

Research Article

Synchronization of a Class of Chaotic Systems with Different Dimensions

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In this paper, two scaling matrices are used to research the synchronization of different dimensional chaotic systems with unknown parameters. Firstly, the definition of synchronization of chaotic systems with different dimensions is introduced. Secondly, based on Lyapunov stability theorem and adaptive control method, an adaptive feedback hybrid controller and parameter adaptive laws are designed to realize synchronization of uncertain chaotic systems with different dimensions. Finally, three numerical experiments are carried out to verify the effectiveness of the proposed method.

1. Introduction

Chaos is a very complex nonlinear motion, which will show randomness, boundedness, unpredictability, and sensitive dependence on the initial value in the evolution process [1]. Chaotic system is different from the general system, and it is greatly affected by the changes in initial value and parameters, through the continuous in-depth understanding of the characteristics of chaotic motion, gradually realizing that the existence of chaos has brought some damage to the system; for example, chaotic motion can cause the circuit to form irregular oscillation and make the chaotic motion gradually away from the target trajectory, which may cause unpredictable consequences. At the same time, chaos also enlarges the influence of small changes and improves their effectiveness of achieving a specific desired state with very high flexibility and low cost. In some research fields, this property may be a valuable advantage. In 1990, Pecora and Carroll first realized chaos synchronization in electronic circuits [2], which set off an upsurge of chaos synchronization control research, making chaos synchronization widely used in many fields, such as communication encryption, engineering optimization, and statistical prediction [3–6].

In recent years, with the in-depth understanding of chaotic systems, some typical synchronization problems have been studied, such as complete synchronization, antisynchronization, projective synchronization, and hybrid synchronization [7–9]. However, in most of the control strategy design process, the system will be affected by various uncertain factors, and it is difficult to directly obtain some accurate mathematical models of the system. Some uncertain parameters of the chaotic system may have a certain impact on the performance of the system. On the contrary, due to the influence of these uncertain factors, the synchronization behavior of the system may be destroyed, and the emergence of unknown quantities in the system makes the synchronization control very complex. Many existing kinds of literatures consider the synchronization between two uncertain chaotic systems in the same dimension; for example, in [10], an adaptive sliding mode disturbance observer is designed to research the synchronization problem of uncertain Dadrás–Momeni chaotic systems with time-varying disturbances. In [11], a finite-time synchronization scheme is proposed for uncertain chaotic systems with external disturbances, parametric uncertainties, Lipschitz nonlinearities, and time delays. In [12], an adaptive

complete synchronization scheme is designed for uncertain financial chaotic systems with unknown parameters.

Synchronization behavior also exists in different dimensional systems in practice, and it is also especially important; for example, the circulatory and respiratory systems with different dimensions exist in synchronization behavior [13]. Similarly, in the communication network, the different dimension structure design and synchronization control of the driving system and the response system will increase the difficulty for the listener to decipher and make the communication more secure [14]. Therefore, it is of great significance to research the synchronization of chaotic systems with uncertain parameters and different dimensions in practical applications. In [15], the synchronization of a particular chaotic system with unknown parameters is studied by constructing the cross-error difference term. Recently, there have been some research studies on achieving the synchronization of chaotic systems by reducing dimension [16, 17], as well as some research studies by increasing dimension and reducing dimension simultaneously [18, 19]. In [20–22], the author further studies the synchronization of different dimensional integer-order or fractional-order chaotic or hyperchaotic systems using scaling matrices.

The main contribution of this paper is to research the scaling matrix synchronization of different dimensional chaotic or hyperchaotic systems with unknown parameters. A new suitable controller and parameter adaptive laws are designed to realize the synchronization of chaotic systems with different dimensions in a certain dimension. Simultaneously, according to the characteristics of the system, a transformation matrix is introduced, which can upgrade or reduce the order of the defined error system to meet the needs of different dimensional systems. Compared with [15–19], the synchronization scheme presented in this paper will not destroy the characteristics of system dimension and will directly synchronize chaotic systems with different dimensions in a certain dimension; on the other hand, the scaling matrix synchronization method used in this paper is not limited to a single synchronization type and can be transformed into a variety of synchronization schemes. In [20–22], although the author further studies the synchronization of chaotic systems with different dimensions by using the scale matrix, most of the results are only applicable to the fully known deterministic chaotic systems. However, they cannot do anything for the uncertain chaotic systems with unknown parameters, and we have done further research on the above work and extended the synchronization problem to the uncertain chaotic or hyperchaotic systems. Because the hyperchaotic system has at least two positive Lyapunov exponents, compared with chaotic system, the higher the degree of instability, the more complex the chaotic behavior. At the same time, according to the characteristics of these systems, appropriate controllers are designed to synchronize and stabilize the chaotic and hyperchaotic systems with different dimensions, which has certain requirements for the design of controllers. These factors bring great difficulties to our research.

This paper is organized as follows. Preliminaries are presented in Section 2. The synchronization theory of different dimension chaotic systems is introduced in detail, and the main conclusions are presented in Section 3. Three examples are presented in Section 4. Finally, in Section 5, concluding comments are given.

2. Preliminaries

Consider the following drive system in the form of

$$\begin{aligned}\dot{\mathbf{x}} &= F(\mathbf{x}), \\ \mathbf{x}(0) &= \mathbf{x}_0,\end{aligned}\quad (1)$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector of the drive system and $F(\mathbf{x}): \mathbf{R}^n \rightarrow \mathbf{R}^n$ defines a vector field in n -dimensional space. The response system is assumed by

$$\dot{\mathbf{y}} = G(\mathbf{y}) + \mathbf{u}, \quad (2)$$

where $\mathbf{y} \in \mathbf{R}^m$ is the state vector of the response system, $G(\mathbf{y}): \mathbf{R}^m \rightarrow \mathbf{R}^m$ defines a vector field in m -dimensional space, and $\mathbf{u} = (u_1, \dots, u_m)^T \in \mathbf{R}^m$ is the control input vector.

Definition 1 (see [20]). The drive system (1) and the response system (2) are said to be synchronized in dimension d , with respect to scaling matrices Θ and Φ , if there exists a controller $\mathbf{u} \in \mathbf{R}^m$ and given matrices $\Theta \in \mathbf{R}^{d \times m}$ and $\Phi \in \mathbf{R}^{d \times n}$ such that the synchronization error $\mathbf{e} = \Theta \mathbf{y} - \Phi \mathbf{x}$ satisfies that

$$\lim_{t \rightarrow \infty} \|\mathbf{e}(t)\| = 0, \quad (3)$$

where d is a certain constant and $\|\cdot\|$ is the vector norm.

In short, if this is the case, then the drive system (1) and the response (2) achieved $\Theta - \Phi$ synchronization.

Remark 1. Θ and Φ are constant scaling matrices.

Remark 2 (see [22]). It is easy to see that depending on our choice of the matrices Θ and Φ , we may have several synchronization types.

- (1) The pair $(\Theta, \Phi) = (I, I)$ yields complete synchronization as $\lim_{t \rightarrow \infty} \|\mathbf{y}(t) - \mathbf{x}(t)\| = 0$.
- (2) The pair $(\Theta, \Phi) = (I, -I)$ yields antisynchronization as $\lim_{t \rightarrow \infty} \|\mathbf{y}(t) + \mathbf{x}(t)\| = 0$.
- (3) The pair $(\Theta, \Phi) = (I, \Lambda)$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$ ($\lambda_i = \pm 1, i = 1, \dots, m$) yields hybrid synchronization as $\lim_{t \rightarrow \infty} \|\mathbf{y}(t) - \Lambda \mathbf{x}(t)\| = 0$.
- (4) The pair $(\Theta, \Phi) = (I, \Phi)$ yields matrix projection synchronization as $\lim_{t \rightarrow \infty} \|\mathbf{y}(t) - \Phi \mathbf{x}(t)\| = 0$.

Therefore, it can be seen from the above that Definition 1 can be applied not only to the synchronization of chaotic or hyperchaotic systems with different dimensions but also to the synchronization of chaotic or hyperchaotic systems with the same dimensions.

3. Scaling Matrix Synchronization of Chaotic Systems with Unknown Parameters

In this part, our main results include three parts: firstly, considering some unknown parameters in chaotic systems (1) and (2), we design new controller and parameter adaptive laws and give sufficient conditions to ensure the $\Theta - \Phi$ synchronization of chaotic systems with different dimensions; secondly, according to the characteristics of the system model, some new sufficient conditions are given to achieve the desired results; finally, some interesting corollaries are given.

Based on [20], we consider the problem of $\Theta - \Phi$ synchronization between n -dimensional drive system and m -dimensional response system with unknown parameters.

It is assumed that the drive system and response system with unknown parameters can be written in the following form:

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x})\boldsymbol{\alpha} + F_1(\mathbf{x}), \\ \dot{\mathbf{y}} &= g(\mathbf{y})\boldsymbol{\beta} + G_1(\mathbf{y}) + \mathbf{u},\end{aligned}\quad (4)$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{\tau_1})^T$ are parameters of drive system, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_{\tau_2})^T$ are parameters of response system, τ_1 and τ_2 are constants, and $f: \mathbf{R}^n \rightarrow \mathbf{R}^{n \times \tau_1}$, $g: \mathbf{R}^m \rightarrow \mathbf{R}^{m \times \tau_2}$, $F_1: \mathbf{R}^n \rightarrow \mathbf{R}^n$, $G_1: \mathbf{R}^m \rightarrow \mathbf{R}^m$ are all continuous functions.

Considering the error system of Definition 1, then the dynamic equation of the error system is

$$\dot{\mathbf{e}} = \Theta \dot{\mathbf{y}} - \Phi \dot{\mathbf{x}}. \quad (5)$$

Assumption 1. In general, $0 < d \leq \max\{m, n\}$. For the convenience of discussion, we will consider the case $d = m$ of dimension in this paper.

Assumption 2. The matrix Θ is row full rank matrix.

Remark 3 (see [23]). If the matrix Θ is row full rank matrix, then define Θ^{-1} as the right inverse of a matrix Θ .

Since there are unknown parameters in the systems, we assume that $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ represent the estimations of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Denote

$$\begin{aligned}R_1 &= (P\Theta\mathbf{y} + \Theta g(\mathbf{y})\hat{\boldsymbol{\beta}}) - (\Phi f(\mathbf{x})\hat{\boldsymbol{\alpha}} + P\Phi\mathbf{x}) \\ &+ \Theta G_1(\mathbf{y}) - \Phi F_1(\mathbf{x}),\end{aligned}\quad (6)$$

where $\Theta \in \mathbf{R}^{m \times m}$ and $\Phi \in \mathbf{R}^{m \times n}$ are the scaling matrices and $P \in \mathbf{R}^{m \times m}$ is the positive-definite control matrix. For the convenience of calculation, let $P = \text{diag}(p_1, p_2, \dots, p_m)$, $p_i > 0$, $i = 1, \dots, m$.

The following adaptive laws are proposed to estimate the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\begin{aligned}\Delta \dot{\boldsymbol{\alpha}} &= (\Phi f(\mathbf{x}))^T \mathbf{e} - \rho(\Delta \boldsymbol{\alpha}), \\ \Delta \dot{\boldsymbol{\beta}} &= -(\Theta g(\mathbf{y}))^T \mathbf{e} - \rho'(\Delta \boldsymbol{\beta}),\end{aligned}\quad (7)$$

where $\Delta \boldsymbol{\alpha} = \boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}$, $\Delta \boldsymbol{\beta} = \boldsymbol{\beta} - \hat{\boldsymbol{\beta}}$, and ρ and ρ' are the constant gain matrices. For the convenience of calculation, let $\rho = \text{diag}(\rho_1, \rho_2, \dots, \rho_{\tau_1})$, $\rho_i > 0$, $i = 1, \dots, \tau_1$ and $\rho' = \text{diag}(\rho'_1, \rho'_2, \dots, \rho'_{\tau_2})$, $\rho'_i > 0$, $i = 1, \dots, \tau_2$.

Then, based on equation (7), the controller can be designed as follows:

$$\mathbf{u} = -\Theta^{-1}R_1. \quad (8)$$

Hence, the corresponding error system is obtained as

$$\begin{aligned}\dot{\mathbf{e}} &= \Theta \dot{\mathbf{y}} - \Phi \dot{\mathbf{x}} \\ &= \Theta(g(\mathbf{y})\boldsymbol{\beta} + G_1(\mathbf{y}) + \mathbf{u}) - \Phi(f(\mathbf{x})\boldsymbol{\alpha} + F_1(\mathbf{x})) \\ &= -P\mathbf{e} + \Theta g(\mathbf{y})\Delta \boldsymbol{\beta} - \Phi f(\mathbf{x})\Delta \boldsymbol{\alpha}.\end{aligned}\quad (9)$$

Theorem 1. For the chaotic systems (4) with unknown parameters, if the controller \mathbf{u} is designed as equation (8) and the parameter adaptive laws are designed as equation (7), then the error system (9) is asymptotically stable at the origin. It can also be said that the response system and drive system in (4) achieved $\Theta - \Phi$ synchronization.

Proof. Select the Lyapunov function as follows:

$$V(t) = \frac{1}{2}(\mathbf{e}^T \mathbf{e} + (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) + (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha})). \quad (10)$$

Then, the derivative of $V(t)$ along equations (7), (8), and (9) can be calculated as

$$\begin{aligned}\dot{V}(t) &= \mathbf{e}^T \dot{\mathbf{e}} + (\Delta \boldsymbol{\beta})^T (\Delta \dot{\boldsymbol{\beta}}) + (\Delta \boldsymbol{\alpha})^T (\Delta \dot{\boldsymbol{\alpha}}) \\ &= \mathbf{e}^T (-P\mathbf{e} + \Theta g(\mathbf{y})\Delta \boldsymbol{\beta} - \Phi f(\mathbf{x})\Delta \boldsymbol{\alpha}) \\ &\quad + (\Delta \boldsymbol{\beta})^T (-(\Theta g(\mathbf{y}))^T \mathbf{e} - \rho'(\Delta \boldsymbol{\beta})) \\ &\quad + (\Delta \boldsymbol{\alpha})^T ((\Phi f(\mathbf{x}))^T \mathbf{e} - \rho(\Delta \boldsymbol{\alpha})). \\ &= -P\mathbf{e}^T \mathbf{e} - \rho'(\Delta \boldsymbol{\beta})^T \Delta \boldsymbol{\beta} - \rho(\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}).\end{aligned}\quad (11)$$

It is clear that $V(t)$ is positive definite and $\dot{V}(t)$ is negative definite in the neighborhood of the zero solution for system (9). According to Lyapunov stability (see Theorem 4.1 in [24]), the synchronous error system (9) is asymptotically stable at the origin. By Definition 1, the synchronization problem of chaotic systems can be transformed into the problem of asymptotic stability of synchronization error system at the origin, so the systems are uniformly asymptotically stable. Therefore, under the action of controller (8) and the parameter adaptive laws (7), the drive system and the response system in (4) achieved $\Theta - \Phi$ synchronization.

This completes the proof of Theorem 1.

We consider the expression (7) of the parameter adaptive laws in Theorem 1 and transform the constant gain matrix into the function of a gain variable, then the parameter adaptive laws in Theorem 1 can be written as

$$\begin{aligned}\Delta \dot{\boldsymbol{\alpha}} &= (\Phi f(\mathbf{x}))^T \mathbf{e} - \rho(t)(\Delta \boldsymbol{\alpha}), \\ \Delta \dot{\boldsymbol{\beta}} &= -(\Theta g(\mathbf{y}))^T \mathbf{e} - \rho'(t)(\Delta \boldsymbol{\beta}),\end{aligned}\quad (12)$$

where $\rho(t): \mathbf{R} \rightarrow \mathbf{R}$ is a gain function.

The adaptive law of gain function is designed as follows:

$$\dot{\rho}(t) = (\Delta\beta)^T \Delta\beta + (\Delta\alpha)^T \Delta\alpha. \quad (13)$$

The following Corollary 1 can be obtained. \square

Corollary 1. For the above chaotic systems (4), keeping the controller in Theorem 1 unchanged and replacing the constant gain matrix with the gain function in the parameter adaptive laws, the system can still achieve $\Theta - \Phi$ synchronization under the action of the controller (8), the parameter adaptive laws (12), and the gain function adaptive law (13).

Proof. The Lyapunov function is selected as

$$V(t) = \frac{1}{2}(\mathbf{e}^T \mathbf{e} + (\Delta\beta)^T (\Delta\beta) + (\Delta\alpha)^T (\Delta\alpha) + \rho^2(t)). \quad (14)$$

The proof method is similar to Theorem 1 and omitted.

Let us denote α_i or β_i be the i th unknown parameter of i th equation in the system. Assume that system (4) is linear at unknown parameters α_i or β_i . Then, systems (4) can be written as

$$\begin{aligned} \dot{\mathbf{x}} &= A_\alpha \mathbf{x} + F_1(\mathbf{x}), \\ \dot{\mathbf{y}} &= B_\beta \mathbf{y} + G_1(\mathbf{y}) + \mathbf{v}, \end{aligned} \quad (15)$$

where A_α and B_β are the diagonal matrix composed of unknown parameters α and β , $A_\alpha = \text{diag}(\alpha_1, \dots, \alpha_n)$, $B_\beta = \text{diag}(\beta_1, \dots, \beta_m)$, and \mathbf{v} is the control input vector.

Note

$$\tilde{\mathbf{e}} = \Omega \mathbf{e}, \quad (16)$$

where $\mathbf{e} = (e_1, \dots, e_m)^T$, $\tilde{\mathbf{e}} = (e_1, \dots, e_n)^T$, and $\Omega = (\Omega_{ij}) \in \mathbf{R}^{n \times m}$ is the transformation matrix. If $n \leq m$,

then $\Omega_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{others} \end{cases}$; if $n > m$, then

$$\Omega_{ij} = \begin{cases} 1 & i = j, i = 1, \dots, m \\ 1 & j = m, i = m + 1, \dots, n. \\ 0 & \text{others} \end{cases} \text{ Meanwhile, denote}$$

$$\begin{aligned} R_2 &= \Theta(B_\beta \mathbf{y} + G_1(\mathbf{y}) - B_{\Delta\beta} \mathbf{e}) \\ &\quad + K\mathbf{e} + \Phi A_{\Delta\alpha} \tilde{\mathbf{e}} - \Phi A_\alpha \mathbf{x} - \Phi F_1(\mathbf{x}), \end{aligned} \quad (17)$$

where $A_{\Delta\alpha} = A_\alpha - A_\alpha = \text{diag}(\Delta\alpha_1, \dots, \Delta\alpha_n)$, $B_{\Delta\beta} = B_\beta - B_\beta = \text{diag}(\Delta\beta_1, \dots, \Delta\beta_m)$, and $K \in \mathbf{R}^{m \times m}$ is the positive-definite control matrix; for the convenience of calculation, let $K = \text{diag}(k_1, k_2, \dots, k_m)$, $k_i > 0$, $i = 1, \dots, m$.

The following new adaptive laws are proposed to estimate the unknown parameters α and β :

$$\begin{aligned} \Delta\dot{\alpha} &= \text{diag}(e_1, \dots, e_n) \Phi^T \mathbf{e} - \psi(\Delta\alpha), \\ \Delta\dot{\beta} &= -\text{diag}(e_1, \dots, e_m) \Theta^T \mathbf{e} - \psi'(\Delta\beta), \end{aligned} \quad (18)$$

where ψ and ψ' are the constant gain matrices. For the convenience of calculation, let $\psi = \text{diag}(\psi_1, \psi_2, \dots, \psi_n)$, $\psi_i > 0$, $i = 1, \dots, n$ and $\psi' = \text{diag}(\psi'_1, \psi'_2, \dots, \psi'_m)$, $\psi'_i > 0$, $i = 1, \dots, m$.

Combined with the adaptive laws of the parameter, the following controller is designed:

$$\mathbf{v} = -\Theta^{-1} R_2. \quad (19)$$

By substituting the controller (19) into the error system (5), it can be obtained:

$$\begin{aligned} \dot{\mathbf{e}} &= \Theta \dot{\mathbf{y}} - \Phi \dot{\mathbf{x}} \\ &= \Theta(B_\beta \mathbf{y} + G_1(\mathbf{y}) + \mathbf{v}) - \Phi(A_\alpha \mathbf{x} + F_1(\mathbf{x})) \\ &= \Theta B_{\Delta\beta} \mathbf{e} - \Phi A_{\Delta\alpha} \tilde{\mathbf{e}} - K\mathbf{e}. \end{aligned} \quad (20)$$

\square

Theorem 2. For the chaotic system (15) with unknown parameters, if the controller \mathbf{v} is designed as equation (19) and the parameter adaptive laws are designed as equation (18), then the error system (20) is asymptotically stable at the origin. It can also be said that the response system and drive system in (15) achieved $\Theta - \Phi$ synchronization.

Proof. Select the Lyapunov function as follows:

$$V(t) = \frac{1}{2}(\mathbf{e}^T \mathbf{e} + (\Delta\beta)^T (\Delta\beta) + (\Delta\alpha)^T (\Delta\alpha)). \quad (21)$$

Then, the derivative of $V(t)$ along equations (18), (19), and (20) can be calculated as

$$\begin{aligned} \dot{V}(t) &= \mathbf{e}^T \dot{\mathbf{e}} + (\Delta\beta)^T (\Delta\dot{\beta}) + (\Delta\alpha)^T (\Delta\dot{\alpha}) \\ &= \mathbf{e}^T (\Theta B_{\Delta\beta} \mathbf{e} - \Phi A_{\Delta\alpha} \tilde{\mathbf{e}} - K\mathbf{e}) \\ &\quad + (\Delta\beta)^T (-\text{diag}(e_1, \dots, e_m) \Theta^T \mathbf{e} - \psi'(\Delta\beta)) \\ &\quad + (\Delta\alpha)^T (\text{diag}(e_1, \dots, e_n) \Phi^T \mathbf{e} - \psi(\Delta\alpha)) \\ &= -K\mathbf{e}^T \mathbf{e} - \psi'(\Delta\beta)^T \Delta\beta - \psi(\Delta\alpha)^T (\Delta\alpha). \end{aligned} \quad (22)$$

Similar to the proof method of Theorem 1, it is easy to obtain that the error system (20) is asymptotically stable at the origin. By Definition 1, the synchronization problem of chaotic systems can be transformed into the problem of asymptotic stability of synchronization error system at the origin. Therefore, under the joint action of the controller (19) and the parameter adaptive laws (18), the drive system and the response system in (15) achieved $\Theta - \Phi$ synchronization. This completes the proof of Theorem 2.

The parameter adaptive laws (18) in Theorem 2 is considered, and the constant gain matrix is transformed into the function of gain variable, and then, the parameter adaptive laws in Theorem 1 can be written as

$$\begin{aligned} \Delta\dot{\alpha} &= \text{diag}(e_1, \dots, e_n) \Phi^T \mathbf{e} - \psi(t)(\Delta\alpha), \\ \Delta\dot{\beta} &= -\text{diag}(e_1, \dots, e_m) \Theta^T \mathbf{e} - \psi(t)(\Delta\beta), \end{aligned} \quad (23)$$

where $\psi(t): \mathbf{R} \rightarrow \mathbf{R}$ is a gain function.

The adaptive law of gain function is designed as follows:

$$\dot{\psi}(t) = (\Delta\beta)^T \Delta\beta + (\Delta\alpha)^T \Delta\alpha. \quad (24)$$

\square

Corollary 2. For the above chaotic systems (15), if the controller in Theorem 2 remains unchanged and the constant gain matrix is replaced by the gain variable function in the parameter adaptive laws, the systems can still achieve $\Theta - \Phi$

synchronization under the joint action of the controller (19), the parameter adaptive laws (23), and the gain function adaptive law (24).

Proof. The Lyapunov function is selected as

$$V(t) = \frac{1}{2} (\mathbf{e}^T \mathbf{e} + (\Delta \boldsymbol{\beta})^T (\Delta \boldsymbol{\beta}) + (\Delta \boldsymbol{\alpha})^T (\Delta \boldsymbol{\alpha}) + \psi^2(t)). \quad (25)$$

The proof method is similar to Theorem 2 and omitted. \square

4. Numerical Simulations

In this section, we give three examples to verify the effectiveness of our proposed method.

Example 1. Consider the chaotic Cai system [25] as the drive system and the hyperchaotic Rossler system [26] as the response system. The system equations are as follows:

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = b_1 x_1 + c_1 x_2 - x_1 x_3, \\ \dot{x}_3 = x_1^2 - d_1 x_3, \end{cases} \quad (26)$$

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 + u_1, \\ \dot{y}_2 = y_1 + a_2 y_2 + y_4 + u_2, \\ \dot{y}_3 = b_2 + y_1 y_3 + u_3, \\ \dot{y}_4 = -c_2 y_3 + d_2 y_4 + u_4, \end{cases} \quad (27)$$

where x_1, x_2, x_3 are the drive system state variables; y_1, y_2, y_3, y_4 are the response system state variables; u_1, u_2, u_3, u_4 are the designed controllers, when the parameters are taken as $a_1 = 20, b_1 = 14, c_1 = 10.6, d_1 = 2.8$ and $a_2 = 0.25, b_2 = 3, c_2 = 0.5, d_2 = 0.05$; and the systems (26) and (27) are chaotic and hyperchaotic. The projection of attractor of chaotic Cai system and hyperchaotic Rossler system is shown in Figures 1 and 2, respectively.

Let make the drive system and the response system achieve synchronization in 4 D . We rewrite the systems (26) and (27) in the following forms:

$$\dot{\mathbf{x}} = f(\mathbf{x})\boldsymbol{\alpha} + F_1(\mathbf{x}), \quad (28)$$

$$\text{where } f(\mathbf{x}) = \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & -x_3 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}, \quad \text{and}$$

$$F_1(\mathbf{x}) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1^2 \end{bmatrix}.$$

$$\dot{\mathbf{y}} = g(\mathbf{y})\boldsymbol{\beta} + G_1(\mathbf{y}) + \mathbf{u}, \quad (29)$$

$$\text{where } g(\mathbf{y}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y_3 & y_4 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}, \quad G_1(\mathbf{x}) =$$

$$\begin{bmatrix} -y_2 - y_3 \\ y_1 + y_4 \\ y_1 y_3 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \text{and we choose}$$

$$\Theta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (30)$$

$$\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then, from (6), it can be concluded that

$$R_1 = (P\Theta\mathbf{y} + \Theta g(\mathbf{y})\hat{\boldsymbol{\beta}}) - (\Phi f(\mathbf{x})\hat{\boldsymbol{\alpha}} + P\Phi\mathbf{x}) + \Theta G_1(\mathbf{y}) - \Phi F_1(\mathbf{x})$$

$$= \begin{bmatrix} y_1 + y_4 + \hat{a}_2 y_2 - \hat{b}_1 x_1 - \hat{c}_1 x_2 + x_1 x_3 - y_2 - y_3 - \hat{a}_1 (x_2 - x_1) + p_1 e_1 \\ y_1 + y_4 + \hat{a}_2 y_2 - \hat{b}_1 x_1 - \hat{c}_1 x_2 + x_1 x_3 + p_2 e_2 \\ y_1 y_3 + \hat{b}_2 - \hat{b}_1 x_1 - \hat{c}_1 x_2 + x_1 x_3 - x_1^2 + \hat{d}_1 x_3 + p_3 e_3 \\ -\hat{c}_2 y_3 + \hat{d}_2 y_4 - \hat{a}_1 (x_2 - x_1) - x_1^2 + \hat{d}_1 x_3 + p_4 e_4 \end{bmatrix}. \quad (31)$$

From (7), the adaptive law of parameters can be designed as follows:

$$\begin{cases} \Delta \dot{a}_1 = (x_2 - x_1)(e_1 + e_4) - \rho_1(\Delta a_1), \\ \Delta \dot{b}_1 = x_1(e_1 + e_2 + e_3) - \rho_2(\Delta b_1), \\ \Delta \dot{c}_1 = x_2(e_1 + e_2 + e_3) - \rho_3(\Delta c_1), \\ \Delta \dot{d}_1 = -x_3(e_3 + e_4) - \rho_4(\Delta d_1), \end{cases} \quad \begin{cases} \Delta \dot{a}_2 = -y_2(e_1 + e_2) - \rho'_1(\Delta a_2), \\ \Delta \dot{b}_2 = -e_3 - \rho'_2(\Delta b_2), \\ \Delta \dot{c}_2 = y_3 e_4 - \rho'_3(\Delta c_2), \\ \Delta \dot{d}_2 = -y_4 e_4 - \rho'_4(\Delta d_2). \end{cases} \quad (32)$$

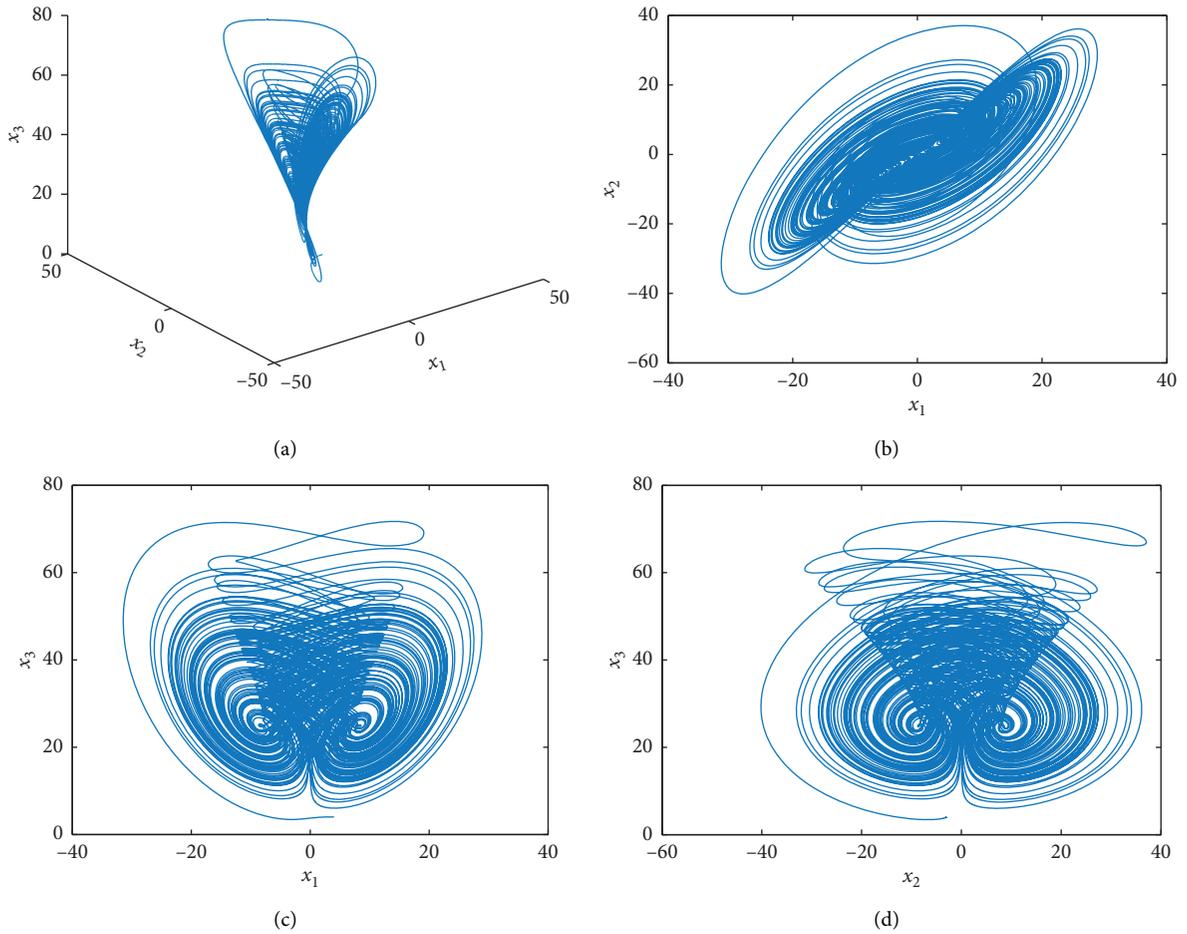


FIGURE 1: Typical dynamic behavior of the chaotic Cai system: (a) projection in (x_1, x_2, x_3) space; (b) projection in (x_1, x_2) space; (c) projection in (x_1, x_3) space; (d) projection in (x_2, x_3) space.

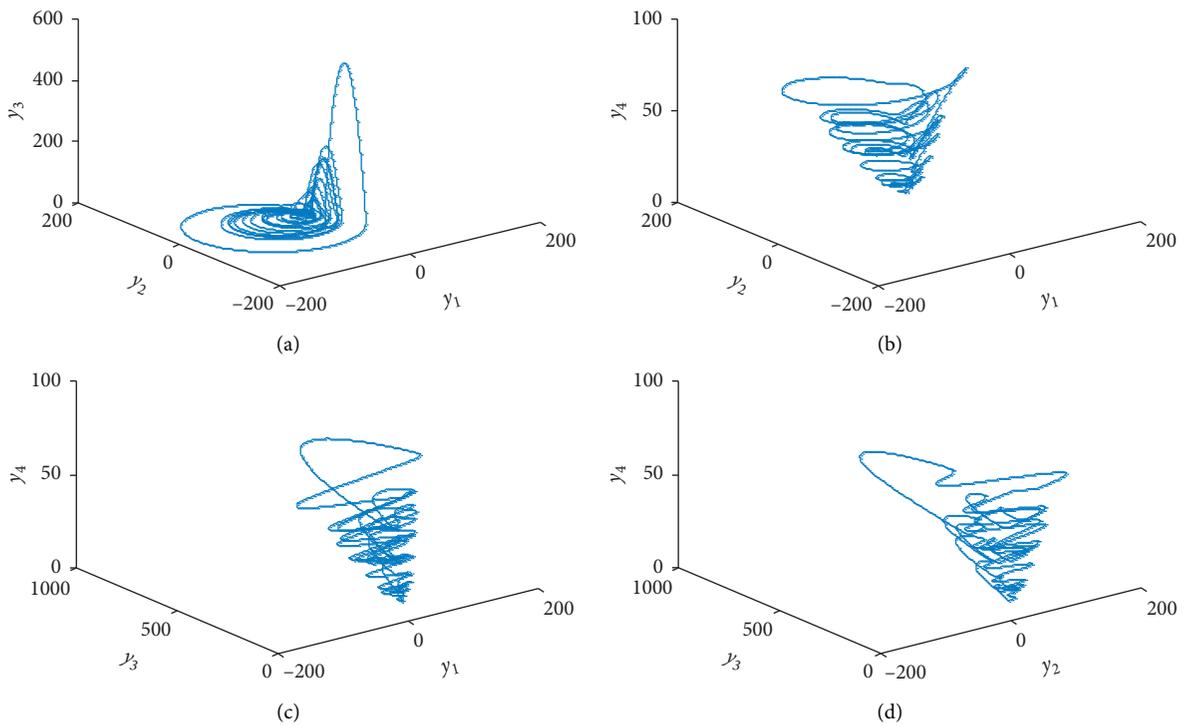


FIGURE 2: Typical dynamical behaviors of the hyperchaotic Rossler system: (a) projection in (y_1, y_2, y_3) space; (b) projection in (y_1, y_2, y_4) space; (c) projection in (y_1, y_3, y_4) space; (d) projection in (y_2, y_3, y_4) space.

From equation (8), the controller can be designed as follows:

$$\begin{cases} u_1 = y_2 + y_3 + \hat{a}_1(x_2 - x_1) - p_1 e_1 + p_2 e_2, \\ u_2 = -y_1 - y_4 - \hat{a}_2 y_2 + \hat{b}_1 x_1 + \hat{c}_1 x_2 - x_1 x_3 - p_2 e_2, \\ u_3 = -y_1 y_3 - \hat{b}_2 + \hat{b}_1 x_1 + \hat{c}_1 x_2 - x_1 x_3 + x_1^2 - \hat{d}_1 x_3 - p_3 e_3, \\ u_4 = \hat{c}_2 y_3 - \hat{d}_2 y_4 + \hat{a}_1(x_2 - x_1) + x_1^2 - \hat{d}_1 x_3 - p_4 e_4. \end{cases} \quad (33)$$

In the controller (33), the error system is transformed into

$$\begin{cases} \dot{e}_1 = -\Delta a_1(x_2 - x_1) + \Delta a_2 y_2 - \Delta b_1 x_1 - \Delta c_1 x_2 - p_1 e_1, \\ \dot{e}_2 = \Delta a_2 y_2 - \Delta b_1 x_1 - \Delta c_1 x_2 - p_2 e_2, \\ \dot{e}_3 = \Delta b_2 - \Delta b_1 x_1 - \Delta c_1 x_2 + \Delta d_1 x_3 - p_3 e_3, \\ \dot{e}_4 = -\Delta c_2 y_3 + \Delta d_2 y_4 - \Delta a_1(x_2 - x_1) + \Delta d_1 x_3 - p_4 e_4. \end{cases} \quad (34)$$

For the numerical simulation, the fourth-order Runge–Kutta integration method has been used to solve the system of differential equations (26) and (27). The initial values of the drive system and response system are $(x_1(0), x_2(0), x_3(0)) = (4, -3, 4)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (5, -6, 3, 3)$; the initial value of parameter estimations are $(\hat{a}_1(0), \hat{b}_1(0), \hat{c}_1(0), \hat{d}_1(0)) = (0.1, 0.1, 0.1, 0.1)$ and $(\hat{a}_2(0), \hat{b}_2(0), \hat{c}_2(0), \hat{d}_2(0)) = (0.1, 0.1, 0.1, 0.1)$. Let two sets of the control matrix and the constant gain matrices be $P = \text{diag}(2, 2, 2, 2)$, $\rho = \text{diag}(1, 2, 1, 2)$, $\rho' = \text{diag}(1, 1, 2, 2)$, $\bar{P} = \text{diag}(0.5, 0.5, 0.5, 0.5)$, $\bar{\rho} = \text{diag}(0.2, 0.5, 0.2, 0.5)$, and $\bar{\rho}' = \text{diag}(0.2, 0.2, 0.5, 0.5)$, respectively. Figures 3(a) and 4(a) show the trajectories of errors. It can be seen that the synchronization errors tend to zero; that is, the driving system (26) and the response system (27) achieve stable synchronization in 4 D . Figures 3(b), 3(c) and 4(b), 4(c) show the trajectories of the unknown parameter estimations, and it can be seen that the estimations of unknown parameter can converge to some constants.

As can be seen from Figures 3 and 4, if the control constant and gain constant are larger, the convergence speed of synchronization will be faster. The values of control constant and gain constant only affect the speed of synchronization.

$$\begin{aligned} P &= \text{diag}(2, 2, 2, 2), \\ \rho &= \text{diag}(1, 2, 1, 2), \\ \rho' &= \text{diag}(1, 1, 2, 2). \end{aligned} \quad (35)$$

$$\begin{aligned} \bar{P} &= \text{diag}(0.5, 0.5, 0.5, 0.5), \\ \bar{\rho} &= \text{diag}(0.2, 0.5, 0.2, 0.5), \\ \bar{\rho}' &= \text{diag}(0.2, 0.2, 0.5, 0.5). \end{aligned} \quad (36)$$

Remark 4. In [15], by adding dimensions to the drive system (26), the synchronization error form $e_i = y_i - x_{i-1}$ ($i = 1, 2, 3, 4$) of cross subtraction is constructed, and the appropriate controller and parameter adaptive laws are designed so that the drive system (26) and the response system (27) with unknown parameters can be synchronized in 4 D . By using

$$\text{Theorem 1 and selecting } \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$\rho = 0, \rho' = 0$, the controller and parameter adaptive laws designed in this paper are the same as that in [15]; that is, the $\Theta - \Phi$ synchronization in this paper can be transformed into the synchronization scheme in [15]. Compared with the conclusion given in [15], the constant gain matrices and gain function are added to the designed parameter adaptive laws, making Theorem 1 and Corollary 1 given in this paper more universal.

Example 2. Consider the hyperchaotic Lorenz system [27] as the drive system and the chaotic Duffing system [28] as the response system. The system equations are as follows:

$$\begin{cases} \dot{x}_1 = \sigma_1(x_2 - x_1), \\ \dot{x}_2 = \sigma_2 x_1 - \sigma_3 x_2 + x_4 - x_1 x_3, \\ \dot{x}_3 = x_1 x_2 - \sigma_4 x_3, \\ \dot{x}_4 = -\sigma_5 x_2 - \sigma_6 x_4, \end{cases} \quad (37)$$

$$\begin{cases} \dot{y}_1 = y_2 + u_1, \\ \dot{y}_2 = y_1 - \tau_1 y_2 - y_1^3 + \tau_2 \cos(t) + u_2, \end{cases} \quad (38)$$

where x_1, x_2, x_3, x_4 are the drive system state variables; y_1, y_2 are the response system state variables; u_1, u_2 are the designed controllers; and when the parameters are taken as $\sigma_1 = 12, \sigma_2 = 23, \sigma_3 = 1, \sigma_4 = 2.1, \sigma_5 = 6, \sigma_6 = 0.2, \tau_1 = 1, \tau_2 = 0.8$, the systems (37) and (38) are hyperchaotic and chaotic. The projection of attractor of hyperchaotic Lorenz system and chaotic Duffing system is shown in Figures 5 and 6, respectively.

In order to synchronize the drive system and the response system in 2 D , we rewrite the systems (37) and (38) as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x})\boldsymbol{\alpha} + F_1(\mathbf{x}), \quad (39)$$

$$\text{where } f(\mathbf{x}) = \begin{bmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & -x_4 \end{bmatrix}, \boldsymbol{\alpha} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix}, \text{ and } F_1$$

$$(\mathbf{x}) = \begin{bmatrix} x_4 \\ -x_2 - x_1 x_3 \\ x_1 x_2 \\ -x_2 x_3 \end{bmatrix}.$$

$$\dot{\mathbf{y}} = g(\mathbf{y})\boldsymbol{\beta} + G_1(\mathbf{y}), \quad (40)$$

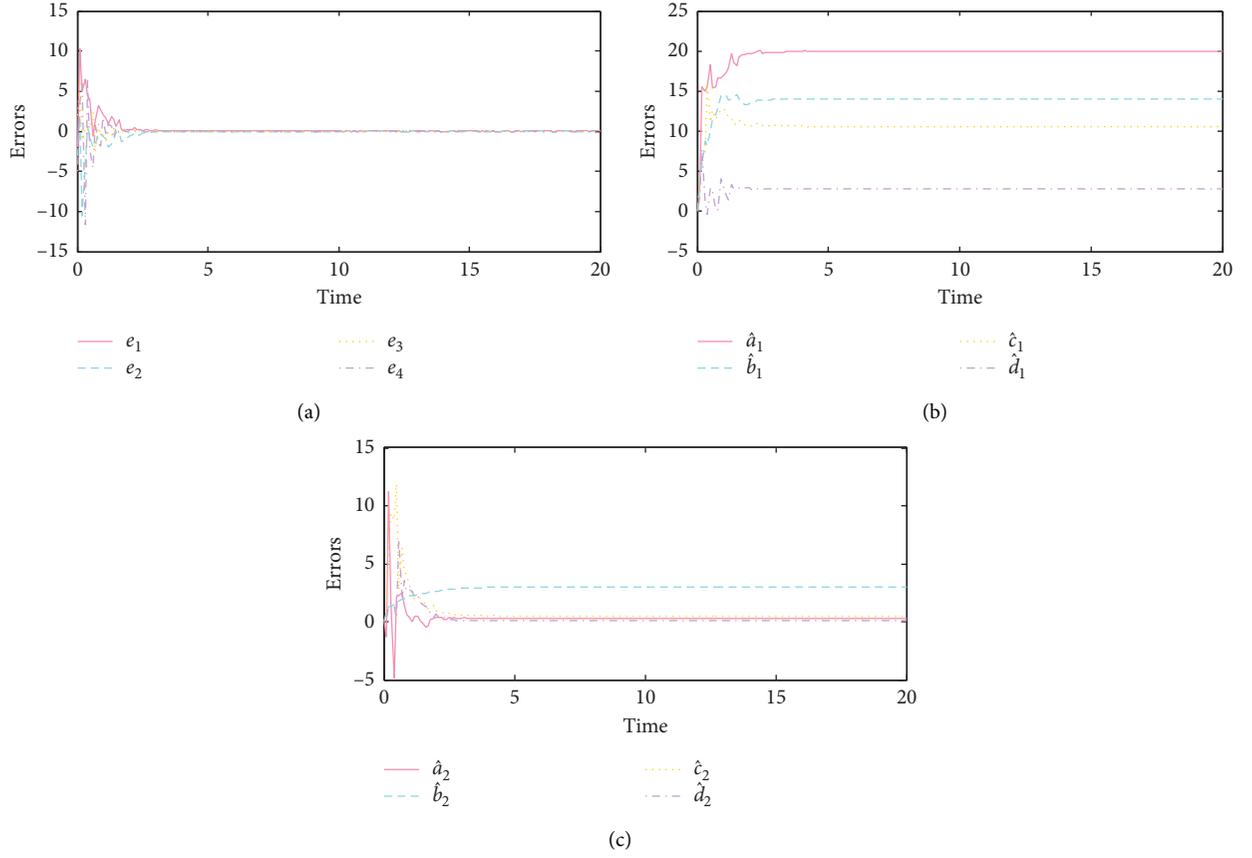


FIGURE 3: (a) Trajectories of the errors; (b) trajectories of the drive system unknown parameter estimates; (c) trajectories of the response system unknown parameter estimates.

where $g(\mathbf{y}) = \begin{bmatrix} 0 & 0 \\ -y_2 & \cos t \end{bmatrix}$, $\boldsymbol{\beta} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$, and $G_1(\mathbf{x}) = \begin{bmatrix} y_2 \\ y_1 - y_1^3 \end{bmatrix}$, and we choose $\Theta = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\Phi = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. (41)

Then, from equation (6), it can be concluded that

$$\begin{aligned}
 R_1 &= (P\Theta\mathbf{y} + \Theta g(\mathbf{y})\hat{\boldsymbol{\beta}}) - (\Phi f(x)\hat{\mathbf{a}} + P\Phi\mathbf{x}) + \Theta G_1(\mathbf{y}) - \Phi F_1(\mathbf{x}) \\
 &= \begin{bmatrix} 2y_2 - x_1x_2 - \hat{\sigma}_1(x_2 - x_1) + \hat{\sigma}_3x_3 + p_1e_1 \\ y_1 - y_1^3 - x_4 + x_3x_1 - \hat{\tau}_1y_2 + \hat{\tau}_2 \cos t - \hat{\sigma}_2x_1 + \hat{\sigma}_3x_2 + \hat{\sigma}_5x_2 + \hat{\sigma}_6x_4 + p_2e_2 \end{bmatrix}.
 \end{aligned} \quad (42)$$

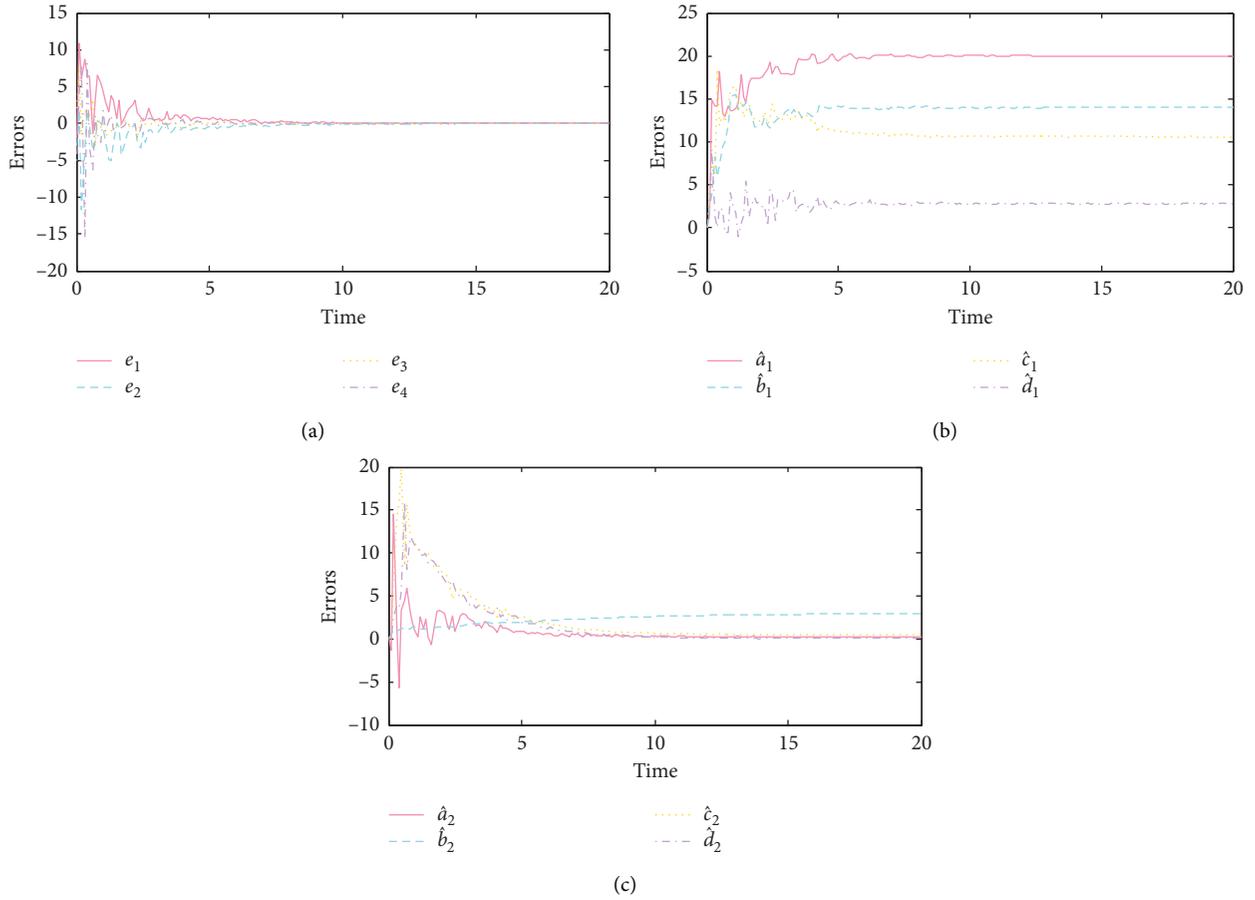


FIGURE 4: (a) Trajectories of the errors; (b) trajectories of the drive system unknown parameter estimates; (c) trajectories of the response system unknown parameter estimates.

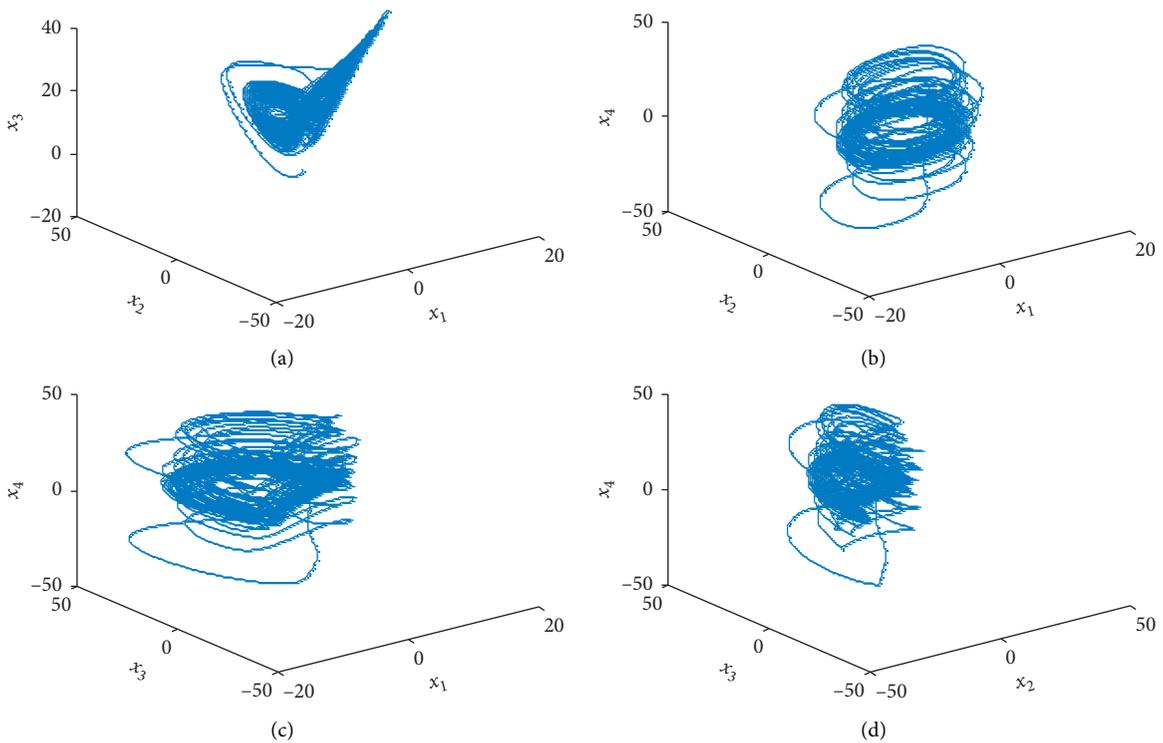


FIGURE 5: Typical dynamical behaviors of the hyperchaotic Lorenz system: (a) projection in (x_1, x_2, x_3) space; (b) projection in (x_1, x_2, x_4) space; (c) projection in (x_1, x_3, x_4) space; (d) projection in (x_2, x_3, x_4) space.

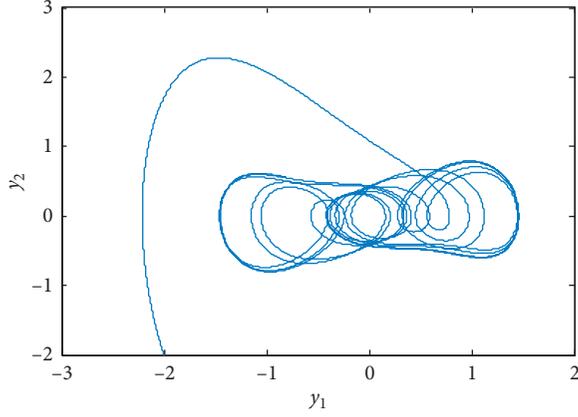


FIGURE 6: Typical dynamic behavior of the chaotic Duffing system: projection in (y_1, y_2) space.

From (12) and (13), the adaptive law of parameters can be designed as follows:

$$\begin{cases} u_1 = -y_2 + \frac{1}{2}x_1x_2 + \frac{1}{2}\hat{\sigma}_1(x_2 - x_1) - \frac{1}{2}\hat{\sigma}_4x_3 - \frac{1}{2}p_1e_1, \\ u_2 = -y_1 + y_1^3 + x_4 - x_1x_3 + \hat{\tau}_1y_2 - \hat{\tau}_2 \cos t + \hat{\sigma}_2x_1 - \hat{\sigma}_3x_2 - \hat{\sigma}_5x_2 - \hat{\sigma}_6x_4 - p_2e_2. \end{cases} \quad (45)$$

In the controller (45), the error system is transformed into

$$\begin{cases} \dot{e}_1 = -\Delta\sigma_1(x_2 - x_1) + \Delta\sigma_4x_3 - p_1e_1, \\ \dot{e}_2 = -\Delta\tau_1y_2 + \Delta\tau_2 \cos t - \Delta\sigma_2x_1 + \Delta\sigma_3x_2 + \Delta\sigma_5x_2 + \Delta\sigma_6x_4 - p_2e_2. \end{cases} \quad (46)$$

For numerical simulation, the fourth-order Runge–Kutta integration method has been used to solve the system of differential (37) and (38). The initial values of the drive system and response system are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-1, -0.6, -1, -0.2)$ and $(y_1(0), y_2(0)) = (-2, -2)$, respectively, and the initial value of parameter estimations is $(\hat{\sigma}_1(0), \hat{\sigma}_2(0), \hat{\sigma}_3(0), \hat{\sigma}_4(0), \hat{\sigma}_5(0), \hat{\sigma}_6(0)) = (10, 20, 3, 2, 1, 2)$ and $(\hat{\tau}_1(0), \hat{\tau}_2(0)) = (0.1, 0.2)$, respectively; the control matrix $P = \text{diag}(3, 2)$, and the initial value of the gain function $\rho(0) = 0.1$. Figure 7(a) shows the trajectories of errors, and it can be seen that the synchronization errors tend to zero; that is, the driving system (37) and the response system (48) achieve stable synchronization in 2 D . Figures 7(b) and 7(c) show the trajectories of the unknown parameter estimations, and it can be seen that the estimations of the unknown parameter can converge to some constants.

Remark 5. In [23], the global fixed-time synchronization of four-dimensional hyperchaotic system (37) and two-dimensional chaotic Duffing system (38) is studied. By using Theorem 1 and selecting $\Theta = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Phi = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$, $\rho =$

$$\begin{cases} \Delta\dot{\sigma}_1 = (x_2 - x_1)e_1 - \rho(t), \\ \Delta\sigma_2 = x_1e_2 - \rho(t), \\ \Delta\sigma_3 = -x_2e_2 - \rho(t), \\ \Delta\sigma_4 = -x_3e_1 - \rho(t), \\ \Delta\sigma_5 = -x_2e_2 - \rho(t), \\ \Delta\sigma_6 = -x_4e_2 - \rho(t), \\ \Delta\hat{\tau}_1 = e_2y_2 - \rho(t), \\ \Delta\hat{\tau}_2 = -e_2 \cos t - \rho(t), \end{cases}$$

(43)

where

$$\dot{\rho}(t) = \Delta\sigma_1^2 + \Delta\sigma_2^2 + \Delta\sigma_3^2 + \Delta\sigma_4^2 + \Delta\sigma_5^2 + \Delta\sigma_6^2 + \Delta\hat{\tau}_1^2 + \Delta\hat{\tau}_2^2. \quad (44)$$

From equation (8), the controller can be designed as follows:

0, $\rho' = 0$, the synchronization type in this paper is the same as that in [23]. However, the authors in [23] focus on the problem of global fixed-time synchronization. When estimating unknown parameters, due to many constraints, the controller designed in [23] is more complex and difficult to meet in practice. Compared with the controller given in [23], the controller designed in this paper is relatively simple. At the same time, compared with the parameter adaptive laws given in [23], the constant gain matrices and the gain function are added in the designed parameter adaptive laws, which make the parameter estimation function given in this paper more effective.

Example 3. Consider the chaotic financial system [29] as the drive system and the hyperchaotic Chen–Lee system [30] as the response system. The system equations are as follows:

$$\begin{cases} \dot{x}_1 = x_3 + (x_2 - r_1)x_1, \\ \dot{x}_2 = 1 - r_2x_2 - x_1^2, \\ \dot{x}_3 = -x_1 - r_3x_3, \end{cases} \quad (47)$$

$$\begin{cases} \dot{y}_1 = \omega_1y_1 - y_2y_3 + v_1, \\ \dot{y}_2 = \omega_2y_2 + y_1y_3 + v_2, \\ \dot{y}_3 = \omega_3y_3 + 0.2y_4 + \frac{1}{3}y_1y_2 + v_3, \\ \dot{y}_4 = 2.2y_1 + 0.05y_4 + 0.5y_2y_3 + v_4, \end{cases} \quad (48)$$

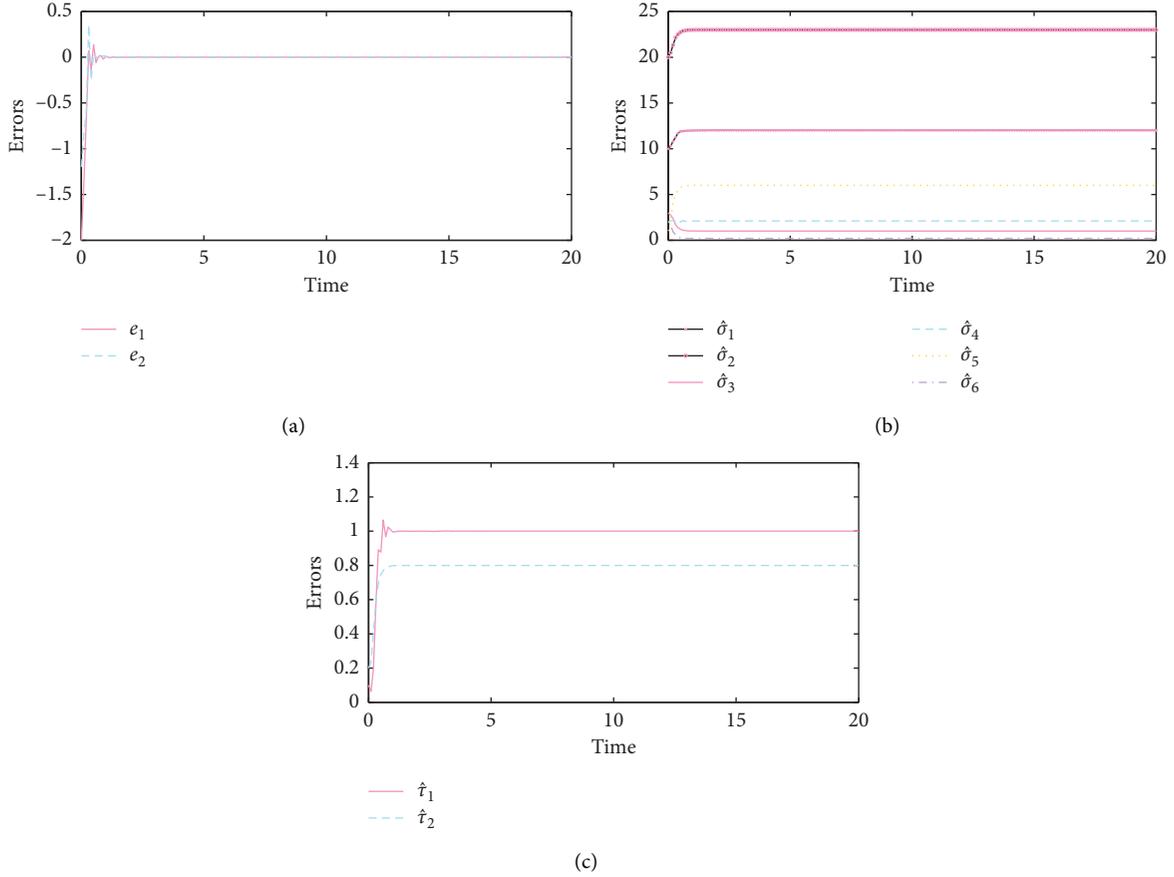


FIGURE 7: (a) Trajectories of the errors; (b) trajectories of the drive system unknown parameter estimates; (c) trajectories of the response system unknown parameter estimates.

where x_1, x_2, x_3 are the drive system state variables, y_1, y_2, y_3, y_4 are the response system state variables, and v_1, v_2, v_3, v_4 are the designed controllers. When the parameters are taken as $r_1 = 0.8, r_2 = 0.2, r_3 = 1.9, \omega_1 = 5, \omega_2 = -10, \omega_3 = -3.8$, the systems (47) and (48) are chaotic and hyperchaotic. The projection of attractor of chaotic financial system and hyperchaotic Chen–Lee system is shown in Figures 8 and 9, respectively.

In order to synchronize the drive system and the response system in 4 D , we rewrite the systems (47) and (48) as follows:

$$\dot{\mathbf{x}} = A_\alpha \mathbf{x} + F_1(\mathbf{x}), \quad (49)$$

$$\text{where } A_\alpha = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \text{ and } F_1(\mathbf{x}) = \begin{bmatrix} x_3 + x_1 x_2 \\ 1 - x_1^2 \\ -x_1 \end{bmatrix}.$$

$$\dot{\mathbf{y}} = B_\beta \mathbf{y} + G_1(\mathbf{y}) + \mathbf{v}, \quad (50)$$

$$\text{where } B_\beta = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad G_1(\mathbf{y}) =$$

$$\begin{bmatrix} -y_2 y_3 \\ y_1 y_3 \\ 0.2 y_4 + 1/3 y_1 y_2 \\ 2.2 y_1 + 0.05 y_4 + 0.5 y_2 y_3 \end{bmatrix}, \text{ and we choose}$$

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad (51)$$

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then, from (17), it can be concluded that

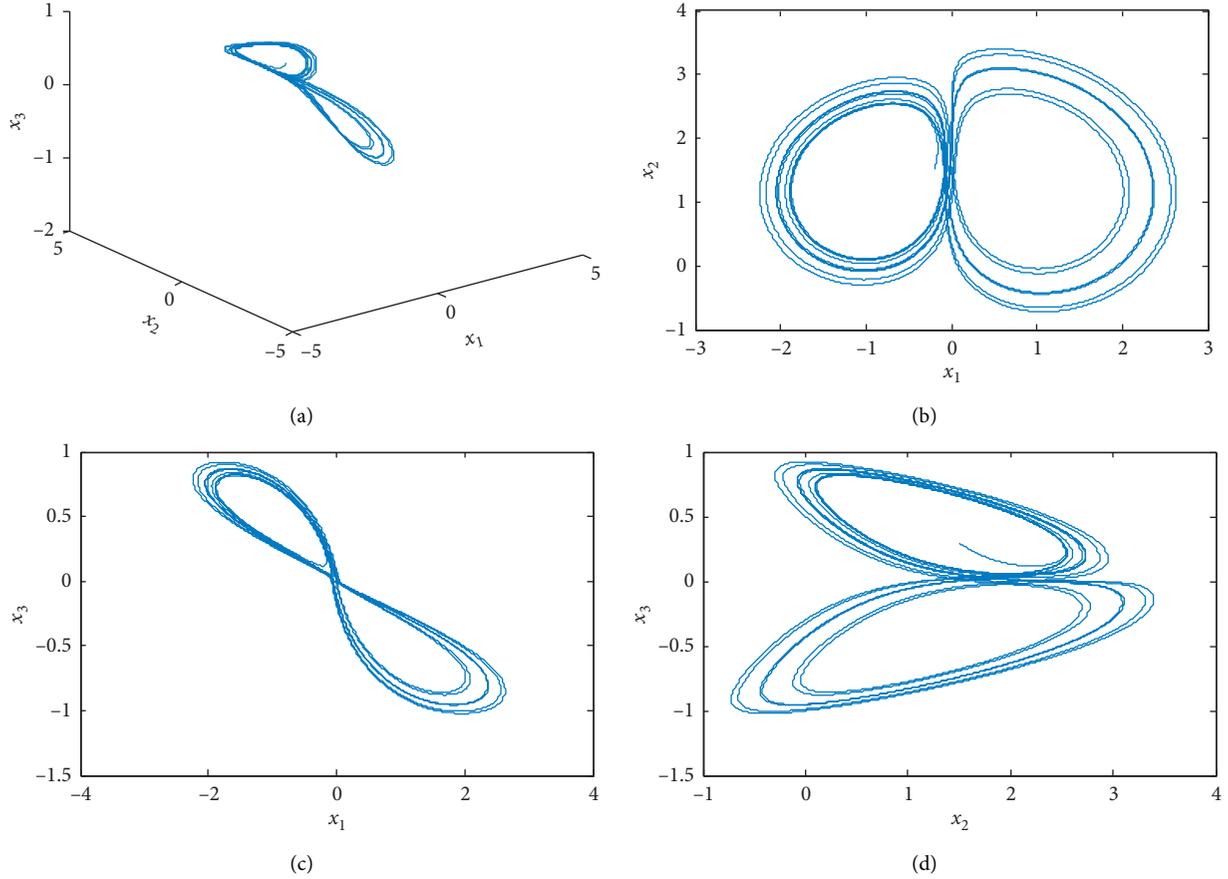


FIGURE 8: Typical dynamic behavior of the chaotic financial system: (a) projection in (x_1, x_2, x_3) space; (b) projection in (x_1, x_2) space; (c) projection in (x_1, x_3) space; (d) projection in (x_2, x_3) space.

$$R_2 = \Theta(B_{\beta}y + G_1(y) - B_{\Delta\beta}e) + Ke + \Phi A_{\Delta\alpha}\bar{e} - \Phi A_{\alpha}x - \Phi F_1(x)$$

$$= \begin{bmatrix} \omega_1 x_1 + r_1 y_1 - y_2 y_3 - x_2 x_1 - x_3 + \hat{\omega}_1 e_1 - \hat{r}_1 e_1 + k_1 e_1 \\ y_1 y_3 - 1 + x_1^2 + \omega_2 x_2 + r_2 y_2 + \hat{\omega}_2 e_2 - \hat{r}_2 e_2 + k_2 e_2 \\ 0.2 y_4 + \frac{1}{3} y_1 y_2 + x_1 + \omega_3 x_3 + r_3 y_3 + \hat{\omega}_3 e_3 - \hat{r}_3 e_3 + k_3 e_3 \\ \omega_1 x_1 + r_1 y_1 + \hat{\omega}_1 e_1 - \hat{r}_1 e_1 + k_1 e_1 - y_2 y_3 - x_2 x_1 \\ -x_3 + 2.2 y_1 + 0.05 y_4 + 0.5 y_2 y_3 - 1 + x_1^2 - y_2 y_3 \\ -x_2 x_1 - x_3 + \omega_1 x_1 + r_1 y_1 + \hat{\omega}_1 e_1 - \hat{r}_1 e_1 + r_2 y_2 - \hat{r}_2 e_2 + k_4 e_4 \end{bmatrix}. \quad (52)$$

From equation (18), the adaptive law of parameters can be designed as follows:

$$\begin{cases} \Delta \hat{r}_1 = e_1^2 + e_1 e_4 - \psi_1(\Delta r_1), \\ \Delta \hat{r}_2 = e_2^2 + e_2 e_4 - \psi_2(\Delta r_2), \\ \Delta \hat{r}_3 = e_3^2 - \psi_3(\Delta r_3), \end{cases}$$

$$\begin{cases} \Delta \dot{\omega}_1 = -e_1^2 - e_1 e_4 - \psi_1'(\Delta \omega_1), \\ \Delta \dot{\omega}_2 = -e_2^2 - \psi_2'(\Delta \omega_2), \\ \Delta \dot{\omega}_3 = -e_3^2 - \psi_3'(\Delta \omega_3). \end{cases} \quad (53)$$

From equation (19), the controller can be designed as follows:

$$\begin{cases} v_1 = y_2 y_3 + x_2 x_1 + x_3 - \omega_1 x_1 - r_1 y_1 - \hat{\omega}_1 e_1 + \hat{r}_1 e_1 - k_1 e_1, \\ v_2 = -y_1 y_3 + 1 - x_1^2 - \omega_2 x_2 - r_2 y_2 - \hat{\omega}_2 e_2 + \hat{r}_2 e_2 - k_2 e_2, \\ v_3 = -0.2 y_4 - \frac{1}{3} y_1 y_2 - x_1 - \omega_3 x_3 - r_3 y_3 - \hat{\omega}_3 e_3 + \hat{r}_3 e_3 - k_3 e_3, \\ v_4 = -2.2 y_1 - 0.05 y_4 - 0.5 y_2 y_3 + 1 - x_1^2 - r_2 y_2 + \hat{r}_2 e_2 - k_4 e_4. \end{cases} \quad (54)$$

In the controller (54), the error system is transformed into

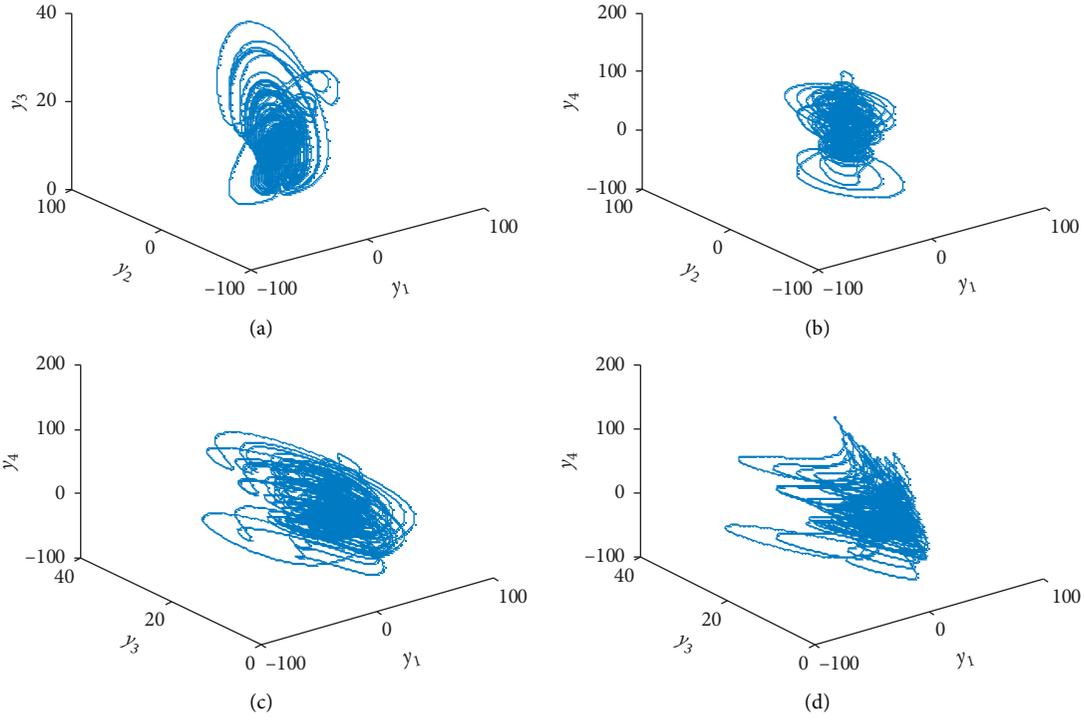


FIGURE 9: Typical dynamical behaviors of the hyperchaotic Chen-Lee system: (a) projection in (y_1, y_2, y_3) space; (b) projection in (y_1, y_2, y_4) space; (c) projection in (y_1, y_3, y_4) space; (d) projection in (y_2, y_3, y_4) space.

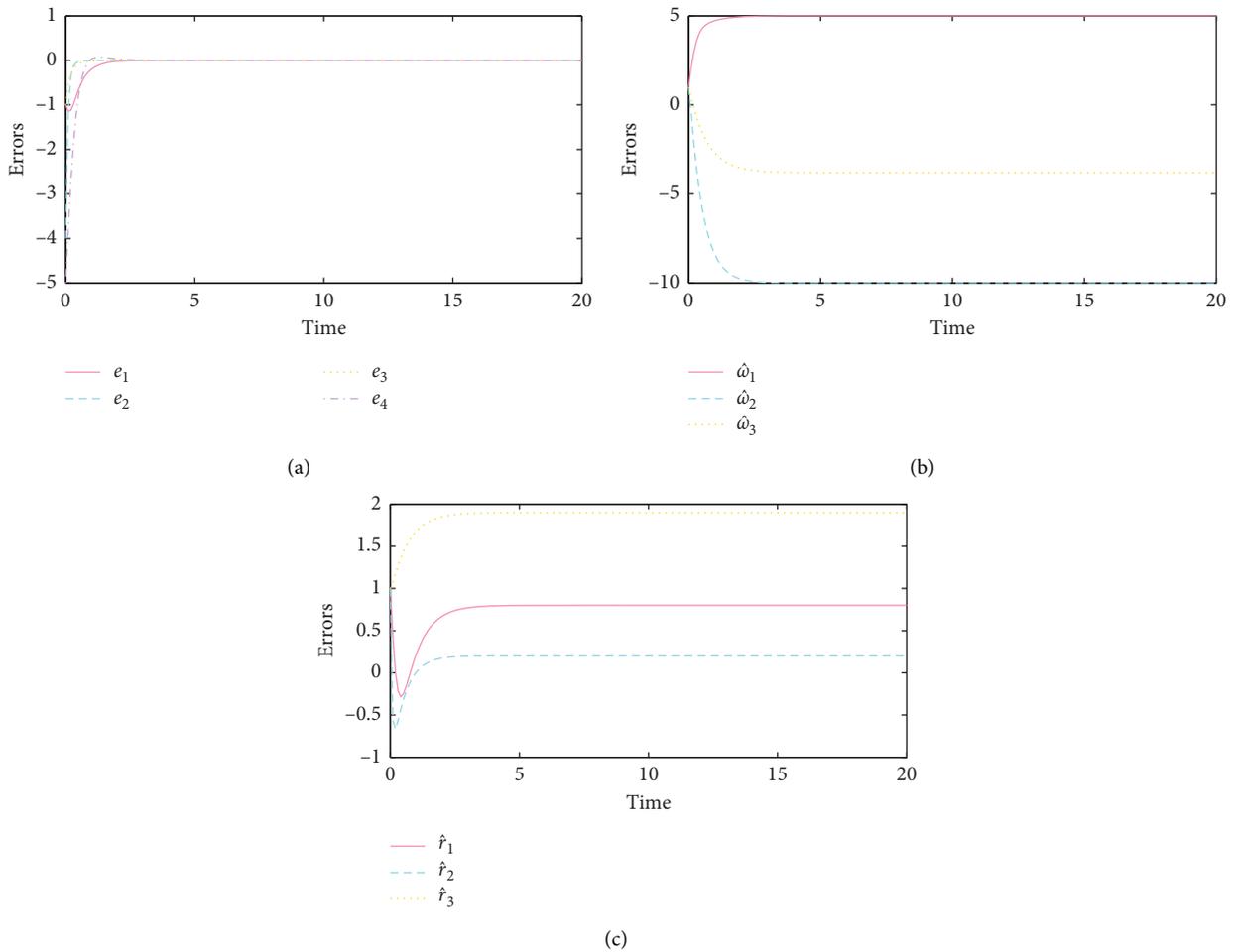


FIGURE 10: (a) Trajectories of the errors; (b) trajectories of the drive system unknown parameter estimates; (c) trajectories of the response system unknown parameter estimates.

$$\begin{cases} \dot{e}_1 = (\Delta\omega_1 - \Delta r_1)e_1 - k_1 e_1, \\ \dot{e}_2 = (\Delta\omega_2 - \Delta r_2)e_2 - k_2 e_2, \\ \dot{e}_3 = (\Delta\omega_3 - \Delta r_3)e_3 - k_2 e_3, \\ \dot{e}_4 = (\Delta\omega_1 - \Delta r_1)e_1 - \Delta r_2 e_2 - k_4 e_4. \end{cases} \quad (55)$$

For numerical simulation, the fourth-order Runge–Kutta integration method has been used to solve the system of differential (47) and (48). The initial values of the drive system and response system are $(x_1(0), x_2(0), x_3(0)) = (2, 6, 4)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (1, 2, 3, 2)$, respectively; the initial value of parameter estimations is $(\hat{r}_1(0), \hat{r}_2(0), \hat{r}_3(0)) = (0.1, 0.1, 0.1)$ and $(\hat{\omega}_1(0), \hat{\omega}_2(0), \hat{\omega}_3(0)) = (0.1, 0.1, 0.1)$, the control matrix $K = \text{diag}(2, 2, 1, 3)$, and the constant gain matrices $\psi = \text{diag}(1.5, 2, 1.5)$ and $\psi' = \text{diag}(1.5, 2, 1.5)$. Figure 10(a) shows the trajectories of errors, and it can be seen that the synchronization errors tend to zero; that is, the driving system (47) and the response system (48) achieve stable synchronization in 4 D . Figures 10(b) and 10(c) show the trajectories of the unknown parameter estimations; it can be seen that the estimations of unknown parameters can converge to some constants.

5. Conclusions

Based on Lyapunov stability theory and adaptive control method, the synchronization of chaotic systems with different dimensions is researched in this paper. Using two scaling matrices, an adaptive feedback hybrid controller and parameter adaptive laws are designed to realize the synchronization of chaotic systems with different dimensions. According to the characteristics of unknown parameters, a more simplified synchronization control scheme is designed. Finally, three examples of chaotic systems are used to demonstrate the effectiveness of the proposed method. Although a new controller is set up according to the characteristics of the system, it is still more complex. It is hoped that a more simple and convenient controller is designed for the synchronization of chaotic system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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