Research Article

Polynomials and General Degree-Based Topological Indices of Generalized Sierpinski Networks

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Sierpinski networks are networks of fractal nature having several applications in computer science, music, chemistry, and mathematics. These networks are commonly used in chaos, fractals, recursive sequences, and complex systems. In this article, we compute various connectivity polynomials such as M -polynomial, Zagreb polynomials, and forgotten polynomial of generalized Sierpinski networks S_k^n and recover some well-known degree-based topological indices from these. We also compute the most general Zagreb index known as (α, β) -Zagreb index and several other general indices of similar nature for this network. Our results are the natural generalizations of already available results for particular classes of such type of networks.

1. Introduction

Networks are studied in graph theory as spacial graphs having nice properties. A fractal is a geometric structure showing capabilities of self-similarity and repetition throughout the structure. Some commonly used fractal-natured networks appear useful in the study of complex systems present in both natural and artificial systems such as computer systems, music, brain, and social networks, allowing further development of the field in network science. These networks are largely used to capture images of complex structures and predict behaviour of biological processes like nerve dendrites and growth pattern of bacteria in any culture. These networks are very close to WK-recursive networks used in the design and implementation of local area networks as well as parallel processing systems. A special case of generalized Sierpinski graphs is the duality of Apollonian network, which displays the prominent scale-free small-world characteristics as observed in various real

networks [1]. Because of diverse properties, these networks have been a subject matter of recent research.

A topological invariant is a mathematical property that remains invariant under the isomorphism of topological structures. In chemical graph theory, one is in search of the most general graph invariant that can capture at most all numerical descriptors of the structure. A new polynomial graph invariant, M -polynomial, was introduced in [2] by Deutsch and Klavžar. This invariant plays similar role in determining degree-based indices as played by Hosoya polynomial in determining distance-based indices.

Applications of the aforementioned polynomial can be traced in mathematical chemistry and pharmacology. Many degree-based topological indices can be derived using successive operations of differentiation and integration. The most interesting application of the M -polynomial is that almost all degree-based graph invariants, which are used to predict physical, chemical, and pharmacological properties

of organic molecules, can be recovered from it (for more information, see [3–9]).

The M -polynomial and related topological indices have been studied for several classes of graphs. In 2015, Deutsch and Klavžar gave M -polynomial, first Zagreb, and second Zagreb indices of polyomino chains, starlike trees, and triangulenes [2]. In 2017, Mobeen gave M -polynomial and several degree-based topological indices of titania nanotubes [10], of triangular boron nanotubes [11], and of Jahangir graph [12]. In 2017, Ajmal gave M -polynomial and topological indices of generalized prism network [13]. The authors computed M -polynomial and related indices of some benzenoid systems in [14], hex-derived network in [15], and V-phenylenic nanotubes and nanotori [16]. Today, the computation of this polynomial is considered as one of the most important areas of research in the fields of molecular computing and degree-based indices.

Several degree-based topological indices, which play important role in mathematical chemistry, can be recovered from the M -polynomial. The most famous degree-based index is the Randić index which was introduced by Milan Randić in 1975 [17]. It is often used in cheminformatics for investigations of organic compounds (for more information, see [18–20]). Later in 1998, working independently, Amic [21] and Bollobas-Erdos [22] proposed the generalized Randić index (for more information, see [23, 24]). Gutman and Trinajstić introduced first Zagreb and second Zagreb indices in 1972 [25]. The augmented Zagreb index was proposed by Furtula in 2010 in [26] and is useful for computing heat of information of alkanes [18, 27]. To know more about topological indices, their computing, and their applications, we refer the reader to [2, 8, 11, 12, 28–33].

This article is organized as follows. In Section 2, we give definitions of graph, topological indices, M -polynomial, and Sierpinski networks. The main results about the Sierpinski networks are given in Section 3; this section also covers the results about the topological indices.

2. Preliminary Notes

Definition 1. A graph G is a pair (V, E) , where V is the set of vertices and E is the set of edges. Any standard text on graph theory can be used as fundamental concepts related to graph and network theory.

The edge e between two vertices u and v is denoted by (u, v) . The degree of a vertex u , denoted by d_u , is the number of edges incident to it. A path from a vertex v to a vertex w is a sequence of vertices and edges that starts from v and stops at w . The number of edges in a path is the length of that path. A graph is said to be *connected* if there is a path between any two of its vertices.

A *molecular graph* is a representation of a chemical compound in terms of graph theory. Specifically, molecular graph is a graph whose vertices correspond to (carbon) atoms of the compound and whose edges correspond to chemical bonds [34]. For our computational models, we consider graphs which are hydrogen suppressed.

In the following, by G we shall mean a connected graph, E its edge set, V its vertex set, $e = (u, v)$ its edge joining the vertices u and v , and d_u degree of its vertex u .

Definition 2 (see [2]). The M -polynomial of G is $M(G; x, y) = \sum_{i \leq j} m_{ij} x^i y^j$, where m_{ij} is the number of edges $e = (u, v)$ of G with $d_u = i$ and $d_v = j$.

Definition 3. The first Zagreb polynomial, second Zagreb polynomial, and forgotten polynomial of G are, respectively, defined as $M_1(G; x) = \sum_{uv \in E(G)} x^{[d_u + d_v]}$, $M_2(G; x) = \sum_{uv \in E(G)} x^{[d_u \times d_v]}$, and $F(G; x) = \sum_{uv \in E(G)} x^{[d_u^2 + d_v^2]}$.

A function I which assigns to every connected graph G a unique number $I(G)$ is called a *graph invariant*. Instead of the function I , it is custom to say the number $I(G)$ as the invariant.

Definition 4 (see [34]). An invariant of a molecular graph which can be used to determine structure-property or structure-activity correlation is called the *topological index*. A topological index is an invariant of the graph.

A topological index is said to be *degree-based* if it depends on degrees of the vertices of the graph.

The following are definitions of some degree-based indices that have connection with the M -polynomial.

Definition 5. The first Zagreb, second Zagreb, and second modified Zagreb indices of G are defined, respectively, by $M_1(G) = \sum_{e \in E} (d_u + d_v)$, $M_2(G) = \sum_{e \in E} d_u \times d_v$, and $MM_2(G) = \sum_{e \in E} 1/(d_u \times d_v)$.

Definition 6. The generalized Randić and reciprocal generalized Randić indices of G are defined, respectively, by $R_\alpha(G) = \sum_{e \in E} (d_u \times d_v)^\alpha$ and $RR_\alpha(G) = \sum_{e \in E} 1/(d_u \times d_v)^\alpha$.

Definition 7. The symmetric division index of G is defined by the relation

$$SSD(G) = \sum_{e \in E} \left\{ \frac{\min(d_u, d_v)}{\max(d_u, d_v)} + \frac{\max(d_u, d_v)}{\min(d_u, d_v)} \right\}. \quad (1)$$

Definition 8. The harmonic, inverse sum, augmented Zagreb, and forgotten indices are defined, respectively, by $H(G) = \sum_{uv \in E(G)} 2/(d_u + d_v)$, $I(G) = \sum_{uv \in E(G)} ((d_u d_v)/(d_u + d_v))$, $A(G) = \sum_{uv \in E(G)} \{(d_u d_v)/(d_u + d_v - 2)\}^3$, and $F(G) = \sum_{uv \in E(G)} [d_u^2 + d_v^2]$.

A remarkable property of the M -polynomial is that all the above degree-based indices can be recovered from it, using the relations given in Table 1.

Another approach to reach at many Zagreb indices is to compute (α, β) -Zagreb index introduced by Azhari et al. in [35]. It can be defined as

TABLE 1: Relations of indices with M -polynomial.

Index	Relation with the M -polynomial
First Zagreb	$M_1(G) = (D_x + D_y)(M(G))_{x=y=1}$
Second Zagreb	$M_2(G) = (D_x \times D_y)(M(G))_{x=y=1}$
Modified second Zagreb	$MM_2(G) = (S_x \times S_y)(M(G))_{x=y=1}$
Generalized Randić	$R_\alpha(G) = (D_x^\alpha \times D_y^\alpha)(M(G))_{x=y=1}$
Reciprocal generalized Randić	$RR_\alpha(G) = (S_x^\alpha \times S_y^\alpha)(M(G))_{x=y=1}$
Symmetric division	$SDD(G) = (D_x S_y + D_y S_x)(M(G))_{x=y=1}$
Harmonic	$H(G) = 2S_x J(M(G))_{x=1}$
Inverse sum	$I(G) = S_x J D_x D_y (M(G))_{x=1}$
Augmented Zagreb	$A(G) = S_x^3 Q_{-2} J D_x^3 D_y^3 (M(G))_{x=1}$
Where	$D_x = x(\partial/\partial x)M(G)$, $D_y = y(\partial/\partial y)M(G)$ $S_x = \int_0^x (M(G; t, y)/t)dt$, $S_y = \int_0^y (M(G; x, t)/t)dt$ $J = M(G; x, x)$, $Q_{-2} = x^{-2}M(G; x, y)$

$$Z_{\alpha,\beta} = \sum_{uv \in E(G)} [(du)^\alpha (dv)^\beta + (du)^\beta (dv)^\alpha]. \quad (2)$$

Sarkar et al. [36] computed $Z_{\alpha,\beta}$ for some derived networks. The authors computed this index for V-phenylenic nanotubes and nanotori molecules [37] and circumcoronene series of benzenoid in [38]. This is somehow more general index, and several other indices can be calculated evaluating it at certain values. The following table establishes relation of other Zagreb indices as particular cases of this general Zagreb index. The redefined Zagreb index was first introduced in 2013 by Ranjini et al. [39] and is defined as

$$\text{ReZM}(G) = \sum_{uv \in E(G)} (du)(dv)[(du) + (dv)], \quad (3)$$

to generalize the Zagreb indices. Table 2 gives mathematical derivations of these indices from (α, β) -Zagreb index [40]. The authors computed this index for some nanotubes and graphs in [41, 42]. Table 2 relates some of Zagreb types indices with (α, β) -Zagreb index.

Definition 9. Let V^t be a set of words of size t on the set of alphabets of V . The letters of word u of length t are denoted by $u_1, u_2, \dots, u_t$. The concatenation of two words u and v is denoted by uv . The graph $S(K_n, t)$, introduced in [43], has vertex set V^t . An edge exists in $\{u, v\}$ if and only if there exists $i \in \{1, 2, \dots, t\}$ such that $u_j = v_i$ if $j < i$ and $u_i \leq v_i$, and $u_j = v_i$ and $v_j = u_i$ if $j > i$. For $n = 3$, we receive Tower of Hanoi graphs. Later in [44] these graphs have been called as *Sierpinski networks*.

Sierpinski networks form an extensively studied class of graphs of fractal nature applicable in topology, mathematics of Tower of Hanoi, computer science, and in other disciplines. A large number of properties like physicochemical properties, thermodynamic properties, chemical activity, biological activity, and so on are determined by the chemical applications of graph theory. In [45], the authors computed molecular topological properties of Sierpinski networks and derived closed formulas for the atom-bond connectivity index, geometric-arithmetic index, and fourth and fifth version of these topological indices for Sierpinski networks. In [46], the authors discussed some matrical aspects of these

TABLE 2: Relations of Indices with (α, β) -Zagreb index.

Index	Derivation from (α, β) -Zagreb index
Refined Zagreb index, ReZM	$Z_{(2,1)}$
General first Zagreb index, M^α	$Z_{(\alpha-1,0)}$
Forgotten index, F(G)	$Z_{(2,0)}$

networks. Recently, the authors computed various versions of degree-based topological indices of some variants of these networks using direct approach [47]. But connectivity polynomials such as the M -polynomial and generalized Zagreb Index have not been computed for this network. Connectivity and number of connected components of this network have been computed recently in [48]. Hamiltonicity of Sierpinski networks has been discussed as well in [48]. Recently in [49], the author used harmonic Sierpinski gasket as a geometric configuration of small antennas. In [50], the authors discussed some energies of the Sierpinski gasket. The authors in [51] computed strong metric dimension of generalized S_k^n having isolated pendant nodes. Total colorings of S_k^n were discussed in [52]. The authors in [53] discussed all kinds of shortest paths in Sierpinski networks with the help of newly designed algorithm. For comprehensive survey and classifications of S_k^n , we refer the readers to [54]. In this paper, we compute M -polynomial and some degree-based topological indices of Sierpinski graphs S_k^n . The Sierpinski network S_3^n has $3^n, n \geq 2$ vertices and $3(3^n - 1)/2$ edges. Figure 1 shows S_3^2 where S_3 is triangle.

Now, we present S_3^4 in Figure 2.

The Sierpinski network S_4^n has $4^n, n \geq 2$ vertices and $4(4^n - 1)/3$ edges. Figure 3 depicts first two instances of Sierpinski network S_4^n for $n = 1$ and $n = 2$.

Note that S_3^n and S_4^n contain only vertices of degrees 2 and 3. Edge partition of $S(n, k)$ Sierpinski network is based on degrees of end vertices of each edge.

3. Main Results

In this section, we give M -polynomial, related indices, and some general Zagreb indices of Sierpinski networks S_k^n .

Theorem 1. The M -polynomial of S_k^n , $n \geq 2$, is $M(S_k^n; x, y) = 2kx^{k-1}y^k + ((k^{n+1} - 5k)/2)x^k y^k$.

FIGURE 1: S_3^2 . FIGURE 2: S_3^4 . FIGURE 3: S_4 and S_4^2 .

Proof. We prove it by finding sets that partition vertices and edges of S_k^n . The disjoint sets of vertices of S_k^n are

$$\begin{aligned} V_{\{k-1\}} &= \{v \in V(G) | d_v = 2\} \longrightarrow |V_{\{1\}}| = k, \\ V_{\{k\}} &= \{v \in V(G) | d_v = 3\} \longrightarrow |V_{\{2\}}| = k^n - k. \end{aligned} \quad (4)$$

The edge partition of S_k^n contains the sets

$$\begin{aligned} E_{\{k-1,k\}} &= \{e = uv \in E(S_k^n) | d_u = 2, d_v = k\} \longrightarrow |E_{\{2,k\}}| = 2k, \\ E_{\{k,k\}} &= \{e = uv \in E(S_k^n) | d_u = 3, d_v = 3\} \longrightarrow |E_{\{k,k\}}| \\ &= \frac{k^{n+1} - 5k}{2}. \end{aligned} \quad (5)$$

Now the M -polynomial of S_k^n is

$$\begin{aligned} M(S_k^n; x, y) &= \sum_{i \leq j} m_{ij} x^i y^j \\ &= \sum_{2 \leq k} m_{2k} x^{k-1} y^k + \sum_{k \leq k} m_{kk} x^k y^k \\ &= \sum_{uv \in E_{\{2,k\}}} m_{2k} x^{k-1} y^k + \sum_{uv \in E_{\{k,k\}}} m_{kk} x^k y^k \\ &= |E_{\{2,k\}}| x^{k-1} y^k + |E_{\{k,k\}}| x^k y^k \\ &= 2k x^{k-1} y^k + \left(\frac{k^{n+1} - 5k}{2} \right) x^k y^k. \end{aligned} \quad (6)$$

□

Theorem 2. The first Zagreb, second Zagreb, modified second Zagreb, Randić, reciprocal generalized Randić, and symmetric division indices of $S_k^n, n \geq 2$, are

- (1) $M_1(S_k^n) = 4k^2 - 2k + k(k^{n+1} - 5k)$
- (2) $M_2(S_k^n) = 2k^2(k-1) + k^2((k^{n+1} - 5k)/2)$
- (3) $mM_2(S_k^n) = (2/(k-1)) + ((k^n - 5)/2k)$
- (4) $R\alpha(S_k^n) = 2k^{\alpha+1}(k-1)^\alpha + k^{2\alpha}((k^{n+1} - 5k)/2)$

- (5) $RR\alpha(S_k^n) = (2/k^{\alpha-1}) + (k^{n+1}/2k^{2\alpha})$
 (6) $SDD(S_k^n) = ((2(k-1)^2 + 2k^2)/(k-1)) + k^{n+1} - 5k$

Proof. Before giving step-by-step proofs of all items, we find $D_x, D_y, S_x,$ and S_y .

$$\begin{aligned}
 D_x(S_k^n) &= 2k(k-1)x^{k-1}y^k + k\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 D_y(S_k^n) &= 2k^2 x^{k-1}y^k + k\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 S_x(S_k^n) &= \frac{2k}{k-1}x^{k-1}y^k + \frac{1}{k}\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 S_y(S_k^n) &= 2x^{k-1}y^k + \frac{1}{k}\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 D_x D_y(M(S_k^n)) &= 2k^2(k-1)x^{k-1}y^k + k^2\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 (D_x + D_y)(M(S_k^n)) &= 4k^2 x^{k-1}y^k + 2k\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 S_x S_y(M(S_k^n)) &= \frac{2}{k-1}x^{k-1}y^k + \left(\frac{k^n-5}{2k}\right)x^k y^k, \\
 D_x^\alpha D_y^\alpha(M(S_k^n)) &= 2k^{\alpha+1}(k-1)^\alpha x^{k-1}y^k + k^{2\alpha}\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k, \\
 S_x^\alpha S_y^\alpha(M(S_k^n)) &= \frac{2x^{k-1}y^k}{k^{\alpha-1}} + \frac{k^{n+1}}{2k^{2\alpha}}x^k y^k, \\
 (D_x S_y + S_x D_y)(M(S_k^n)) &= \left(\frac{2(k-1)^2 + 2k^2}{k-1}\right)x^{k-1}y^k + 2k\left(\frac{k^{n+1}-5k}{2}\right)x^k y^k.
 \end{aligned} \tag{7}$$

Now the indices are

- (1) $M_1(S_k^n) = (D_x + D_y)(M(S_k^n))|_{x=y=1} = k^{n+2} - k^2 - 2k$
 (2) $M_2(S_k^n) = (D_x D_y)(M(S_k^n))|_{x=y=1} = 2k^2(k-1) + k^2((k^{n+1}-5k)/2)$
 (3) $mM_2(S_k^n) = (S_x S_y)(M(S_k^n))|_{x=y=1} = (2/(k-1)) + ((k^n-5)/2k)$
 (4) $R\alpha(S_k^n) = (D_x^\alpha D_y^\alpha)(M(S_k^n))|_{x=y=1} = 2k^{\alpha+1}(k-1)^\alpha + k^{2\alpha}((k^{n+1}-5k)/2)$
 (5) $RR\alpha(S_k^n) = (S_x^\alpha S_y^\alpha)(M(S_k^n))|_{x=y=1} = (2/(k-1)^\alpha)(k^{\alpha-1}) + ((k^{n+1}-5k)/2k^{2\alpha})$
 (6) $SDD(S_k^n) = (D_x S_y + S_x D_y)(M(S_k^n))|_{x=y=1} = ((2(k-1)^2 + 2k^2)/(k-1)) + k^{n+1} - 5k$ $\square$

Theorem 3. The harmonic, inverse sum, and augmented indices of $S_k^n, n \geq 1$, are

- (1) $H(S_k^n) = ((4k)/(2k-1)) + ((k^{n+1}-5k)/(2k))$
 (2) $I(S_k^n) = ((2k^2(k-1))/(2k-1)) + (k(k^{n+1}-5k)/4)$

- (3) $A(S_k^n) = ((2k^4(k-1)^3)/((2k-3)^3)) + (k^6/(2k-2)^3)((k^{n+1}-5k)/2)$

Proof

- (1) $H(S_k^n) = 2S_x J(S_k^n; x, y)|_{x=1} = [2S_x(2kx^{2k-1} + ((k^{n+1}-5k)/2)x^2k)]_{x=1} = 2[((2k)/(k-1))x^{2k-1} + (1/2k)((k^{n+1}-5k)/2)x^2k]_{x=1} = ((4k)/(2k-1)) + ((k^{n+1}-5k)/(2k))$
 (2) $I(S_k^n) = [S_x J D_x D_y(S_k^n)]_{x=1} = [S_x J D_x(D_y(S_k^n))]_{x=1} = [S_x J D_x 2k^2 x^{k-1}y^k + k((k^{n+1}-5k)/2)x^k y^k]_{x=1} = [S_x J 2k^2(k-1)x^{k-1}y^k + k^2((k^{n+1}-5k)/2)x^k y^k]_{x=1} = [S_x 2k^2(k-1)x^2k-1 + k^2((k^{n+1}-5k)/2x^2k)]_{x=1} = [(2k^2(k-1)2k-1)/(x^{2k-1})) + k((k^{n+1}-5k)/4)x^{2k}]_{x=1} = ((2k^2(k-1))/(2k-1)) + k((k^{n+1}-5k)/4)$
 (3) $A(S_k^n) = [S_x^3 Q_{-2} J D_x^3 D_y^3(M(S_k^n))]_{x=1} = [S_x^3 Q_{-2} J(2k^4(k-1)^3)x^{2k-1} + k^6 x^{2k}(k^{n+1}-5k/2)]_{x=1} = [S_x^3 Q_{-2}(2k^4(k-1)^3)x^{2k-3} + k^6 x^{2k-2}(k^{n+1}-5k/2)]_{x=1}$

FIGURE 4: $F(S_k^n)$.

$$= [S_x^3 ((2k^4 (k-1)^3) x^{2k-3} / (2k-3)^3) + k^6 ((x^{2k-2}) / (2k-2)^3) (k^{n+1} - 5k/2)]_{x=1} = [((2k^4 (k-1)^3) / (2k-3)^3) x^{2k-3} + k^6 ((x^{2k-2}) / (2k-2)^3) (k^{n+1} - 5k/2)]_{x=1} = [((2k^4 (k-1)^3) / (2k-3)^3) + k^6 (1 / (2k-2)^3) ((k^{n+1} - 5k) / 2)]$$

□

$$2k^2 |E_2(S_k^n)| = 2k((k-1)^2 + (k)^2) + 2k^2((k^{n+1} - 5k) / 2)$$

□

Theorem 4. The first Zagreb polynomial, second Zagreb polynomial, forgotten polynomial, and forgotten index of $S_k^n, n \geq 2$, are

- (1) $M_1(S_k^n; x) = 4k^2 x^{2k-1} + 2k((k^{n+1} - 5k) / 2) x^2 k$
- (2) $M_2(S_k^n; x) = 2k^2 (k-1) x^{2k-1} + k^2 ((k^{n+1} - 5k) / 2) x^{2k}$
- (3) $F(S_k^n; x) = 2kx^{(k-1)^2 + (k)^2} + ((k^{n+1} - 5k) / 2) x^{2k^2}$
- (4) $F(S_k^n) = 2k((k-1)^2 + (k)^2) + 2k^2((k^{n+1} - 5k) / 2)$

Proof

- (1) $M_1(S_k^n; x) = \sum_{uv \in E(S_k^n)} x^{[d_u + d_v]} = \sum_{uv \in E_1(S_k^n)} x^{[d_u + d_v]} + \sum_{uv \in E_2(S_k^n)} x^{[d_u + d_v]} = |E_1(S_k^n)| x^{2k-1} + |E_2(S_k^n)| x^{2k} = 2kx^{2k-1} + ((k^{n+1} - 5k) / 2) x^{2k}$
- (2) $M_2(S_k^n; x) = \sum_{uv \in E(S_k^n)} x^{[d_u \times d_v]} = \sum_{uv \in E_1(S_k^n)} x^{[d_u \times d_v]} + \sum_{uv \in E_2(S_k^n)} x^{[d_u \times d_v]} = |E_1(S_k^n)| x^{2k-1} + |E_2(S_k^n)| x^{2k} = 2k^2 (k-1) x^{2k-1} + k^2 ((k^{n+1} - 5k) / 2) x^{2k}$
- (3) $F(S_k^n; x) = \sum_{uv \in E(S_k^n)} x^{[d_u^2 + d_v^2]} = \sum_{uv \in E_1(S_k^n)} x^{[d_u^2 + d_v^2]} + \sum_{uv \in E_2(S_k^n)} x^{[d_u^2 + d_v^2]} = |E_1(S_k^n)| x^{(k-1)^2 + (k)^2} + |E_2(S_k^n)| x^{2k^2} = 2kx^{(k-1)^2 + (k)^2} + ((k^{n+1} - 5k) / 2) x^{2k^2}$
- (4) $F(S_k^n) = \sum_{uv \in E(S_k^n)} [d_u^2 + d_v^2] = \sum_{uv \in E_1(S_k^n)} [d_u^2 + d_v^2] + \sum_{uv \in E_2(S_k^n)} [d_u^2 + d_v^2] = (k-1)^2 + (k)^2 |E_1(S_k^n)| +$

Theorem 5. For all $S_k^n, n \geq 1$, we have $Z_{\alpha, \beta} = 2(k-1)^\alpha k^{\beta+1} + ((k^n - 5) / 2) k^{\alpha+\beta+1}$.

$$\text{REZ}\alpha(S_k^n) = Z_{2,1} = k^2 \left[2(k-1)^2 + \left(\frac{k^n - 5}{2} \right) k^2 \right],$$

$$M^\alpha(S_k^n) = Z_{\alpha-1,0} = \left[2k(k-1)^{\alpha-1} + \left(\frac{k^n - 5}{2} \right) k^\alpha \right], \quad (8)$$

$$F(S_k^n) = Z_{2,0} = k \left[2(k-1)^2 + \left(\frac{k^n - 5}{2} \right) k^2 \right].$$

Proof. To prove this, we again have to refer to Table 1 where edge partitions of $S_k^n, n \geq 1$ are given. Then, by using relation in (2), we have

$$\begin{aligned} Z_{\alpha, \beta} &= \sum_{uv \in E(G)} [(du)^\alpha (dv)^\beta + (du)^\beta (dv)^\alpha] \\ &= 2k(k-1)^\alpha (k)^\beta + \left(\frac{k^{n+1} - 5k}{2} \right) k^\alpha k^\beta \\ &= 2k(k-1)^\alpha (k)^\beta + \left(\frac{k^n - 5}{2} \right) k^{\alpha+\beta+1}. \end{aligned} \quad (9)$$

Remaining three parts of this theorem are just easy consequences of the result of $Z_{\alpha, \beta}$. □

FIGURE 5: Inverse sum index of (S_n^k) . FIGURE 6: $mM_2(S_n^k)$.

4. Conclusions and Discussion

In the present article, we computed some connectivity polynomials and many degree-based descriptors of Sierpinski networks specifically the closed form of M -polynomial and generalized Zagreb index. In this section, we give graphical representation of our results. Figures 4–7 are graphic index analysis of Sierpinski networks. These graphs are self explanatory to some extent and clearly demonstrate that values of all indices depend directly on the parameters n and k . We pose natural open problem to compute all above indices and polynomials for more generalized kinds of Sierpinski networks $S(G, K)$

where G is any graph. We here give short analysis of the indices depending upon n and k . For analysis, we use 3D graphs developed by GeoGebra. First 3D surface is plotted for forgotten index (see Figure 4). Slices along different axes represent the forgotten index for different values of n and k .

Next Figure 5 is plotted for inverse sum index. It can be seen that the surface has some part as of a saddle where along one parameter its curvature is opposite to the other parameter.

Figure 6 is the plotted surface for modified second Zagreb index of (S_n^k) . Here the surface is parabolic cylinder and curves on the surface show the curve for modified

FIGURE 7: $\text{ReZM}(S_n^k)$.

second Zagreb index for different combinations of parameters n and k . These curves are parabolic in nature which depict the behaviour of the respective index with respect to particular parameters involved.

Similarly, Figure 7 represents the surface plotted for redefined Zagreb index of (S_n^k) . Different graphs shown on the surface represent critical behaviour of this index for the parameters n and k . It is trivially observed that for some combinations, it is a straight line, and for some, it is parabolic. These facts can be used as raw inputs in networking and fractals where these networks play a significant role. All surfaces for other computed indices can be easily plotted and interpreted similarly. Physicochemical properties of the molecules modelled on the S_n^k networks can be forecasted using the results given above.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The manuscript was drafted and written by MMM and ZH. The concepts were initially developed by JBL, CF, and MA. All authors have read and approved the final manuscript.

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