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Research Article

Fractional Rogue Waves with Translational Coordination, Steep Crest, and Modified Asymmetry

Bo Xu,[1,2] **Yufeng Zhang**,[1] **and Sheng Zhang**[3]

[1]*School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China*
[2]*School of Educational Sciences, Bohai University, Jinzhou 121013, China*
[3]*School of Mathematical Sciences, Bohai University, Jinzhou 121013, China*

Correspondence should be addressed to Bo Xu; bxu@bhu.edu.cn, Yufeng Zhang; zyfxz@cumt.edu.cn, and Sheng Zhang; szhangchina@126.com

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To construct fractional rogue waves, this paper first introduces a conformable fractional partial derivative. Based on the conformable fractional partial derivative and its properties, a fractional Schrödinger (NLS) equation with Lax integrability is then derived and first- and second-order fractional rogue wave solutions of which are finally obtained. The obtained fractional rogue wave solutions possess translational coordination, providing, to some extent, the degree of freedom to adjust the position of the rogue waves on the coordinate plane. It is shown that the obtained first- and second-order fractional rogue wave solutions are steeper than those of the corresponding NLS equation with integer-order derivatives. Besides, the time the second-order fractional rogue wave solution undergoes from the beginning to the end is also short. As for asymmetric fractional rogue waves with different backgrounds and amplitudes, this paper puts forward a way to construct them by modifying the obtained first- and second-order fractional rogue wave solutions.

1. Introduction

Rogue waves are a kind of local waves, which are first found in the ocean. In the high field intensity, rogue waves may be triggered as an additional phenomenon [1]. Though different from tsunami, the rogue waves that occur in the ocean have extraordinary destructiveness. Recently, rogue waves especially optical ones gained high attention [2–14]. It is Solli et al. [2] who reported the optical rogue waves generated through a generalized NLS equation. In 2010, Yan [4] obtained financial rogue waves by means of a nonlinear option pricing model.

The celebrated focusing and defocusing NLS equations:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0, \tag{1}$$

which are the basic models of NLS-type equations; some generalized NLS equations can be reduced to equation (1). As two examples, the first one we introduce is the generalized NLS equation with gain in the following form used in nonlinear fiber optics [15]:

$$i\psi_z = \frac{\beta(z)}{2}\psi_{\tau\tau} - \gamma(z)|\psi|^2\psi + i\frac{g(z)}{2}\psi, \tag{2}$$

where $\beta(z)$, $\gamma(z)$, and $g(z)$ are the supposed functions of propagation distance z, representing the group velocity dispersion, nonlinearity, and distributed gain, respectively. And, the second one is the integrable coupled NLS equations [6]:

$$iq_{1t} + q_{1xx} + 2\left(|q_1|^2 + |q_2|^2\right)q_1 = 0, \tag{3}$$

$$iq_{2t} + q_{2xx} + 2\left(|q_1|^2 + |q_2|^2\right)q_2 = 0. \tag{4}$$

Besides, the $(2+1)$-dimensional Davey–Stewartson-I (DS-I) equations [7]:

$$iA_t = A_{xx} + A_{yy} + \left(\varepsilon|A|^2 - 2Q\right)A, Q_{xx} - Q_{yy} = \varepsilon\left(|A|^2\right)_{xx}, \quad (\varepsilon = \pm 1), \tag{5}$$

which are regarded as an extension of equation (1) to the two-dimensional space, which can be converted into an equivalent form of equation (1) by using the transformation of variables $\tau = 2t$ and $\xi = \pm\sqrt{3}x/2 \pm y/2$.

Fractional calculus has been applied to many fields [16–30], which shows its irreplaceable role. In 2014, Fujioka et al. [17] described fractional optical solitons by an extended NLS equation with fractional dispersion and fractional nonlinearity. In 2020, Liu and Chen [31] derived the time-space fractional cylindrical Kadomtsev–Petviashvili (cKP) and modified cKP equations which better described the propagation of ion-acoustic waves in ultrarelativistic plasmas. In order to fully understand the propagation characteristics and periodicity of dust acoustic solitary waves in dusty plasmas, Zhang et al. [32] derived the modifed Zakharov–Kuznetsov (mZK) equation and obtained some exact solutions of the time-fractional mZK equation. From a mathematical point of view [33], there are essential differences between rogue wave solutions and soliton solutions. Although rogue wave solutions are unstable, they are also different from nonautonomous solitons. The NLS-type equations not only have soliton solutions but also have rogue wave solutions. In this paper, we would like to derive a fractional-focusing NLS equation:

$$iD_t^{\alpha}u + \frac{1}{2}D_x^{2\alpha}u + |u|^2u = 0, \quad 0 < \alpha = \frac{p}{q} \le 1, \tag{6}$$

and then, we construct its first- and second-order fractional rogue wave solutions; here, p and q are coprime positive integers, and q is an odd number so that $(x-a)^{1-\alpha} \in \mathbb{R}$ embedded into the fractional partial derivatives D_t^{α} and $D_x^{2\alpha} = D_x^{\alpha}D_x^{\alpha}$ hold (see Definition 1). Note that when $\alpha = 1$, equation (6) reduces to the integer-order focusing NLS equation [5] which is the first mathematical model for rogue waves.

The rest of this paper is organized as follows. In Section 2, we present the conformable fractional partial derivative and its properties. In Section 3, we derive the fractional NLS equation and give its Lax representation. In Section 4, we bilinearize the fractional NLS (6) and then construct its first- and second-order fractional rogue wave solutions and show their nonlinear dynamic evolution. In Section 5, we discuss the construction of asymmetric fractional first- and second-order rogue waves by modifying the obtained fractional rogue wave solutions and conclude this paper at the same time.

2. Definition and Some Basic Properties

We first present the definition of the conformable fractional partial derivative [34].

Definition 1 [34]. Let function $u(x,t)$: $\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$,arbitrary constant $a \in \mathbb{R}$, and fractional order $\alpha \in (0,1]$; then, the conformable fractional partial derivative is defined as follows:

$$D_x^{\alpha}u(x,t) = u_x^{(\alpha)}(x,t) = \lim_{\varepsilon\longrightarrow 0}\frac{u\left[x+\varepsilon(x-a)^{1-\alpha},t\right]-u(x,t)}{\varepsilon}, \quad 0 < \alpha = \frac{p}{q} \le 1, \tag{7}$$

where the positive integers p and q are coprime and q is an odd number. As a supplement of Definition 1, we let $u_x^{(\alpha)}(x,t)|_{x=a} = \lim_{x\longrightarrow a^+}u_x^{(\alpha)}(x,t)$.

When equation (7) holds, we say $u(x,t)$ is α differentiable with respect to x. If $x \in [a,\infty)$ and $u(x,t) \in \mathbb{R}$ is not related to t and $\alpha \in (0,1]$, then $D_x^{\alpha}u(x)$ becomes the generalized version [35] of the conformable fractional derivative [36] proposed by Khalil et al.

We next list some basic properties of the conformable fractional partial derivative defined in Definition 1. The proofs of these properties are similar to the ones in [35, 36], we omit them here.

Property 1. If $u(x,t)$ is α differentiable with respect to x, then

$$D_x^{\alpha}u(x,t) = (x-a)^{1-\alpha}u_x(x,t). \tag{8}$$

Property 2. If $u(x,t)$ and $v(x,t)$ are α differentiable with respect to x, then

$$D_x^{\alpha}[ku(x,t) + lv(x,t)] = kD_x^{\alpha}u(x,t) + lD_x^{\alpha}v(x,t), \tag{9}$$

$$D_x^{\alpha}[u(x,t)v(x,t)] = v(x,t)D_x^{\alpha}u(x,t) + u(x,t)D_x^{\alpha}v(x,t), \tag{10}$$

$$D_x^{\alpha}\frac{u(x,t)}{v(x,t)} = \frac{v(x,t)D_x^{\alpha}u(x,t) - u(x,t)D_x^{\alpha}v(x,t)}{v^2(x,t)}, \quad v(x,t) \ne 0, \tag{11}$$

where k and l are all constants.

Property 3. If $u[v(x,t)]$ and $v(x,t)$ are α differentiable with respect to $v(x,t)$ and x, respectively, then

$$D_x^{\alpha}u[v(x,t)] = u_v(x,t)D_x^{\alpha}v(x,t), \quad x \ne a, \tag{12}$$

$$D_x^{\alpha}u[v(x,t)]|_{x=a} = u_v(x,t)\lim_{x\longrightarrow a^+}D_x^{\alpha}v(x,t), \quad x = a. \tag{13}$$

Property 4. Fractional derivatives of some common functions have results:

$$D_x^{\alpha}c = 0, \tag{14}$$

$$D_x^{\alpha}(x-a)^r = r(x-a)^{r-\alpha}, \tag{15}$$

$$D_x^\alpha e^{(x-a)^\alpha} = \alpha e^{(x-a)^\alpha}, \tag{16}$$

$$D_x^\alpha \sin\ (x-a)^\alpha = \alpha\ \cos\ (x-a)^\alpha, \tag{17}$$

$$D_x^\alpha \cos\ (x-a)^\alpha = -\alpha\ \sin\ (x-a)^\alpha, \tag{18}$$

where c and $r$are constants.

As the end of this section, we finally show in Figure 1 four different conformable fractional partial derivatives of $u(x,t) = (x-a)^2 + t$ with respect to x by setting $a = 1$ and $t = 0$.

It can be seen that the smaller the fractional order α is, the larger the value of its derivative function $D_x^\alpha u(x,t) = 2(x-a)^{2-\alpha}$is. This naturally raises the question of whether we can construct such rogue wave solutions whose widths and crests are dependent on the fractional order so that they are steeper and more destructive than those of the corresponding integer-order case. The following work will clarify the existence of such a fractional rogue wave solution for equation (6). In addition, we will construct, in Section 4, another fractional rogue wave solution of equation (6), which takes less time than the integer order one from the occurrence to the end. Steep crest or short duration is closer to one of the characteristics of rogue waves, that is, it appears and disappears very suddenly. Besides, it is worth noting that the rogue wave solutions to construct in this paper possess coordinate parameters. Such rogue wave solutions are no longer limited to the coordinate origin as usual, and their positions are selective. Besides, after a modification, such fractional rogue wave solutions can also show asymmetry.

3. Lax Representation of the Fractional NLS Equation

For fractional NLS equation (6), we have the following Theorem 1.

Theorem 1. *Fractional NLS equation (6) is Lax integrable, which has the Lax representation:*

$$[L, D_t^\alpha - M] = 0, \tag{19}$$

with

$$L = i\begin{pmatrix} D_x^\alpha & -u \\ -u^* & -D_x^\alpha \end{pmatrix}, M = i\begin{pmatrix} D_x^{2\alpha} + \frac{1}{2}uu^* & -uD_x^\alpha - \frac{1}{2}u_x^{(\alpha)} \\ -u^* D_x^\alpha - \frac{1}{2}u_x^{*(\alpha)} & -D_x^{2\alpha} - \frac{1}{2}uu^* \end{pmatrix}, \tag{20}$$

where * denotes the complex conjugate.

Proof. A direct computation tells

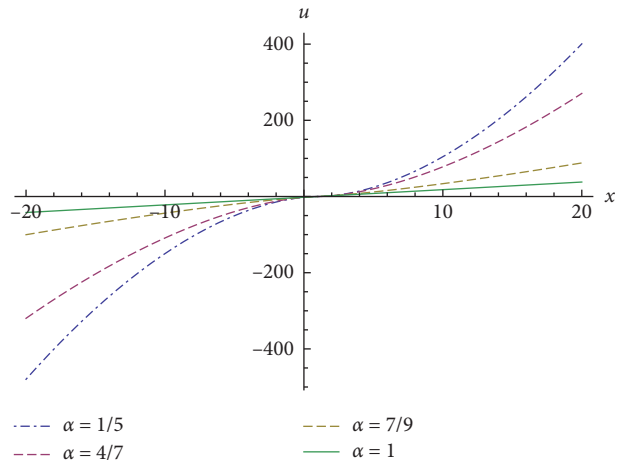

Figure 1: Conformable fractional partial derivatives of u with different orders.

$$[L, -M] = \begin{pmatrix} 0 & -\frac{1}{2}u_x^{(2\alpha)} - u^2u^* \\ \frac{1}{2}u_x^{*(2\alpha)} + u^{*2}u & 0 \end{pmatrix}. \tag{21}$$

At the same time, it is easy to verify the relation:

$$[L, D_t^\alpha] = i\begin{pmatrix} 0 & u_t^{(\alpha)} \\ u_t^{*(\alpha)} & 0 \end{pmatrix}. \tag{22}$$

Substituting equations (21) and (22) into equation (19) yields

$$\begin{pmatrix} 0 & iu_t^{(\alpha)} \\ iu_t^{*(\alpha)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}u_x^{(2\alpha)} + u^2u^* \\ -\frac{1}{2}u_x^{*(2\alpha)} - u^{*2}u & 0 \end{pmatrix} = 0, \tag{23}$$

which is exactly equivalent to equation (6). The proof is ended. □

4. Fractional Rogue Wave Solutions of the Fractional NLS Equation

This section consists of three parts for the rogue wave solutions of equation (6), one part is the bilinear forms of equation (6) and the other three parts are the first- and second-order fractional rogue wave solutions and their nonlinear dynamic analysis.

4.1. Bilinearization. For the bilinear forms of fractional NLS equation (6), the following Theorem 2 reaches.

Theorem 2. *Suppose that*

$$u = \frac{g}{f}e^{i(x-a)^\alpha/\alpha}, \tag{24}$$

where f and g are real- and complex-valued functions of x and t, respectively, then fractional NLS equation (6) has the following bilinear forms:

$$\left(iH_t^{\alpha} + \frac{1}{2}H_x^{2\alpha}\right) g \cdot f = 0, \tag{25}$$

$$H_x^{2\alpha} f \cdot f = gg^* - f^2, \tag{26}$$

where H_x^{α} and H_t^{α} are the fractional version [24] of Hirota's bilinear operator [37]:

$$H_x^{m\alpha} H_t^{n\alpha} f \cdot g = \left(D_x^{\alpha} - D_{x'}^{\alpha}\right)^m \left(D_t^{\alpha} - D_{t'}^{\alpha}\right)^n \cdot f(x,t) g(x',t')\big|_{x'=x,t'=t}, \quad m,n \in \mathbb{N}. \tag{27}$$

Proof. Firstly, by the transformation

$$u = v(x,t) e^{i(x-a)^{\alpha}/\alpha}, \tag{28}$$

we write equation (6) as

$$iD_t^{\alpha} v + \frac{1}{2}D_x^{2\alpha} v + \left(|v|^2 - 1\right)v = 0, \quad 0 < \alpha = \frac{p}{q} \le 1. \tag{29}$$

Secondly, we suppose that

$$v = \frac{g}{f}, \tag{30}$$

then equation (29) can be reduced to

$$i\left(fg_t^{\alpha} - f_t^{\alpha} g\right) + \frac{1}{2}\left(fg_x^{2\alpha} - 2f_x^{\alpha} g_x^{\alpha} + f_x^{2\alpha} g\right) - \frac{g}{f}\left\{2\left[ff_x^{2\alpha} - \left(f_x^{\alpha}\right)^2\right] - gg^* + f^2\right\} = 0. \tag{31}$$

Further, letting

$$2\left[ff_x^{2\alpha} - \left(f_x^{\alpha}\right)^2\right] - gg^* + f^2 = 0. \tag{32}$$

With the help of the fractional bilinear operator (27), we finally can rewrite equations (31) and (32) as equations (25) and (26), respectively. Thus, considering equations (28) and (30), we have completed the proof.

Based on the fractional bilinear forms (25) and (26), the fractional N-soliton solution of equation (6) can be obtained. However, the starting point of this paper is to construct rogue wave solutions, and we omit them. In what follows, we shall employ equation (31) to construct fractional rouge wave solutions of equation (6). □

4.2. First-Order Fractional Rogue Wave Solution. For the first-order fractional rogue wave solution, we have Theorem 3 below.

Theorem 3. *Fractional NLS equation (6) has the following first-order fractional rogue wave solution:*

$$u = e^{i\left((x-a)^{\alpha}/\alpha\right)}\left[1 - \frac{8i\alpha(t-a)^{\alpha} + 4\alpha^2}{4\left[(x-a)^{\alpha}\right]^2 + 4\left[(t-a)^{\alpha}\right]^2 + \alpha^2}\right], \tag{33}$$

and the maximum crest value of which is $|u|_{\max} = 3$.

Proof. We first suppose that

$$f = a_1\left[(x-a)^{\alpha}\right]^2 + a_2\left[(t-a)^{\alpha}\right]^2 + a_3, \tag{34}$$

$$g = \left(a_{41} + ia_{42}\right)\left[(x-a)^{\alpha}\right]^2 + \left(a_{51} + ia_{52}\right)\left[(t-a)^{\alpha}\right]^2 + \left(a_{61} + ia_{62}\right)(x-a)^{\alpha} + \left(\left(a_{71} + ia_{72}\right)(t-a)^{\alpha} + a_{81} + ia_{82}\right), \tag{35}$$

where a_1, a_2, a_3, a_{41}, a_{42}, a_{51}, a_{52}, a_{61}, a_{62}, a_{71}, a_{72}, a_{81}, and $a_{82}$are all undetermined real constants. Substituting equations (34) and (35) into equation (31), canceling the common denominator, and then collecting the coefficients of the same powers $[(x-a)^{\alpha}]^j[(t-a)^{\alpha}]^s$ $(j,s = 0,1,2,\ldots)$ of the real part and imaginary part, we drive a set of algebraic equations about these undermined real constants. Further, solving the derived set of algebraic equations yields

$$\begin{aligned} a_2 &= a_1, \\ a_3 &= \frac{a_1\alpha^2}{4}, \\ a_{41} &= a_1, \\ a_{51} &= a_1, \\ a_{72} &= -2a_1\alpha, \\ a_{81} &= -\frac{3}{4}a_1\alpha^2, \end{aligned} \tag{36}$$

where a_1 is a nonzero real constant, and all the other undetermined real constants not mentioned here are all zeros. Finally, from equations (28), (30), and (34)–(36), we arrive at first-order rogue wave solution (33).

Since

$$|u|^2 = 1 + \frac{8\left\{4\left[(t-a)^{\alpha}\right]^2 + \alpha^2 - 4\left[(x-a)^{\alpha}\right]^2\right\}}{\left\{4\left[(x-a)^{\alpha}\right]^2 + 4\left[(t-a)^{\alpha}\right]^2 + \alpha^2\right\}^2} \tag{37}$$

and

$$\frac{\sqrt{4\left[(t-a)^{\alpha}\right]^2 + \alpha^2} + 2\left|(x-a)^{\alpha}\right|}{4\left[(t-a)^{\alpha}\right]^2 + 4\left[(x-a)^{\alpha}\right]^2 + \alpha^2} \frac{\sqrt{4\left[(t-a)^{\alpha}\right]^2 + \alpha^2} - 2\left|(x-a)^{\alpha}\right|}{4\left[(x-a)^{\alpha}\right]^2 + 4\left[(t-a)^{\alpha}\right]^2 + \alpha^2} \le 1, \tag{38}$$

we then have $|u|^2 \le 9$, and hence, complete the proof. □

4.3. Second-Order Fractional Rogue Wave Solution. For the second-order fractional rogue wave solution, we have Theorem 4 below.

Theorem 4. *Fractional NLS equation (6) has the following second-order fractional rogue wave solution:*

$$u = e^{i(x-a)^{\alpha}/\alpha}\left(1 - \frac{h}{f}\right), \tag{39}$$

and the crest value at the point (a, a) *is* $|u| = 5$. *Here,*

$$\begin{aligned} f = {} & 16\left[(x-a)^{\alpha}\right]^6 + 16\left[(t-a)^{\alpha}\right]^6 + 48\left[(x-a)^{\alpha}\right]^4\left[(t-a)^{\alpha}\right]^2 + 48\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^4 \\ & + 12\alpha^2\left[(x-a)^{\alpha}\right]^4 + 108\alpha^2\left[(t-a)^{\alpha}\right]^4 - 72\alpha^2\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2 + 27\alpha^4\left[(x-a)^{\alpha}\right]^2 + 99\alpha^4\left[(t-b)^{\alpha}\right]^2 + \frac{9\alpha^6}{4}, \end{aligned} \tag{40}$$

$$\begin{aligned} h = {} & 48\alpha^2\left[(x-a)^{\alpha}\right]^4 + 240\alpha^2\left[(t-a)^{\alpha}\right]^4 + 288\alpha^2\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2 + 72\alpha^4\left[(x-a)^{\alpha}\right]^2 \\ & + 216\alpha^4\left[(t-a)^{\alpha}\right]^2 + i(t-a)^{\alpha}\left\{96\alpha\left[(x-a)^{\alpha}\right]^4 - 96\alpha\left[(t-a)^{\alpha}\right]^4 - 192\alpha\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2\right. \\ & \left. + 144\alpha^3\left[(x-a)^{\alpha}\right]^2 - 48\alpha^3\left[(t-a)^{\alpha}\right]^2 + 90\alpha^5\right\} - 9\alpha^6. \end{aligned} \tag{41}$$

Proof. Inspired by the work [38] and equation (39), we suppose that

$$\begin{aligned} f = {} & a_1\left[(x-a)^{\alpha}\right]^6 + a_2\left[(t-a)^{\alpha}\right]^6 + a_3\left[(x-a)^{\alpha}\right]^4\left[(t-a)^{\alpha}\right]^2 + a_4\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^4 \\ & + a_5\left[(x-a)^{\alpha}\right]^4 + a_6\left[(t-a)^{\alpha}\right]^4 + a_7\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2 + a_8\left[(x-a)^{\alpha}\right]^2 + a_9\left[(t-b)^{\alpha}\right]^2 + a_{10}, \end{aligned} \tag{42}$$

$$\begin{aligned} g = {} & f + a_{11}\left[(x-a)^{\alpha}\right]^4 + a_{12}\left[(t-a)^{\alpha}\right]^4 + a_{13}\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2 + a_{14}\left[(x-a)^{\alpha}\right]^2 + a_{15}\left[(t-a)^{\alpha}\right]^2 \\ & - \mathrm{i}(t-a)^{\alpha}\left\{a_{16}\left[(x-a)^{\alpha}\right]^4 + a_{17}\left[(t-a)^{\alpha}\right]^4 + a_{18}\left[(x-a)^{\alpha}\right]^2\left[(t-a)^{\alpha}\right]^2 + a_{19}\left[(x-a)^{\alpha}\right]^2 + a_{20}\left[(t-a)^{\alpha}\right]^2 + a_{21}\right\} + a_{22}, \end{aligned} \tag{43}$$

where $a_j\,(j = 1, 2, \ldots, 22)$ are all real constants to be further undetermined. By a series of operations, such as substituting equations (40) and (41) into equation (31), canceling the common denominator, separating the real and imaginary parts, and collecting the coefficients of the same powers $\left[(x-a)^{\alpha}\right]^j\left[(t-a)^{\alpha}\right]^s\,(j, s = 0, 1, 2, \ldots)$, then a set of algebraic equations about these undermined constants is derived. Solving this set of algebraic equations, we can determine equations (40) and (41) by

$$\begin{aligned} a_1 &= 16, \\ a_2 &= 16, \\ a_3 &= 48, \\ a_4 &= 48, \\ a_5 &= 12\alpha^2, \\ a_6 &= 108\alpha^2, \\ a_7 &= -72\alpha^2, \\ a_8 &= 27\alpha^4, \\ a_9 &= 99\alpha^4, \end{aligned} \tag{44}$$

$$\begin{aligned} a_{10} &= \frac{9\alpha^6}{4}, \\ a_{11} &= -48\alpha^2, \\ a_{12} &= -240\alpha^2, \\ a_{13} &= -288\alpha^2, \\ a_{14} &= -72\alpha^4, \\ a_{15} &= -216\alpha^4, \end{aligned} \tag{45}$$

$$\begin{aligned} a_{16} &= 96\alpha, \\ a_{17} &= 96\alpha, \\ a_{18} &= 192\alpha, \\ a_{19} &= -144\alpha^3, \\ a_{20} &= 48\alpha^3, \\ a_{21} &= -90\alpha^5, \\ a_{22} &= 9\alpha^6. \end{aligned} \tag{46}$$

By using equations (28), (30), and (42)–(46), we can easily obtain equation (39). When $x = a$ and $t = a$, equations (40) and (41) give $f = 9\alpha^6/4$ and $h = -9\alpha^6$, respectively. We then obtain $|u| = 5$. The proof is ended. □

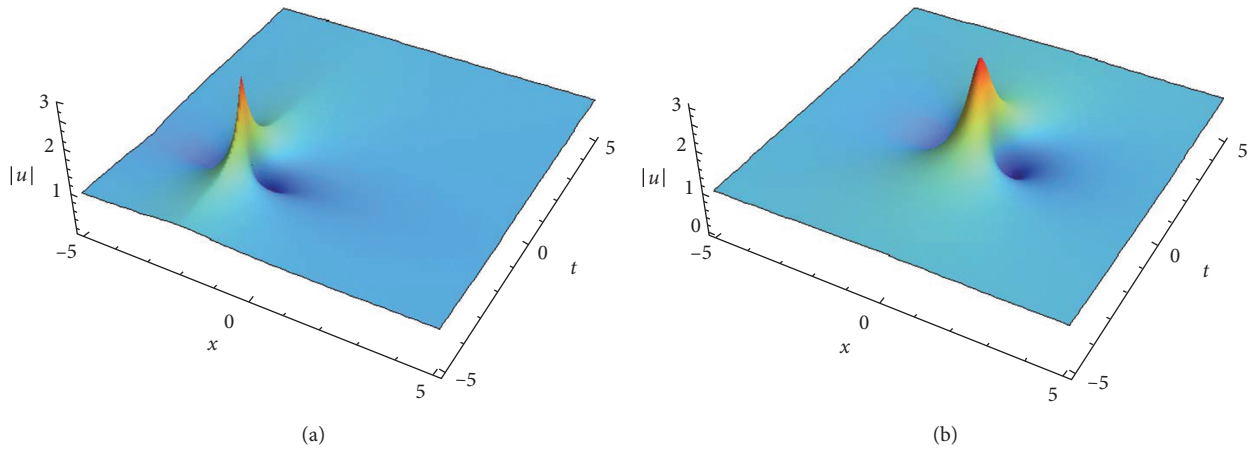

FIGURE 2: First-order fractional rogue wave solution (color online) (33) and its comparison with the integer order. (a) $\alpha = 5/7$ and $a = -2$; (b) $\alpha = 1$ and $a = 0$.

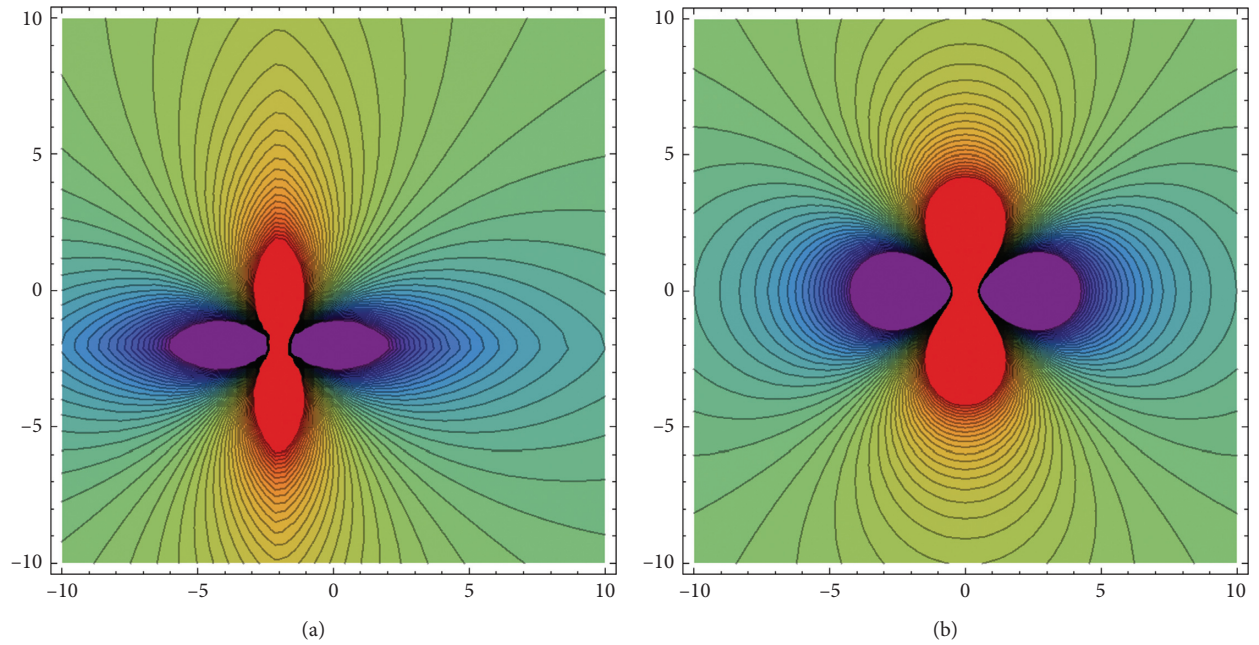

FIGURE 3: Contours of first-order fractional rogue wave solution (color online) (33) and its comparison with the integer order. (a) $\alpha = 5/7$ and $a = -2$. (b) $\alpha = 1$ and $a = 0$.

4.4. Nonlinear Dynamic Evolution and Comparison. In this section, we show first- and second-order fractional rogue wave solutions (33) and (39) for the nonlinear dynamic evolution. At the same time, for the sake of comparison, we also show the rogue wave solutions of corresponding NLS equation (6) with the integer order.

In Figures 2–5 , first-order rogue wave solution (33) with translational coordination and its comparison reference without translational coordination are shown. It can be seen that the width of fractional rogue wave (33) along the x-axis is much narrower than that of the corresponding integer-order rogue wave.

A similar difference also occurs in second-order rogue wave solution (39) with translational coordination and its comparison reference without translational coordination, but the widths of fractional rogue wave (39) along x-axis and t-axis are all much narrower than those of the corresponding integer-order rogue wave, see Figures 6–9 . This shows that,

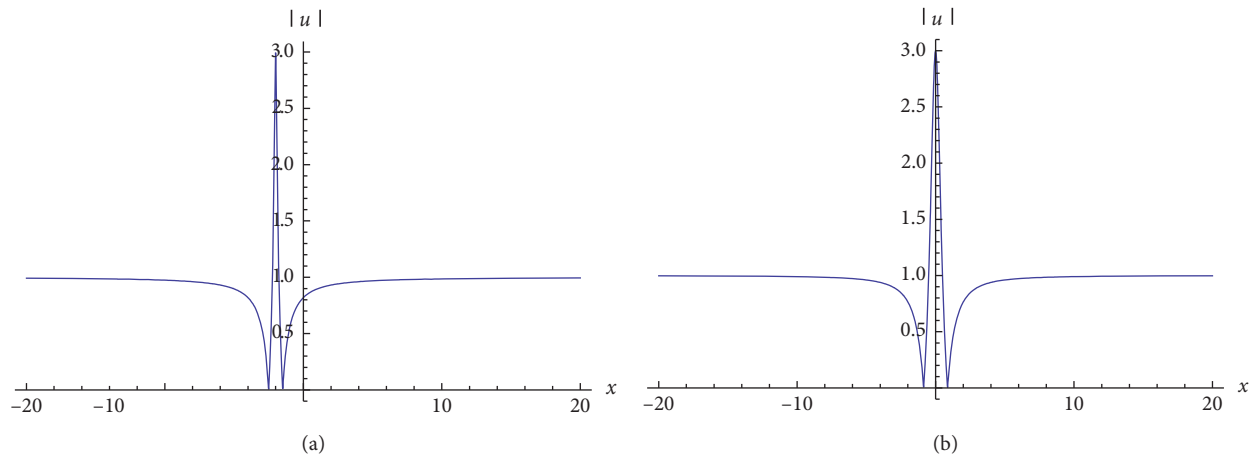

FIGURE 4: Amplitudes of first-order fractional rogue wave solution (33) and its comparison with the integer order. (a) $\alpha = 5/7$, $a = -2$, and $t = -2$. (b) $\alpha = 1$, $a = 0$, and $t = 0$.

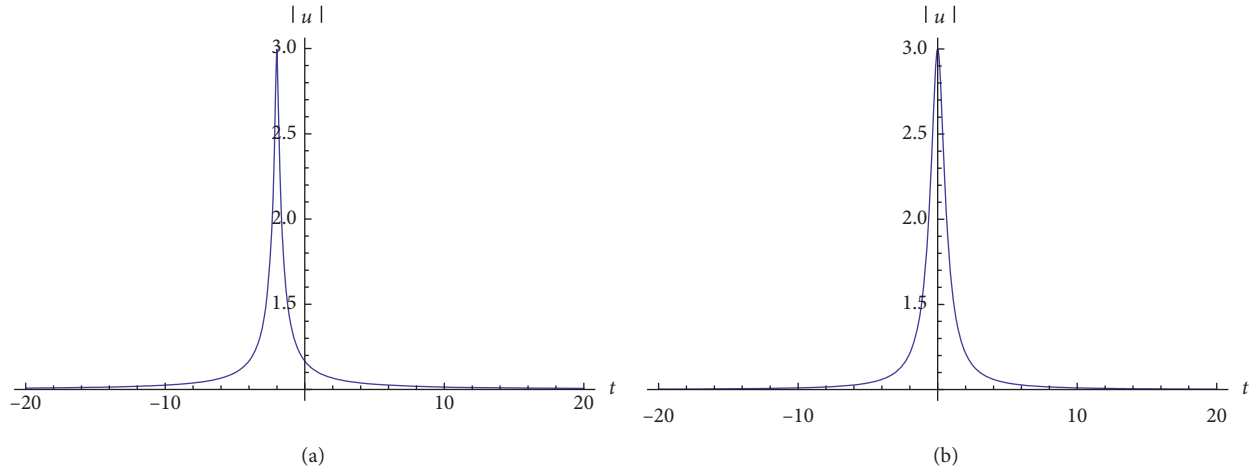

FIGURE 5: Dynamic evolution of first-order fractional rogue wave solution (33) and its comparison with the integer order. (a) $\alpha = 5/7$, $a = -2$, and $x = -2$. (b) $\alpha = 1$, $a = 0$, and $x = 0$.

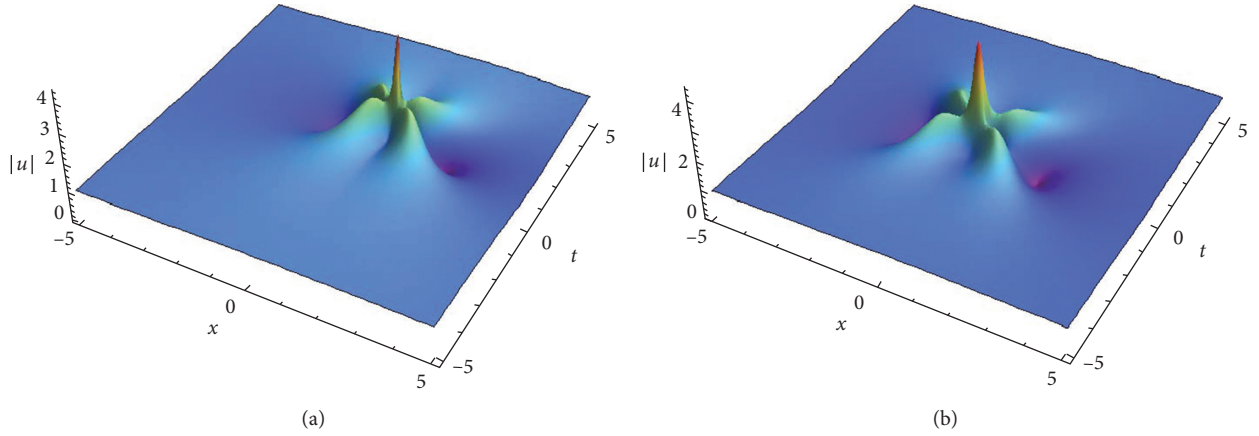

FIGURE 6: Second-order fractional rogue wave solution (color online) (39) and its comparison with the integer order. (a) $\alpha = 9/7$ and $a = 1$; (b) $\alpha = 1$ and $a = 0$.

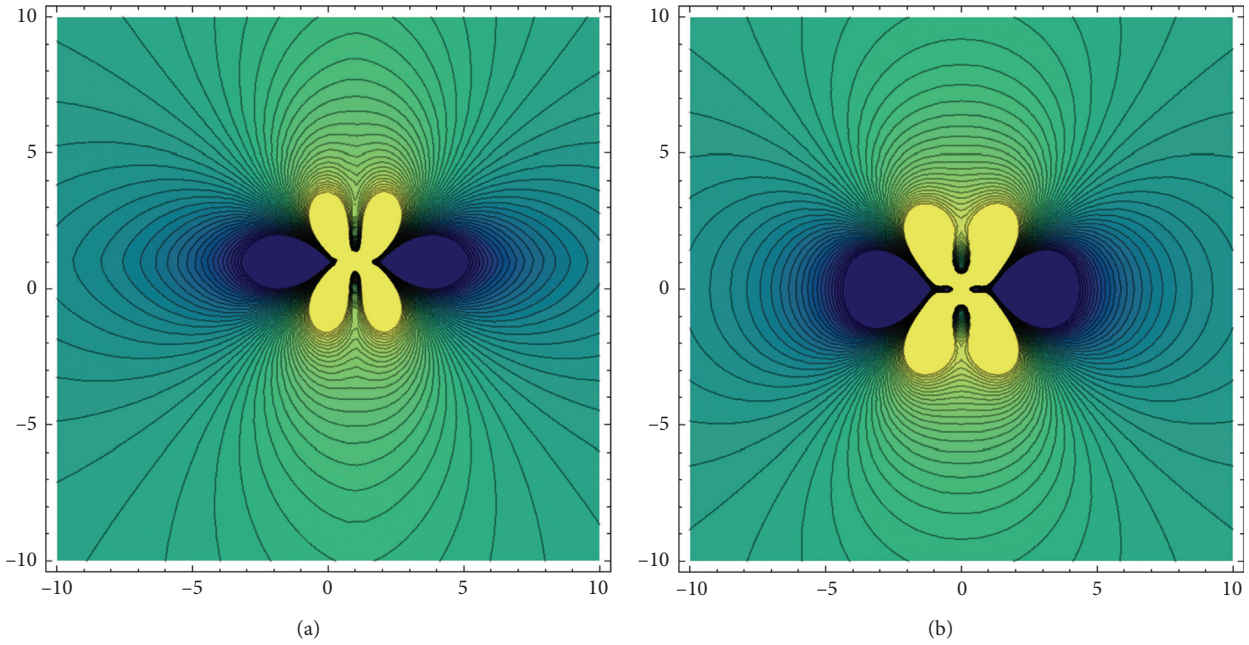

Figure 7: Contours of first-order fractional rogue wave solution (color online) (39) and its comparison with the integer order. (a) $\alpha = 9/7$ and $a = 1$; (b) $\alpha = 1$ and $a = 0$.

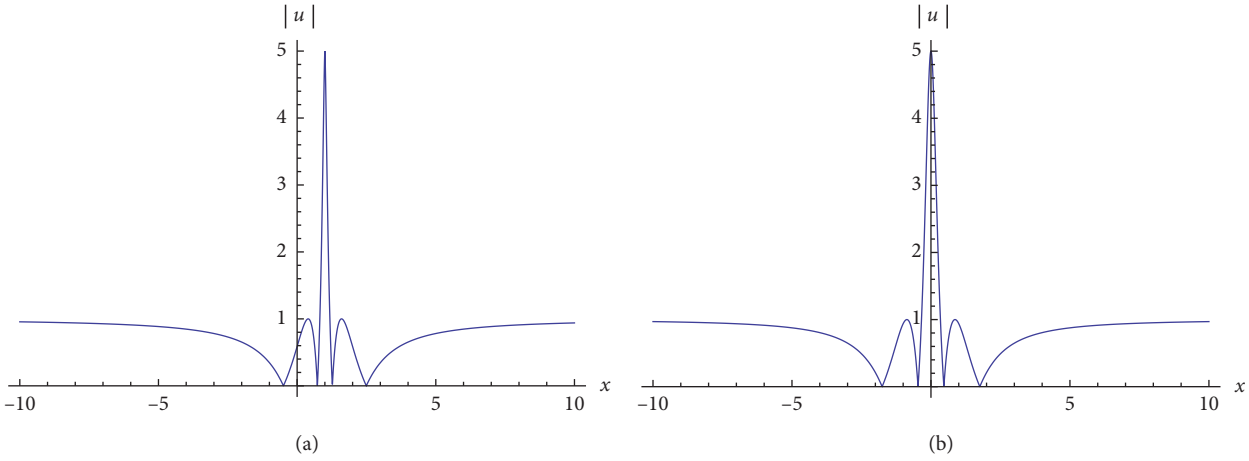

Figure 8: Amplitudes of first-order fractional rogue wave solution (39) and its comparison with the integer order. (a) $\alpha = 9/7$, $a = 1$, and $t = 1$; (b) $\alpha = 1$, $a = 0$, and $t = 0$.

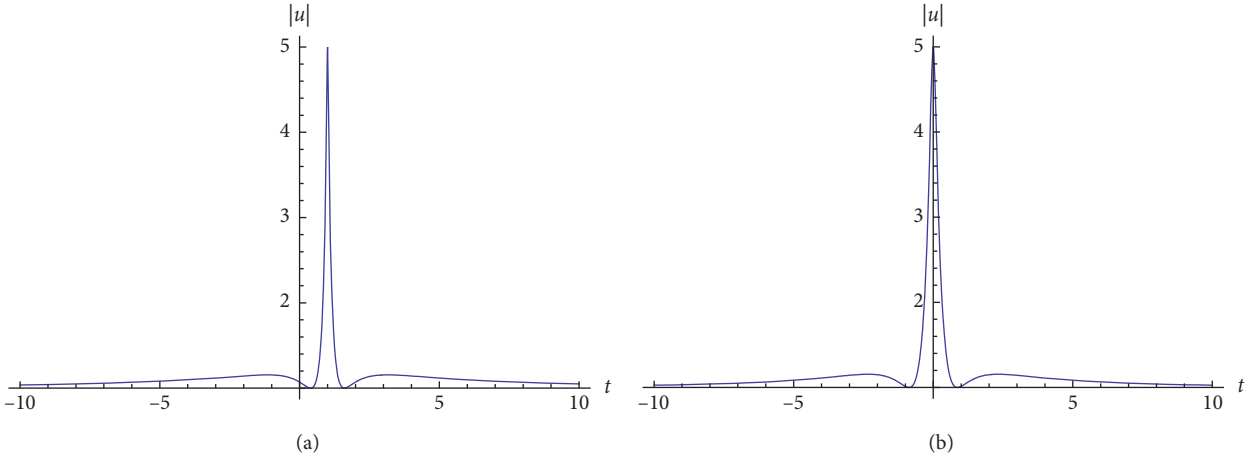

Figure 9: Dynamic evolution of first-order fractional rogue wave solution (39) and its comparison with the integer order. (a) $\alpha = 9/7$, $a = 1$, and $x = 1$; (b) $\alpha = 1$, $a = 0$, and $x = 0$.

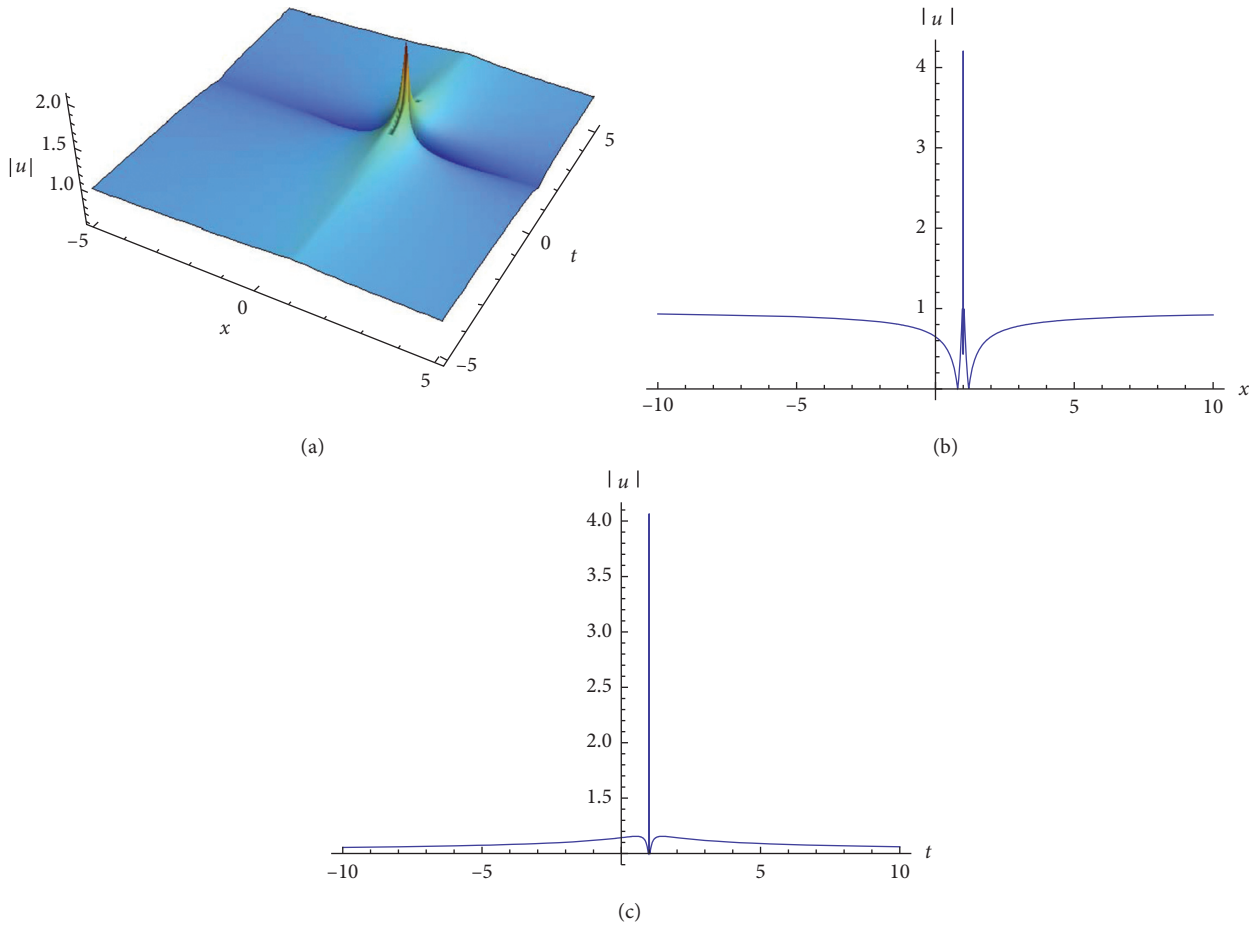

Figure 10: Second-order fractional rogue wave solution (color online) (39) with $\alpha = 1/3$ and $a = 1$.

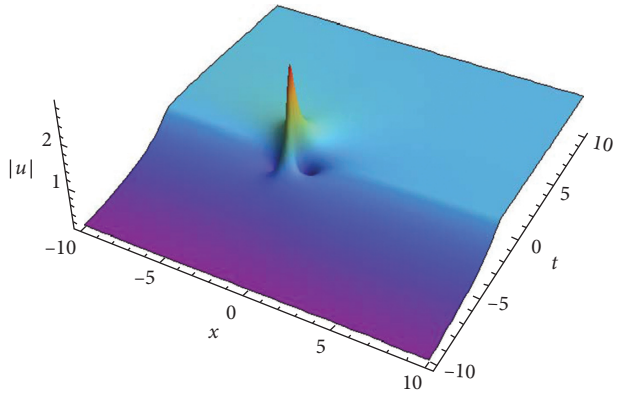

Figure 11: Asymmetric first-order fractional rogue wave (color online) with $\alpha = 9/10$ and $a = -2$ modified from solution (33).

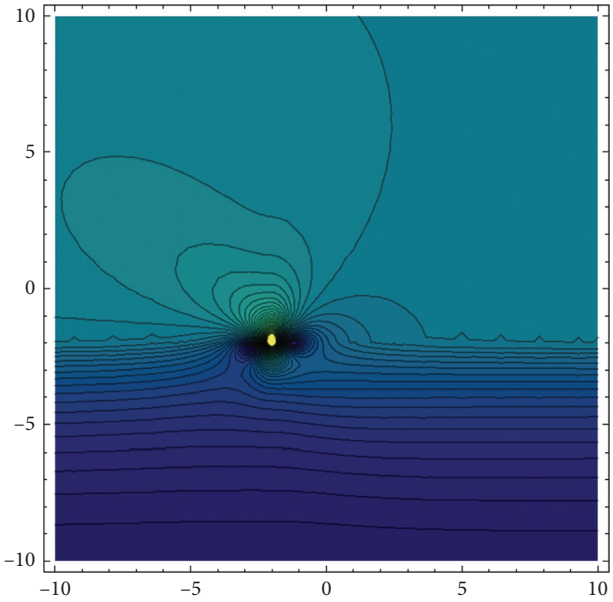

Figure 12: Contours of the asymmetric first-order fractional rogue wave (color online) withα = 9/10 and a = −2 modified from solution (33).

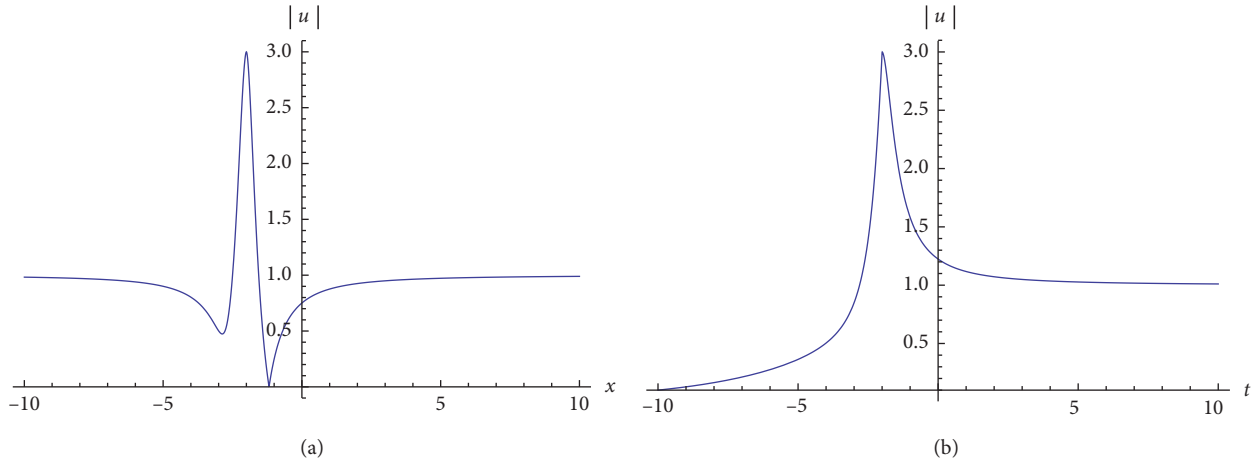

Figure 13: Amplitude and dynamic evolution of the asymmetric first-order fractional rogue wave with α = 9/10 and a = −2 modified from solution (33). (a) t = −2. (b) x = −2.

in addition to the steep crest, the time from the beginning to the end of the fractional rouge wave is also short. With the decrease of fractional order α, the high and subhigh crests of second-rogue wave solution (39) are getting closer and closer until almost together, always keeping $|u| = 5$ at the point $(x, t) = (1, 1)$(see Figure 10).

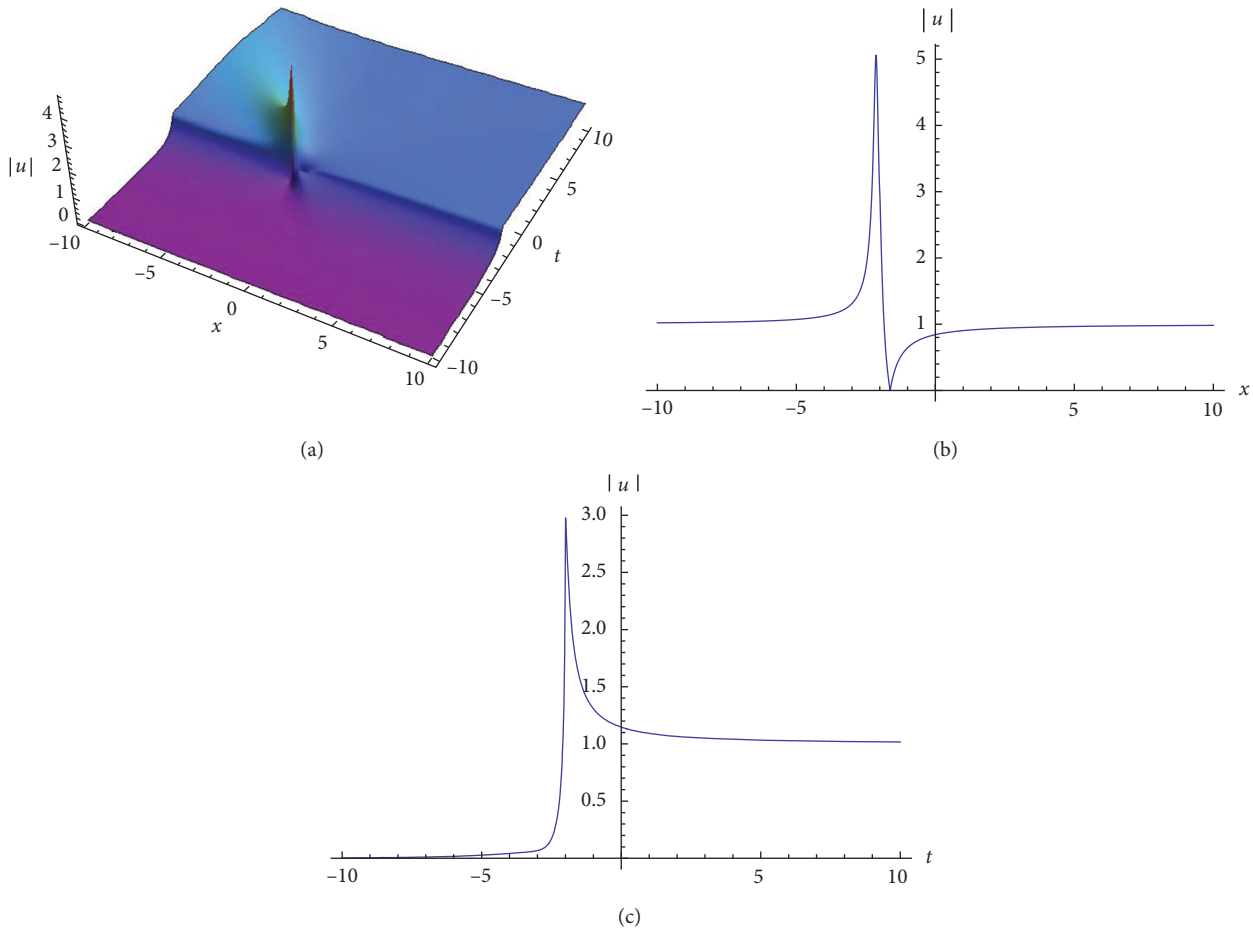

Figure 14: Asymmetric first-order fractional rogue wave (color online) and its amplitude and dynamic evolution with $\alpha = 5/8$ and $a = -2$ modified from solution (33).

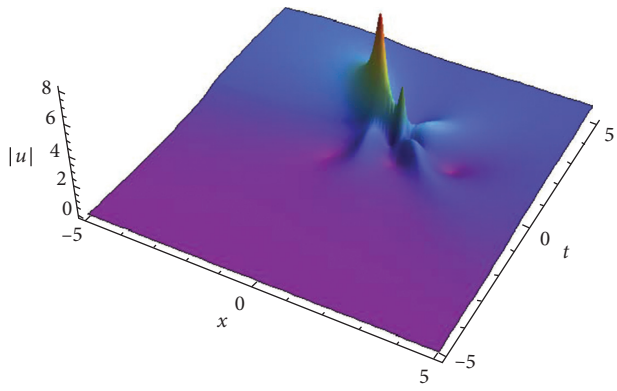

Figure 15: Asymmetric second-order fractional rogue wave (color online) with $\alpha = 5/6$ and $a = 1$ modified from solution (39).

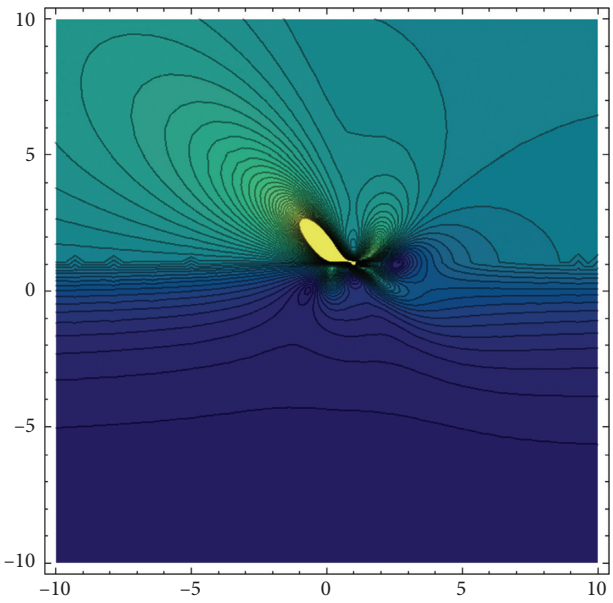

Figure 16: Contours of the asymmetric second-order fractional rogue wave (color online) with $\alpha = 5/6$ and $a = 1$ modified from solution (39).

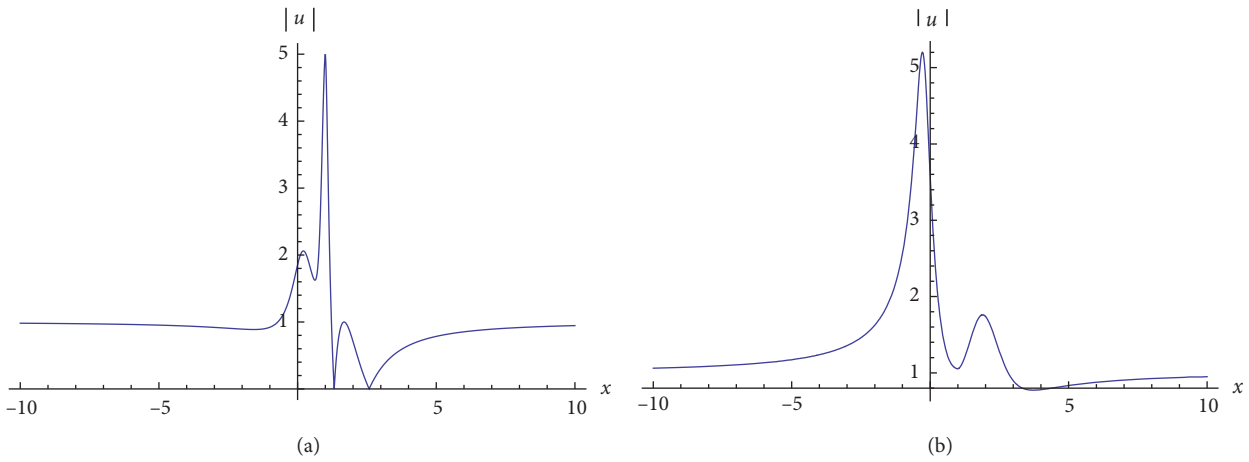

Figure 17: Amplitudes of the symmetric second-order fractional rogue wave with $\alpha = 5/6$ and $a = 1$ modified from solution (39). (a) $t = 1$. (b) $t = 2$.

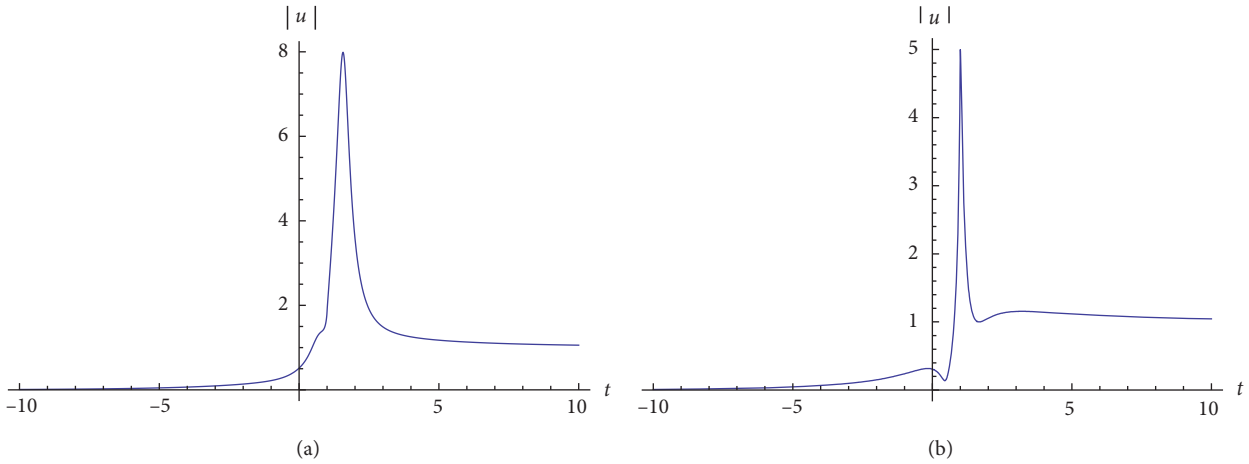

Figure 18: Dynamic evolution of the asymmetric second-order fractional rogue wave with $\alpha = 5/6$ and $a = 1$ modified from solution (39). (a) $x = 0$. (b) $x = 1$.

5. Discussion and Conclusion

In summary, we have derived fractional NLS equation (6) with Lax integrability by introducing conformable fractional partial derivative (7). Based on transformed equation (31), first- and second-order fractional rogue wave solutions (33) and (39) with translational coordination of equation (6) are obtained. The translation coordinate provides a certain degree of freedom for adjusting the position of fractional rogue wave solutions (33) and (39) on the coordinate plane.

First-order fractional rogue wave solution (33) can also obtained by using the limits $\lim_{k\longrightarrow 0} v_k^{(2\alpha)}$ as did in [14] and equation (28); here,

$$v = \frac{1 - 2\sqrt{4-k^2}/2 - k^2 \pm ik\sqrt{4-k^2}\cos\left[k(x-a)^\alpha/\alpha\right]e^{\pm k\sqrt{4-k^2}(t-a)^\alpha/2\alpha} + 2 - k^2 \mp ik\sqrt{4-k^2}/2 - k^2 \pm ik\sqrt{4-k^2}e^{\pm k\sqrt{4-k^2}(t-a)^\alpha/\alpha}}{1 - \sqrt{4-k^2}\cos\left[k(x-a)^\alpha/\alpha\right]e^{\pm k\sqrt{4-k^2}(t-a)^\alpha/2\alpha} + e^{\pm k\sqrt{4-k^2}(t-a)^\alpha/\alpha}}, \tag{47}$$

which is a fractional homoclinic solution of equation (29). Using the limits of similar fractional homoclinic solutions like equation (47) to construct second-order fractional rogue wave solution (39) is worthy of study.

In addition to steepness and instantaneity, asymmetry is also a remarkable characteristic of the rogue wave. It is also worth exploring to construct asymmetric rogue waves by means of fractional models. In Figures 11–18 , two asymmetry (see Figure 13(a) for an example) first- and second-order fractional rogue waves with different backgrounds (see Figures 13(b), 14(c), and 18(b) for $|u|_{t\longrightarrow-\infty} = 0$, while $|u|_{t\longrightarrow\infty} = 1$) and large amplitudes (see Figures 14(b) and 18(a) for $|u| \approx 5$ and $|u| \approx 8$) are shown based on solutions (33) and (39), and the assumption that the value of $(x-a)^{1-\alpha}$ in Definition 1 always returns a complex number for all x<a when q is an even number. Computer simulation results through Mathematica 8.0 show that, with the decrease of fractional order α, one of the four subhigh waves in the second-order fractional rogue waves gradually exceeds the highest crest. This is different from the usual second-order rogue waves. However, under this case, such first- and second-order rogue wave solutions do not satisfy equation (6). This is because that when $(x-a)^{1-\alpha}$ with $x<a$ is a complex number, f is no longer a real valued function as previously assumed, and f and g^* are not the solutions of equation (31). In fact, when $x<a$, it cannot transform equation (6) into equation (29), but $x>a$ is valid. In spite of this, it may be important that this paper provides a way to construct asymmetric fractional rogue waves by modifying the obtained first- and second-order fractional rogue wave solutions.

Data Availability

The data used in this paper are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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