$$
$$\dot{V}(w)=w\left[F_{\underline{V_2}}(w)\right]. \quad (25)$$

Therefore,

$$\mathbb{H}_{\rho}(u,w)=\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}+\rho\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\right)\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\right)|_{\rho\in(-1,1)}. \tag{26}$$

Type II:

Let $A(u)^*$ and $V(w)^*$ be two functional forms, satisfying all the conditions mentioned above, where

$$A(u)^*|_{(\rho_1>0)}=(1-u)^{1-\rho_1}u^{\rho_1},$$
$$V(w)^*|_{(\rho_2>0)}=(1-w)^{1-\rho_2}w^{\rho_2}. \tag{27}$$

Then, the corresponding BTIITLIFW-FGM (Type-II) can be derived from

$$\mathbb{H}_{\rho,\rho_1,\rho_2}(u,w)=uw\left[1+\rho A(u)^*V(w)^*\right]. \tag{28}$$

Then,

$$\begin{aligned}\mathbb{H}_{\rho,\rho_1,\rho_2}(u,w)=&\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\\&\times\left\{1+\rho\left[\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\right)^{1-\rho_1}\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1\rho_1}\right.\right.\\&\left.\left.\times\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\right)^{1-\rho_2}\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2\rho_2}\right]\right\}.\end{aligned} \tag{29}$$

Type-III:

Let $\ddot{A}(u)|_{u\in(0,1)}$ and $\ddot{V}(w)|_{w\in(0,1)}$ be two functional forms, satisfying all the abovementioned conditions where

$$\ddot{A}(u)=u\log(\dot{u}+t1)|_{\dot{u}+u=1},\ \ddot{V}(w)=w\log(\dot{w}+t1)|_{\dot{w}+w=1}. \tag{30}$$

In this case, one can also derive a closed form expression for the associated CDF of the BT**II**TLIFW-FGM (Type-III) from

$$\mathbb{H}_{\rho}(u,w)=uw\left[\rho\ddot{A}(u)\ddot{V}(u)+1\right], \tag{31}$$

as follows

$$\begin{aligned}\mathbb{H}_{\rho}(u,w)=&\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\\&\times\left\{1+\rho\left[\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\log\left(2-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{u}-\theta_1 u\right)\right]\right\}^{\lambda_1}\right)\right.\right.\\&\left.\left.\times\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\log\left(2-\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{w}-\theta_2 w\right)\right]\right\}^{\lambda_2}\right)\right]\right\}.\end{aligned} \tag{32}$$

3.4. Via Renyi's Entropy. Let $m\in(0,1)=F_{\underline{V}_1}(x_1)$ and $w\in(0,1)=F_{\underline{V}_2}(x_2)$. Then, the Renyi's entropy copula can be expressed as

$$\mathbb{H}(u,w)=x_2u+x_1w-x_1x_2. \tag{33}$$

Then, the associated BTIITLIFW can be directly derived from $C(m,w)=C(F_{\underline{P}_1}(x_1),F_{\underline{P}_2}(x_2))$ as

$$\mathbb{H}(u,w)=x_2\left\{1-\exp\left[-2\exp\left(\frac{\alpha_1}{x_1}-\theta_1 x_1\right)\right]\right\}^{\lambda_1}+x_1\left\{1-\exp\left[-2\exp\left(\frac{\alpha_2}{x_1}-\theta_2 x_1\right)\right]\right\}^{\lambda_2}-x_1x_2. \tag{34}$$

4. Statistical and Reliability Measures

4.1. Quantile Function. For RV Z has CDF of the TIITLIFW distribution, the quantile function Z_q of the TIITLIFW distribution is given by the following equation:

$$Z_q = \frac{1}{2\theta}\left\{-d(q) + \sqrt{d^2(q) + 4\theta\alpha}\right\}, \quad 0 < q < 1, \tag{35}$$

where

$$d(q) = \log\left[\frac{-1}{2}\log\left(1 - (1-q)^{1/\lambda}\right)\right]. \tag{36}$$

4.2. Moments and Generating Functions. The r_{th} moment of the TIITLIFW distribution is obtained using the formula

$$\mu_r'(z) = \int_0^\infty z^r f(z)\mathrm{d}z, \tag{37}$$

hence using equation (7) we obtain

$$\mu_r'(z) = \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k} \int_0^\infty \left(\theta z^{r-k} + \alpha z^{r-k-2}\right)\exp[-(j+1)\theta z]\mathrm{d}z. \tag{38}$$

Setting $y = (j+1)\theta z$, it follows that

$$\mu_r'(z) = \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\Gamma(r-k+1)}{\theta_*^{r-k+1}} + \frac{\alpha\Gamma(r-k-1)}{\theta_*^{r-k-1}}\right]\Big|_{(\theta_* = [(j+1)\theta])}, \tag{39}$$

where

$$\Gamma(1+\nabla) = \int_0^\infty z^\nabla \exp(-z)\mathrm{d}z \tag{40}$$

is a gamma function. In particular, if $r = 1$ and $r = 2$, we obtain the mean and variance of the TIITLIFW distribution. The MGF of the TIITLIFW is given by

$$\begin{aligned} M_Z(t) &= E(\exp(tZ)) \\ &= \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\Gamma(1-k)}{[(j+1)\theta - t]^{1-k}} + \frac{\alpha\Gamma(-k-1)}{[(j+1)\theta - t]^{-k-1}}\right]. \end{aligned} \tag{41}$$

The mathematical form of the Galton skewness and Moors kurtosis of TIITLIFW distribution can be computed using the quantile function and well-known relationships. The first four moments, skewness, and kurtosis of the TIITLIFW distribution for different values of parameters are represented in Table 1. Table 1 shows that the skewness is always positive, and kurtosis is always greater than three.

Figure 3 shows two plots of the skewness of the TIITLIFW distribution. Figure 4 presents two plots of the kurtosis of the TIITLIFW distribution. Based on Figures 3 and 4, we conclude that the TIITLIFW model can have many useful skewness and kurtosis shapes.

4.3. Incomplete Moments. The s^{th} lower and upper incomplete moments of Z are defined by

$$\begin{aligned} v_{s,Z}(u) &= E\left(Z^s|_{(Z<u)}\right) = \int_0^u f(z)z^s\mathrm{d}z, \\ O_{s,Z}(u) &= E\left(Z^s|_{(Z>u)}\right) = \int_u^\infty z^s f(z)z^s\mathrm{d}z, \end{aligned} \tag{42}$$

respectively, for any real $= s > 0$. The s^{th} lower incomplete moment of the TIITLIFW distribution is

$$\begin{aligned} v_{s,Z}(t) &= \int_0^t z^s f(z)\mathrm{d}z = \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\int_0^t\left(\theta z^{s-k} + \alpha z^{s-k-2}\right)\exp[-(j+1)\theta z]\mathrm{d}z \\ &= \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\gamma(s-k+1,(j+1)\theta t)}{\theta_*^{s-k+1}} + \frac{\alpha\gamma(s-k-1,(j+1)\theta t)}{\theta_*^{s-k-1}}\right], \end{aligned} \tag{43}$$

where

$$\gamma(s_1, s_2) = \int_0^{s_2} z^{s_1 - 1}\exp(-z)\mathrm{d}z \tag{44}$$

is the lower incomplete gamma function. Similarly, the s^{th} upper incomplete moment of the TIITLIFW distribution is

$$\begin{aligned} O_{s,Z}(t) &= \int_t^\infty z^s f(z)\mathrm{d}z \\ &= \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\int_t^\infty\left(\theta z^{s-k} + \alpha z^{s-k-2}\right)\exp[-(j+1)\theta z]\mathrm{d}z \\ &= \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\Gamma(s-k+1,(j+1)\theta t)}{\theta_*^{s-k+1}} + \frac{\alpha\Gamma(s-k-1,(j+1)\theta t)}{\theta_*^{s-k-1}}\right], \end{aligned} \tag{45}$$

where

$$\Gamma(1+s,t) = \int_t^\infty z^s\exp(-z)\mathrm{d}z \tag{46}$$

is the upper incomplete gamma function.

4.4. Mean Deviation, Lorenz, and Bonferroni Curves. For RV Z with PDF, $f(z)$, distribution function $F(z)$, mean $\mu = E(Z)$, and $M = \text{Median}(Z)$, the mean deviation about the mean and median, respectively, is given by

$$\begin{aligned} \delta_{1,\mu}(z) &= \int_0^\infty |z-\mu| f(z)\mathrm{d}z = 2\mu F(\mu) - 2\mu + 2O_1(\mu), \\ \delta_{2,M}(z) &= \int_0^\infty |z-M| f(z)\mathrm{d}z = 2MF(M) - M - \mu + 2O_1(M), \end{aligned} \tag{47}$$

where

$$\begin{aligned} O_{1,Z}(d) &= \int_d^\infty z f(z)\mathrm{d}z \\ &= \sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left\{\frac{\theta\Gamma(-k+2,(k+1)\theta d)}{\theta_*^{-k+2}} + \frac{\alpha\Gamma(-k,(j+1)\theta d)}{\theta_*^{-k}}\right\}. \end{aligned} \tag{48}$$

Table 1: Moments, skewness, and kurtosis of the TIITLIFW distribution.

α	θ	λ	μ_1'	μ_2'	μ_3'	μ_4'	Skewness	Kurtosis
1	1	1	1.93	1.050	1.86	8.6	1.740	7.87
1	3	4	0.61	0.010	0.0009	0.0005	0.880	4.37
2	7	2	0.58	0.004	0.0003	9.90×10^{-5}	1.074	5.23
2	10	5	0.45	0.001	9.6×10^{-6}	1.85×10^{-6}	0.489	3.51
3	1	8	1.65	0.045	0.006	0.007	0.568	3.61
3	3	10	0.96	0.004	8.9×10^{-5}	6.09×10^{-5}	0.312	3.24
4	7	17	0.72	0.0006	9.2×10^{-7}	9.96×10^{-7}	0.066	3.02
4	10	20	0.61	0.0003	1.5×10^{-10}	2.00×10^{-7}	3.55×10^{-5}	3.002

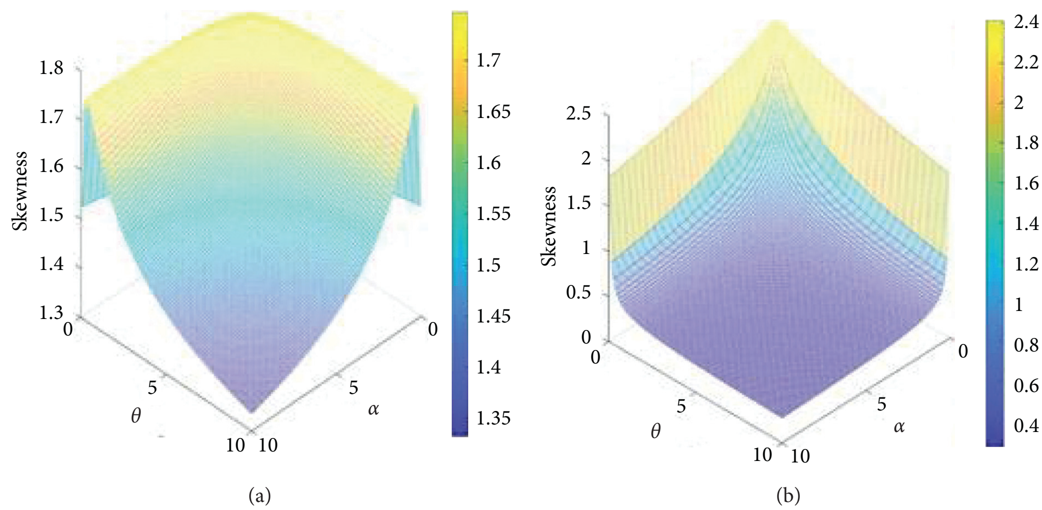


Figure 3: Plots of the skewness of the TIITLIFW distribution. (a) Skewness for $\lambda = 1.1$. (b) Skewness for $\lambda = 7$.

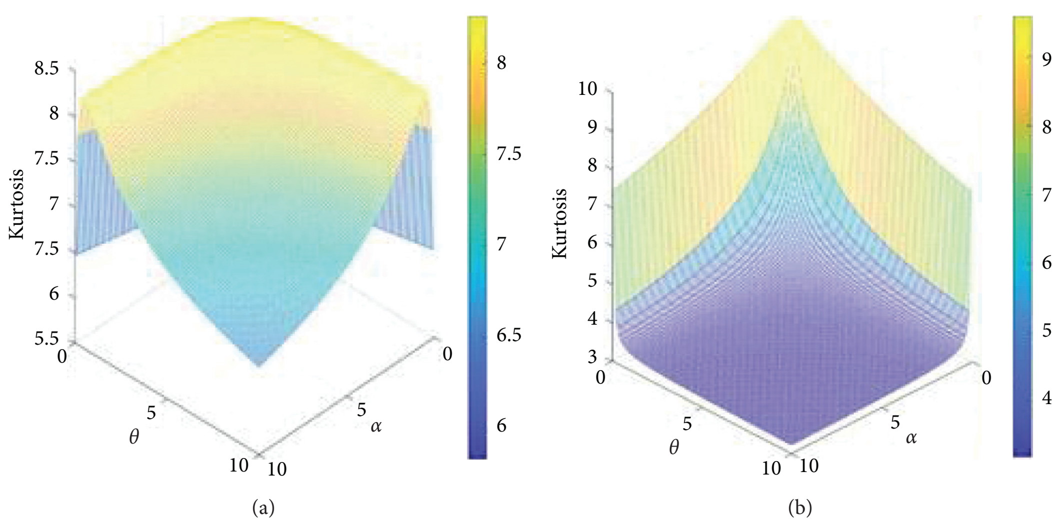


Figure 4: Plots of the kurtosis of the TIITLIFW distribution. (a) Kurtosis for $\lambda = 1.1$. (b) Kurtosis for $\lambda = 7$.

The Lorenz curve for a positive RV Z is defined as

$$L(p) = \frac{1}{\mu}\int_0^q z f(z)dz = \frac{1}{\mu}v_{1,Z}(q). \tag{49}$$

Then, we have

$$L(p) = \frac{1}{\mu}\sum_{i,j,k=0}^{\infty}\nabla_{i,j,k}\left[\frac{\theta\gamma(-k+2,(j+1)\theta q)}{\theta_*^{-k+2}} + \frac{\alpha\gamma(-k,(j+1)\theta q)}{\theta_*^{-k}}\right], \tag{50}$$

where $q = G^{-1}(p)$. Also, Bonferroni curve is defined by

$$B(p) = \frac{1}{\mu p}\int_0^q z f(z)\mathrm{d}z = \frac{v_{1,Z}(q)}{\mu p}. \quad (51)$$

Then,

$$B(p) = \frac{1}{\mu p}\sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\gamma(-k+2,(j+1)\theta q)}{\theta_*^{-k+2}} + \frac{\alpha\gamma(-k,(j+1)\theta q)}{\theta_*^{-k}}\right]. \quad (52)$$

4.5. The Mean and Strong Mean Inactivity Times Functions. The mean inactivity time (MIT) can be derived from

$$m(t) = E\left((t-Z)|_{Z\le t}\right) = t - \frac{1}{F(t)}\int_0^t z f(z)\mathrm{d}z. \quad (53)$$

Then, the MIT can be derived as

$$m(t) = t - \frac{1}{F(t)}\sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\gamma(-k+2,(j+1)\theta t)}{\theta_*^{-k+2}} + \frac{\alpha\gamma(-k,(j+1)\theta t)}{\theta_*^{-k}}\right]. \quad (54)$$

The strong mean inactivity time (SMIT) is a new reliability function given by

$$S(t) = \frac{1}{F(t)}\int_0^t 2zF(z)\mathrm{d}z = t - \frac{1}{F(t)}\int_0^t z^2 f(z)\mathrm{d}z. \quad (55)$$

Therefore, the SMIT can be expressed as

$$S(t) = t - \frac{1}{F(t)}\sum_{i,j,k=0}^{\infty} \nabla_{i,j,k}\left[\frac{\theta\gamma(-k+3,(j+1)\theta t)}{\theta_*^{-k+3}} + \frac{\alpha\gamma(1-k,(j+1)\theta t)}{\theta_*^{1-k}}\right]. \quad (56)$$

5. Non-Bayesian Estimation Methods under Uncensored Schemes

5.1. The MLE Method. Let $Z_1, Z_2, \ldots, Z_n$ be a random sample of size n from TIITLIFW. The log-likelihood function for the vector of parameters $\underline{V}$ can be written as

$$\log(\mathcal{L}) = n\log(2\lambda) + \sum_{i=1}^{n}\log\left(\theta + \frac{\alpha}{z_{[i:n]}^2}\right) - \alpha\sum_{i=1}^{n} z_{[i:n]}^{-1} - \theta\sum_{i=1}^{n} z_{[i:n]} - 2\sum_{i=1}^{n}\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right) + (\lambda-1)\sum_{i=1}^{n}\log\left\{1-\exp\left[-2\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right)\right]\right\}. \quad (57)$$

The associated score function is given by

$$U_n(\underline{V}) = \left[\frac{\partial(\log(\mathcal{L}))}{\partial\lambda}, \frac{\partial(\log(\mathcal{L}))}{\partial\alpha}, \frac{\partial(\log(\mathcal{L}))}{\partial\theta}\right]^T. \quad (58)$$

The $\log(\mathcal{L})$ can be maximized by solving the following nonlinear likelihood equations $U_n(\underline{V}|\alpha) = \partial\log(\mathcal{L})/\partial\alpha, U_n(\underline{V}|\theta) = \partial\log(\mathcal{L})/\partial\theta$ and $U_n(\underline{V}|\lambda) = \partial\log(\mathcal{L})/\partial\lambda$. Then,

$$\begin{aligned}
U_n(\underline{V}|\alpha) &= \frac{1}{\alpha+\theta z_{[i:n]}^2} - \sum_{i=1}^{n} z_{[i:n]}^{-1} - 2\sum_{i=1}^{n} z_{[i:n]}^{-1}\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right) \\
&\quad + 2(\lambda-1)\sum_{i=1}^{n}\frac{\exp\left[-2\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)\right]\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)z_{[i:n]}^{-1}}{1-\exp\left[-2\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)\right]}, \\
U_n(\underline{V}|\theta) &= \sum_{i=1}^{n}\frac{z_i^2}{\alpha+\theta z_{[i:n]}^2} - \sum_{i=1}^{n} z_{[i:n]} - 2\sum_{i=1}^{n} z_{[i:n]}\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right) \\
&\quad + 2(\lambda-1)\sum_{i=1}^{n}\frac{z_{[i:n]}\exp\left[-2\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)\right]\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)}{1-\exp\left[-2\exp\left(\left(\alpha/z_{[i:n]}\right)-\theta z_{[i:n]}\right)\right]}, \\
U_n(\underline{V}|\lambda) &= \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left(1-\exp\left[-2\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right)\right]\right).
\end{aligned} \quad (59)$$

The maximum likelihood estimation (MLE) of $\underline{V}$, say $\widehat{\underline{V}}$, is obtained by solving the system of nonlinear equations

$$U_n(\underline{V}|\alpha) = U_n(\underline{V}|\theta) = U_n(\underline{V}|\lambda) = 0. \quad (60)$$

5.2. The CVME Method. The CVMEs of the parameters α, θ, and λ [12] are obtained by minimizing the following expression with respect to the parameters α, θ, and λ, respectively, where

$$\text{CVME}_{(\underline{V})} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n}\left[F_{\underline{V}}\left(z_{[i:\,n]}\right) - p_{(i,n)}\right]^2, \tag{61}$$

where

$$p_{(i,n)} = \frac{2i-1}{2n} \tag{62}$$

refers to the empirical estimate of the CDF at $z_{[i:\,n]}$ computed from a certain sample and

$$\text{CVME}_{(\underline{V})} = \sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - p_{(i,n)}\right)^2. \tag{63}$$

Then, CVMEs of the parameters α, θ, and λ are obtained by solving the following nonlinear system:

$$\begin{aligned}
&\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - p_{(i,n)}\right)D_{(\alpha)}\left(z_{[i:\,n]};\underline{V}\right) = 0,\\
&\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - p_{(i,n)}\right)D_{(\theta)}\left(z_{[i:\,n]};\underline{V}\right) = 0,\\
&\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - p_{(i,n)}\right)D_{(\lambda)}\left(z_{[i:\,n]};\underline{V}\right) = 0,
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
D_{(\alpha)}\left(z_{[i:n]};\underline{V}\right) &= \frac{\partial F_{\underline{V}}\left(z_{[i:n]}\right)}{\partial \beta},\\
D_{(\theta)}\left(z_{[i:n]};\underline{V}\right) &= \frac{\partial F_{\underline{V}}\left(z_{[i:n]}\right)}{\partial b},\\
D_{(\lambda)}\left(z_{[i:n]};\underline{V}\right) &= \frac{\partial F_{\underline{V}}\left(z_{[i:n]}\right)}{\partial a}.
\end{aligned} \tag{65}$$

5.3. The OLSE and WLSE Method. Let $F_{\underline{V}}(z_{[i:\,n]})$ denote the CDF of the TIITLIFW model and $z_{1:\,n} < z_{2:\,n} < \cdots < z_{n:\,n}$ be the n ordered RS. The OLSEs [13] are obtained by minimizing

$$\text{OLSE}(\underline{V}) = \sum_{i=1}^{n}\left[F_{\underline{V}}\left(z_{[i:n]}\right) - q_{(i,n)}\right]^2, \tag{66}$$

where

$$q_{(i,n)} = \frac{i}{n+1}. \tag{67}$$

Then,

$$\text{OLSE}(\underline{V}) = \sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:n]}} - \theta z_{[i:n]}\right)\right]\right\}^{\lambda} - q_{(i,n)}\right)^2. \tag{68}$$

The OLSEs are obtained by solving

$$\begin{aligned}
0 &= \sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - q_{(i,n)}\right)D_{(\alpha)}\left(z_{[i:\,n]};\underline{V}\right),\\
0 &= \sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - q_{(i,n)}\right)D_{(\theta)}\left(z_{[i:\,n]};\underline{V}\right),\\
0 &= \sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}} - \theta z_{[i:\,n]}\right)\right]\right\}^{\lambda} - q_{(i,n)}\right)D_{(\lambda)}\left(z_{[i:\,n]};\underline{V}\right),
\end{aligned} \tag{69}$$

where $D_{(\alpha)}(z_{[i:\,n]};\underline{V})$, $D_{(\theta)}(z_{[i:\,n]};\underline{V})$, and $D_{(\lambda)}(z_{[i:\,n]};\underline{V})$ are as defined above. The WLSE is obtained by minimizing the function WLSE $(\underline{V})$ with respect to α, θ, and λ. Then,

$$\text{WLSE}(\underline{V}) = \sum_{i=1}^{n}\tau_{(i,n)}\left[F_{\underline{V}}\left(z_{[i:\,n]}\right) - q_{(i,n)}\right]^2, \tag{70}$$

where

$$\tau_{(i,n)} = \frac{\left[(1+n)^2(2+n)\right]}{[i(1+n-i)]}. \tag{71}$$

The WLSEs are obtained by solving

$$0 = \sum_{i=1}^{n}\left[\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}}-\theta z_{[i:\,n]}\right)\right]\right\}^{\lambda}-q_{(i,n)}\right)\tau_{(i,n)}D_{(\alpha)}\left(z_{[i:\,n]};\underline{V}\right)\right],$$
$$0 = \sum_{i=1}^{n}\left[\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}}-\theta z_{[i:\,n]}\right)\right]\right\}^{\lambda}-q_{(i,n)}\right)\tau_{(i,n)}D_{(\theta)}\left(z_{[i:\,n]};\underline{V}\right)\right], \tag{72}$$
$$0 = \sum_{i=1}^{n}\left[\left(1-\left\{1-\exp\left[-2\exp\left(\frac{\alpha}{z_{[i:\,n]}}-\theta z_{[i:\,n]}\right)\right]\right\}^{\lambda}-q_{(i,n)}\right)\tau_{(i,n)}D_{(\lambda)}\left(z_{[i:\,n]};\underline{V}\right)\right].$$

5.4. The ADE Method. The ADEs are obtained by minimizing the function

$$ADE_{\left(z_{[i:\,n]},z_{[-i+1+n:\,n]}\right)}(\underline{V}) = -n - n^{-1}\sum_{i=1}^{n}\left[(2i-1)\times\left\{\log F_{(\underline{V})}\left(z_{[i:\,n]}\right)+\log\overline{F_{(\underline{V})}}\right\}\right], \tag{73}$$

where

$$\overline{F_{(\underline{V})}}\left(z_{[-i+1+n:\,n]}\right) = \left[1-F_{(\underline{V})}\left(z_{[-i+1+n:\,n]}\right)\right]. \tag{74}$$

Then, the parameter estimates are derived by solving the nonlinear equations

$$0 = \frac{\partial\left[\mathrm{ADE}_{\left(z_{[i:n]},z_{[-i+1+n:n]}\right)}(\underline{V})\right]}{\partial\alpha},$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{\left(z_{[i:n]},z_{[-i+1+n:n]}\right)}(\underline{V})\right]}{\partial\theta}, \tag{75}$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{\left(z_{[i:n]},z_{[-i+1+n:n]}\right)}(\underline{V})\right]}{\partial\lambda}.$$

5.5. The ADE (R-T) Method. The ADEs (R-T) are obtained by minimizing

$$\mathrm{ADE}(R-T)_{\left(z_{[i:\,n]},z_{[-i+1+n:\,n]}\right)}(\underline{V}) = \frac{1}{2}n - 2\sum_{i=1}^{n}F_{(\underline{V})}\left(z_{[i:\,n]}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\overline{F_{(\underline{V})}}\left(z_{[-i+1+n:\,n]}\right)\right\}. \tag{76}$$

Table 2: Simulation study where the parameters $\alpha = 2$, $\theta = 3$, and $\lambda = 1.1$.

	n	α		θ		λ	
		BIAS	RMSE	BIAS	RMSE	BIAS	RMSE
MLE	20	3.960664661	7.371292530	6.502976058	12.27860585	5.994434410	19.60201039
CVM		1.620584171	4.069729958	2.213270730	6.529766076	12.59993180	30.87329914
OLS		2.710178460	5.609086527	5.167076155	10.57746096	4.269781273	11.26286107
WLS		0.606986631	1.328889644	1.240119335	2.801331754	0.303093751	1.238119094
ADE		0.877995110	1.077412768	−1.63464215	1.761963513	0.438186918	0.623443953
ADE (R-T)		0.805406079	0.893762293	−1.64368531	1.709771020	0.467885067	0.699341007
MLE	60	0.490490730	1.308428645	0.795829577	2.486467250	0.682538359	2.697087211
CVM		0.225350690	1.215083979	0.201813532	2.282594960	2.660481953	7.613754145
OLS		0.690519124	2.090460467	1.195212075	3.941427479	1.765869927	6.040806374
WLS		0.601764048	1.247109825	1.289725679	2.611933294	-0.00644429	0.873971382
ADE		0.373785697	0.452084104	−2.16735411	2.181478121	0.423126388	0.611877218
ADE (R-T)		0.457139728	0.511691568	−2.12760556	2.139745408	0.42017769	0.547818729
MLE	100	0.018172775	0.507215114	−0.025370613	1.037306445	0.319774071	0.782082919
CVM		0.538024886	1.56715411	0.8451798220	2.808614624	2.721488864	10.63346516
OLS		0.244321092	0.908633723	0.4496964790	1.755602945	0.856444063	3.727821407
WLS		0.372181773	1.093968544	0.6547917990	2.093192893	0.296432069	1.102596681
ADE		0.299928998	0.363326475	−2.294706687	2.303336884	0.553574110	0.880204347
ADE (R-T)		0.299368704	0.357439216	−2.293811686	2.301440994	0.322106629	0.456459819
MLE	200	0.059490160	0.379563707	0.086240490	0.799301698	0.100750015	0.513932285
CVM		0.206137566	0.755763241	0.343870896	1.480081518	0.459569237	2.626198546
OLS		0.189569092	0.705969135	0.303683878	1.359136200	0.634967556	3.901570961
WLS		0.531225762	0.991440159	1.025489159	1.903738414	-0.14192911	0.631061741
ADE		0.150472878	0.210401078	−2.43351747	2.437339304	0.380366426	0.721144948
ADE (R-T)		0.171083470	0.238382214	−2.41188394	2.416440980	0.214422238	0.556727524
MLE	500	0.051955667	0.242120349	0.106991720	0.514449659	0.0088330000	0.288297422
CVM		0.016797852	0.343498639	−0.00934092	0.679792300	0.1496096990	0.459788656
OLS		-0.01498810	0.294877728	−0.03730224	0.609548552	0.1311953210	0.481402194
WLS		0.278499022	0.571907974	0.537259780	1.106740303	-0.10950015	0.406177306
ADE		0.092722301	0.159113157	−2.48691428	2.489745756	-0.13488436	0.375397440
ADE (R-T)		0.075620057	0.146911912	−2.50639867	2.508592698	-0.10385764	0.461934338

Then, the estimates follow by solving the nonlinear equations

$$0 = \frac{\partial \mathrm{ADE}(R-T)_{(z_{[i:n]}, z_{[-i+1+n:n]})}(\underline{V})}{\partial \alpha},$$
$$0 = \frac{\partial \mathrm{ADE}(R-T)_{(z_{[i:n]}, z_{[-i+1+n:n]})}(\underline{V})}{\partial \theta}, \quad (77)$$
$$0 = \frac{\partial \mathrm{ADE}(R-T)_{(z_{[i:n]}, z_{[-i+1+n:n]})}(\underline{V})}{\partial \lambda}.$$

6. Comparing the Non-Bayesian Estimation Methods under Uncensored Schemes via a Simulation Study

A numerical simulation study is conducted to compare the non-Bayesian estimation methods. The simulation study is based on $N = 1000$ which generated datasets from the TIITLIFW distribution where $n = 20, 60, 100, 200$, and 500 and $\alpha = 2, \theta = 3$, and $\lambda = 1.1$. The comparison is performed based on the bias $(\mathrm{BIAS}_{(\underline{V})})$ and root mean - standard error $(\mathrm{RMSE}_{(\underline{V})})$.

From Table 2, we note the following:

1) The $\mathrm{BIAS}_{(\underline{V})}$ tends to 0 as n increases and tends to ∞ which means that all estimators are nonbiased.

2) The $\mathrm{RMSE}_{(\underline{V})}$ tends to 0 as n increases and tends to ∞ which means incidence of consistency property.

7. Comparing Models under Uncensorship

We illustrate the flexibility and the performance of the TIITLIFW distribution as compared to some alternative models using two real data applications. The goodness-of-fit statistics for this distribution are compared with other competitive distributions. The MLEs of the distribution parameters are determined numerically. To compare the distributions, we consider the measures of goodness-of-fit, such as Akaike information criterion (C_1), consistent Akaike information criterion (C_2), and Bayesian information criterion (C_3) statistic. The better distribution to fit the data corresponds to smaller values of these statistics.

We consider two uncensored datasets for comparing competitive models. The first data present the remission times (in months) of a random sample of 128 bladder cancer patients [14]: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31,

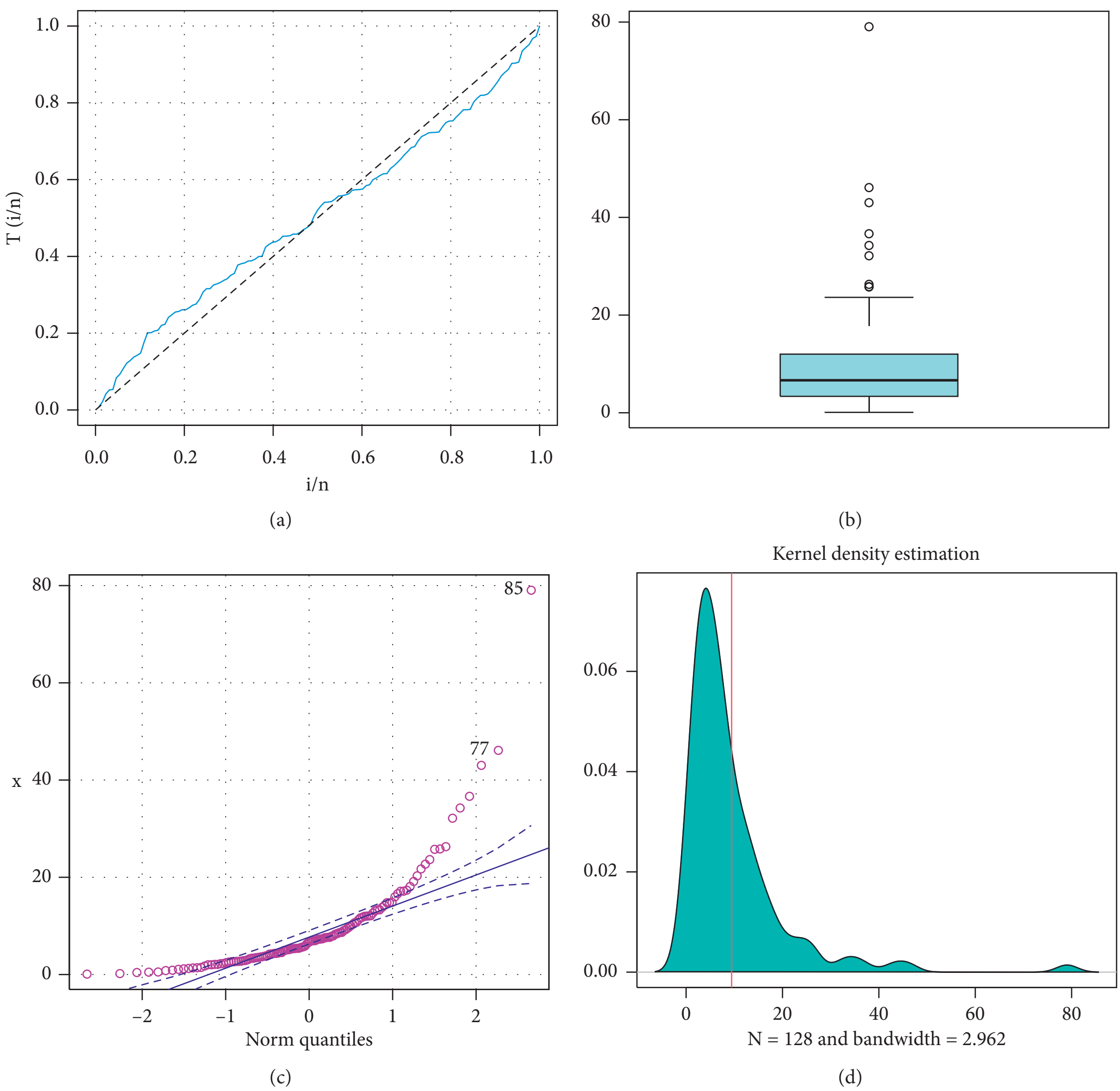


Figure 5: The TTT plot (a), box plot (b), QQ plot (c), and KDE plot (d) for cancer data.

0.81, 2.69, 4.23, 5.41, 0.90, 2.69, 4.18, 5.34, 7.59, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 5.71, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 10.66, 15.96, 36.66, 1.05, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, and 22.69. The second data present the lifetimes of 38 devices provided by [14]: 0.1, 1, 1, 1, 2, 3, 6, 40, 45, 46, 47, 50, 55, 0.2, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 1, 82, 83, 84, 84, 84, 85, 85, 1, 60, 63, and 86, 86. According to [15], the total time in test (TTT) plots, box plots, quantile-quantile (QQ) plots, and kernel density estimation (KDE) plots are shown in Figure 5 and Figures 6(a) and 6(c) for bladder cancer data and lifetimes data, respectively. Based on Figures 6(a) and 6(c), the HRF of the bladder cancer data is "upside down" or "reversed **U**-shape." Based on Figures 6(a) and 6(c), the HRF of the lifetimes data is "**U**-shape." The box plots (middle panels) are presented along with its corresponding normal quantile-quantile plot (right panels) in Figures 5 and 6 for discovering the outliers and normality.

The following competitive models are considered in the comparison: the exponentiated IFW (Exp-IFW) [12], IFW [3], exponentiated generalized IW (ExpG-IFW) [16], generalized IW (G-IW) [17], IW [18], and [2].

Tables 3 and 4 present the MLEs for the bladder cancer data and lifetimes data. Tables 5 and 6 show the statistics criteria for the bladder cancer data and lifetime data. From Tables 5 and 6, it is clear that the TIITLIFW distribution provides the best fits for the two datasets. Figures 7 and 8 show the estimated PDFs (EPDFs) (left panel) and the estimated HRF (EHRFs) (right panel) for bladder cancer data and lifetimes data, respectively. Figures 9 and 10 show the profile of the log-likelihood function for bladder cancer data and lifetimes data, respectively. From

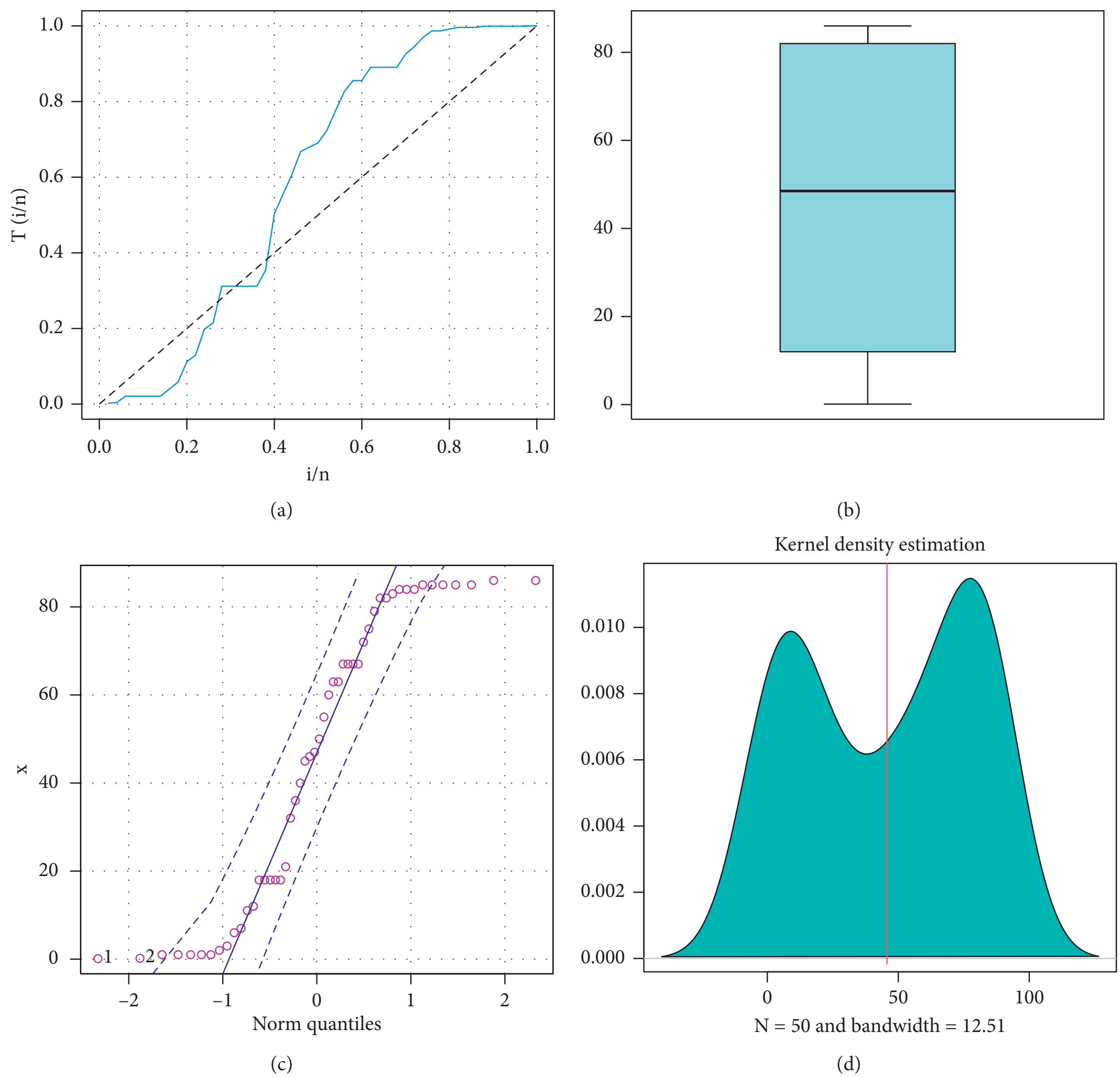


Figure 6: The TTT plot (a), box plot (b), QQ plot (c), and KDE plot (d) for the lifetimes data.

Table 3: The MLEs for bladder cancer data.

Model	Estimates			
	α	θ	λ	β
TIITLIFW	0.070	0.468	0.26	
Exp-IFW	0.080	0.169	2.47	
IFW	0.126	0.143		
ExpG-IFW	1.006	0.500	1.05	2
G-IW	0.750	0.340	1.79	
IW	16.14	0.464		
FW	0.054	0.915		

Table 4: The MLEs for lifetimes data.

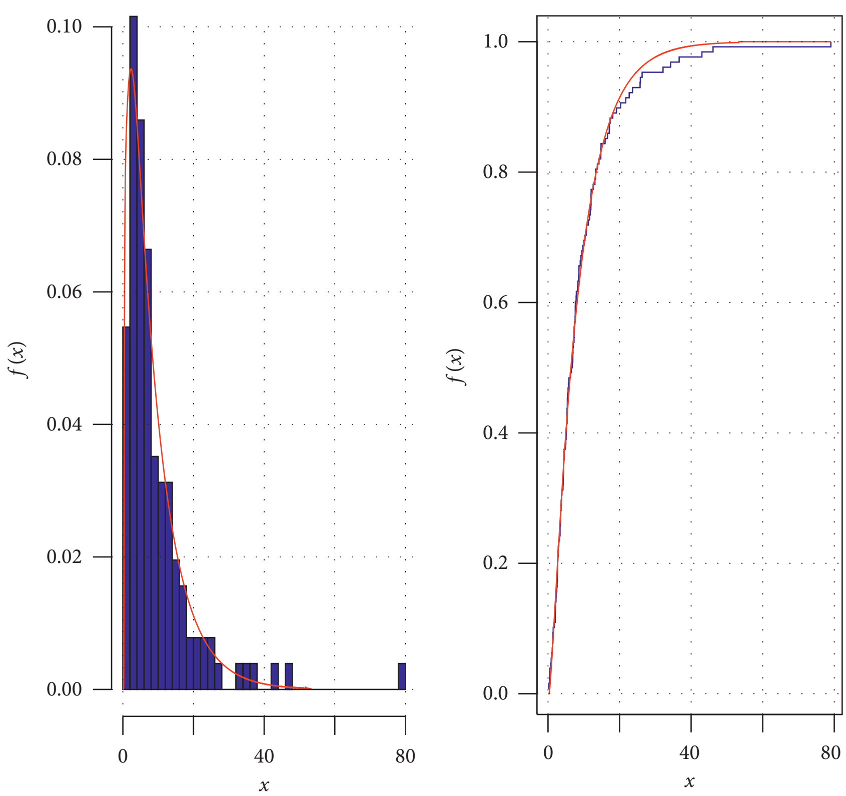
Model	Estimates			
	α	θ	λ	β
TIITLIFW	0.108	0.025	1.198	
Exp-IFW	0.099	0.030	2.187	
IFW	0.165	0.024		
ExpG-IFW	1.008	0.610	2.14	0.75
G-IW	0.596	0.274	1.27	
IW	1.043	0.397		
FW	0.012	0.70		

Table 5: Statistics for bladder cancer data.

Model	$-2\log(\mathscr{L})$	C_1	C_2	C_3
TIITLIFW	413.85	833.695	833.889	842.251
Exp-IFW	423.46	852.909	853.104	861.470
IFW	453.61	911.220	911.310	916.920
ExpG-IFW	488.05	984.090	984.420	995.500
G-IW	495.18	996.360	996.560	1004.92
IW	500.12	1004.25	1004.33	1009.94
FW	525.53	1055.07	1055.16	1060.77

Table 6: Statistics for lifetimes data.

Model	$-2\log(\mathscr{L})$	C_1	C_2	C_3
TIITLIFW	233.51	473.027	473.549	478.763
Exp-IFW	233.52	473.029	473.551	478.765
IFW	242.57	488.914	489.169	492.738
ExpG-IFW	250.81	505.620	505.880	509.448
G-IW	254.92	517.839	518.727	525.487
IW	287.48	580.951	581.473	586.687
FW	281.07	566.140	566.396	569.964



(a) (b)

Figure 7: The EPDF (a) and the EHRF (b) for bladder cancer data.

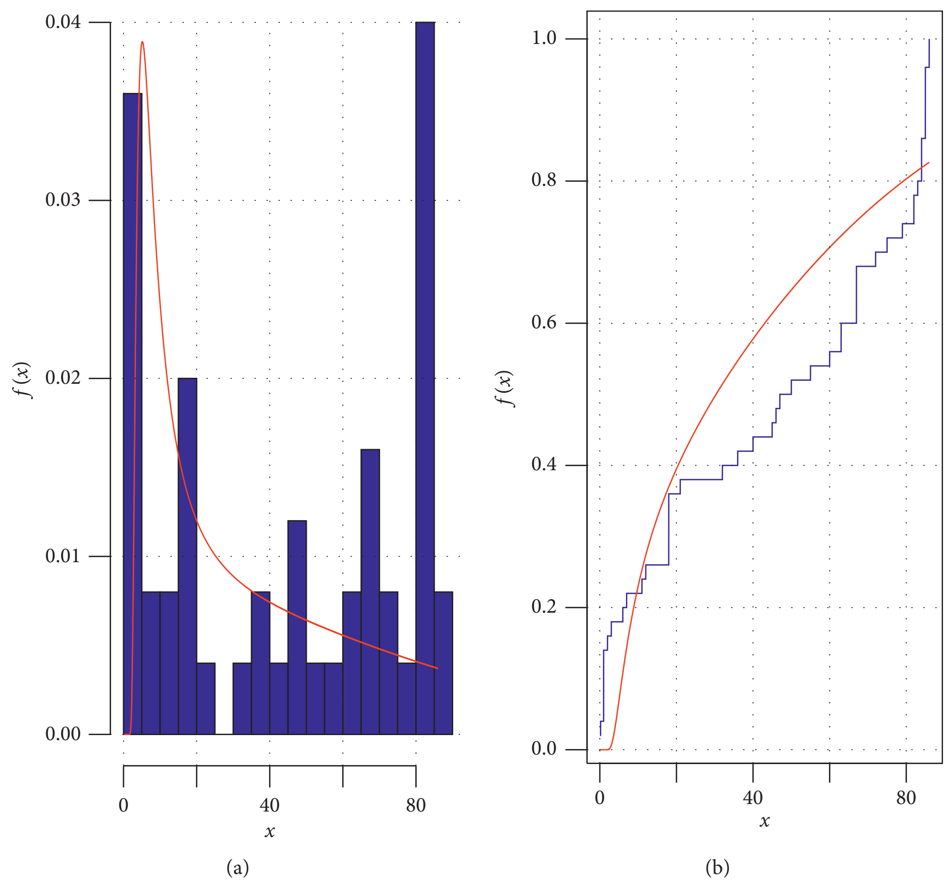


Figure 8: The EPDF (a) and the EHRF (b) for lifetimes data.

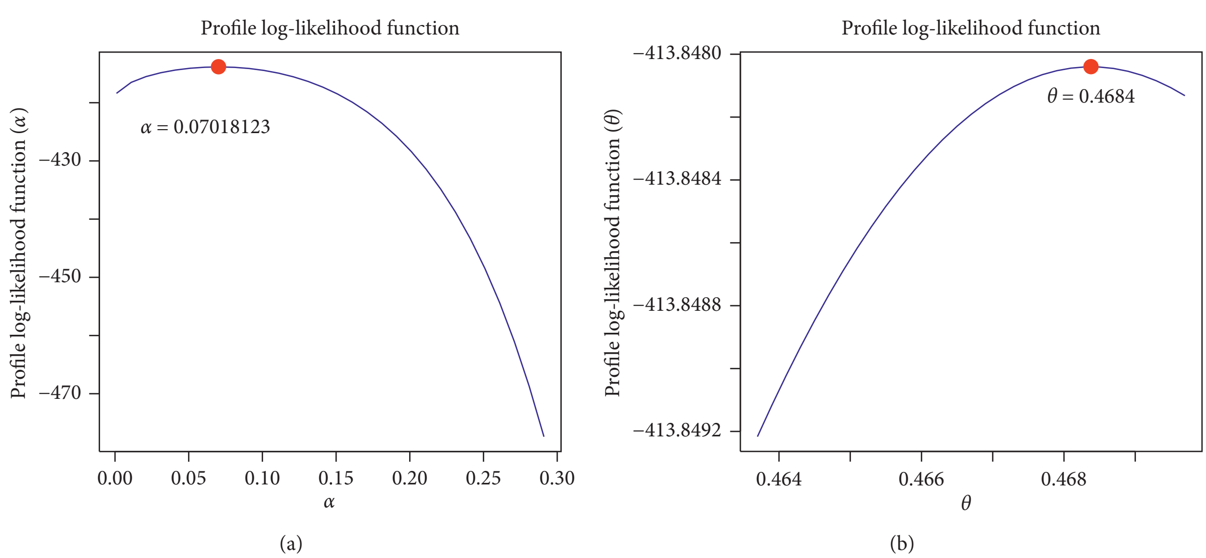
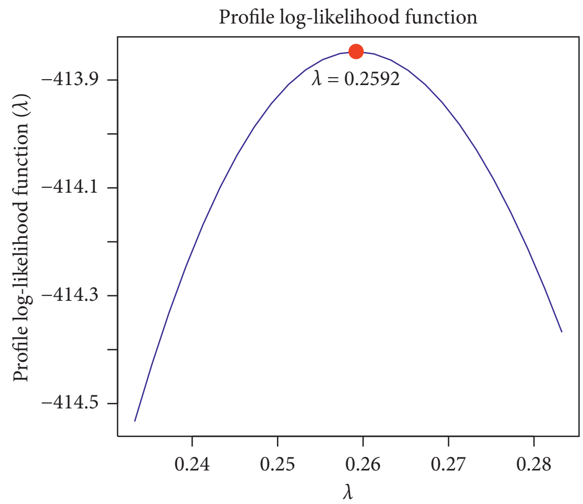
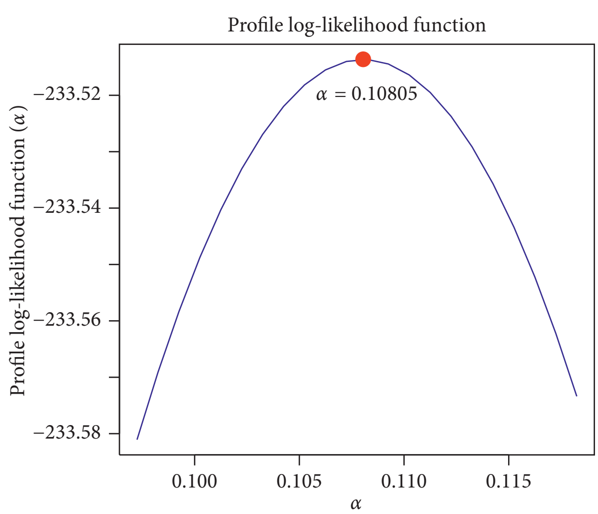


Figure 9: Continued.

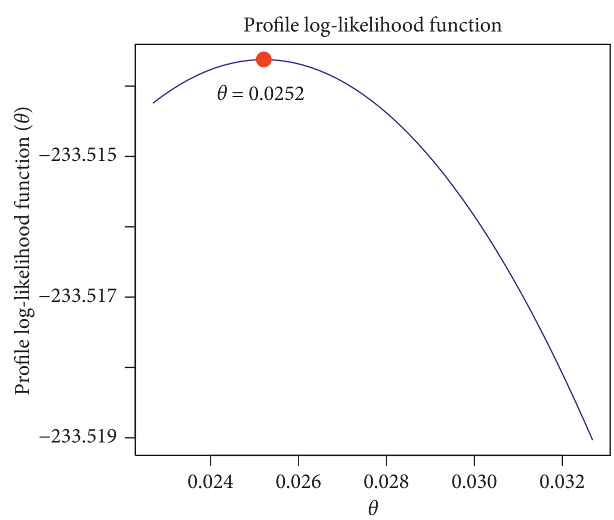


(c)

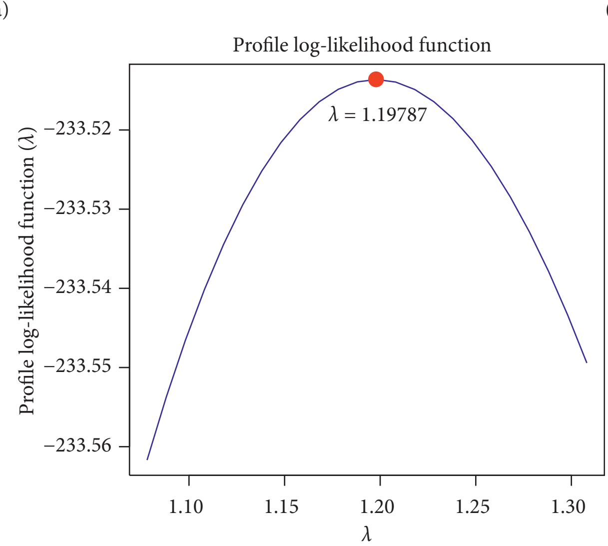
Figure 9: The profile of the log-likelihood function for bladder cancer data.



(a)



(b)



(c)

Figure 10: The profile of the log-likelihood function for lifetimes data.

Figures 7 and 8, we conclude that the new model can achieve a good fit.

8. Concluding Remarks

A three-parameter lifetime distribution, so-called the TIITLIFW distribution, is introduced as an extension of the inverse flexible Weibull distribution. Some explicit expressions for mathematical quantities of the TIITLIFW distribution are derived. The hazard rate function allows constant, decreasing, increasing, upside down bathtub, or bathtub-shaped forms. We consider six different estimation methods to estimate the parameters of the TIITLIFW distribution. The performance of these proposed estimation methods is conducted via some simulations. A real data application proves that the TIITLIFW model provides consistently better fits compared to some other well-known competitive models.

Data Availability

The data used to support the findings in this study are included within the paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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