Hindawi
Complexity
Volume 2021, Article ID 6671499, 31 pages
https://doi.org/10.1155/2021/6671499

Research Article

Complex Dynamics of a Stochastic Two-Patch Predator-Prey Population Model with Ratio-Dependent Functional Responses

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Received 2 November 2020; Revised 10 December 2020; Accepted 24 December 2020; Published 16 January 2021

Academic Editor: Heng Liu

This paper investigates a stochastic two-patch predator-prey model with ratio-dependent functional responses. First, the existence of a unique global positive solution is proved via the stochastic comparison theorem. Then, two different methods are used to discuss the long-time properties of the solutions pathwise. Next, sufficient conditions for extinction and persistence in mean are obtained. Moreover, stochastic persistence of the model is discussed. Furthermore, sufficient conditions for the existence of an ergodic stationary distribution are derived by a suitable Lyapunov function. Next, we apply the main results in some special models. Finally, some numerical simulations are introduced to support the main results obtained. The results in this paper generalize and improve the previous related results.

1. Introduction

The dynamic relationship between predators and their preys has been universal in mathematical ecology. In the nature world, foraging behaviour is a common phenomenon. Ecological species have the ability to adapt through learning (see [1]). An individual will adjust its behaviour by learning in response to a change of the environment in order to survive and acquire the most food. In [1], the authors studied the two-patch predator-prey population model

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(r_1 - a_1x_1) - \dfrac{s_1x_1\nu y}{1 + h_1s_1x_1}, \\ \dfrac{dx_2}{dt} = x_2(r_2 - a_2x_2) - \dfrac{s_2x_2(1-\nu)y}{1 + h_2s_2x_2}, \\ \dfrac{dy}{dt} = y\left[-m_1\nu - m_2(1-\nu) + \dfrac{s_1x_1e_1\nu}{1 + h_1s_1x_1} + \dfrac{s_2x_2e_2(1-\nu)}{1 + h_2s_2x_2}\right], \end{cases} \tag{1}$$

with nonnegative initial conditions. Here, x_i denotes the density of prey in patch i $(i = 1, 2)$, and y represents the density of predators. ν $(0 \le \nu \le 1)$ is the proportion of time that predators stay in patch 1 on average; r_i $(i = 1, 2)$ is the intrinsic growth rate of prey in patch i; a_i is the intraspecific competition coefficient of the prey in patch i; s_i is the attacking rate of the predators in patch i; e_i is the expected biomass of the prey converted to predators in patch i; m_i is the per capita mortality rate of predators in patch i; and h_i is the handling time of the predation in patch i, respectively.

It is well known that the functional response between the predator and prey plays an important role in the population dynamics. In model (1), the authors assumed that an individual predator consumes the prey with functional response $(x/1 + shx)$, which depends only on the prey. However, when predators have to search for food and, therefore, have to share or compete for food, a ratio-dependent functional response is more reasonable (see [2]). Based on the Holling-type II function, Arditi and Ginzburg [3] first proposed a ratio-dependent functional response of form $(\alpha x/x + \beta y)$. Here, α is the encounter rate with prey by a searching predator, and β is the half saturation constant for the prey. Kuang and Beretta [4] investigated the predator-prey model with ratio-dependent functional response

$$\begin{cases} \dfrac{dx}{dt} = x\left[r - ax - \dfrac{\alpha y}{x + \beta y}\right], \\ \dfrac{dy}{dt} = y\left[-d + \dfrac{e\alpha x}{x + \beta y}\right], \end{cases} \tag{2}$$

with nonnegative initial conditions. Here, x and y represent population sizes of the prey and predator at time t, respectively. All parameters are positive constants. r and a, respectively, stand for the prey intrinsic growth rate and the intraspecific competition rate of the prey. d is the death rate of the predator population. α, β, and e, respectively, represent the encounter rate, half capturing saturation constant, and conversion rate that predator y preys on prey x.

Note that population model (1) with the functional responses only depend on prey density. However, the ratio-dependent functional response depends not only on the prey but also on the predator. Thus, the ratio-dependent function of the prey and predator is more suitable to substitute for the model. Therefore, based on models (1) and (2), a two-patch predator-prey population model with ratio-dependent functional responses is expressed in the following form:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(r_1 - a_1x_1) - \dfrac{\alpha_1 v x_1 y}{x_1 + \beta_1 y}, \\ \dfrac{dx_2}{dt} = x_2(r_2 - a_2x_2) - \dfrac{\alpha_2(1-v)x_2 y}{x_2 + \beta_2 y}, \\ \dfrac{dy}{dt} = y\left[-m_1 v - m_2(1-v) + \dfrac{e_1\alpha_1 v x_1}{x_1 + \beta_1 y} + \dfrac{e_2\alpha_2(1-v)x_2}{x_2 + \beta_2 y}\right], \end{cases} \tag{3}$$

where x_i denotes the density of prey in patch i $(i = 1, 2)$ and y represents the density of predators. All meanings of the parameters are exact as or similar to those for model (1) except the following. Here, α_i, β_i, and e_i $(i = 1, 2)$ are the encounter rate, the half-saturation constant, and the conversion rate that y preys on x_i, respectively.

From [5], it can be seen that stochasticity or variability plays an important role in understanding the dynamics of predator-prey populations. Note that noise in models can lead to several interesting dynamical effects, which are not anticipated by their deterministic counterpart. Thus, in order to simulate population dynamics, environmental fluctuations should be considered in modeling. In general, environmental fluctuations can be simulated by a colored noise. From [6], it can be seen that if the colored noise is not strongly correlated, then one can approximate the colored noise by a white noise $\dot{w}(t)$. In fact, the white noise $\dot{w}(t)$ is formally regarded as the derivative of a Brownian motion $w(t)$, i.e., $\dot{w}(t) = (dw(t)/dt)$ (see [7]). As a result, it is more objective to modeling stochastic population models with white noise in mathematical biology. Recently, many authors have paid their attention to stochastic prey-predator models with white noise, see [8–15] and the references therein. Reference [8] investigated the stability of a stochastic one-predator-two-prey population model with time delay, while [13] considered the stability of a stochastic two-predator one-prey population model with time delay. References [10, 11, 15] discussed the dynamic behaviors of stochastic population models with the Allee effect. Reference [12] is concerned with a stochastic three-species food web model with omnivory and ratio-dependent functional response.

To the best of our knowledge, so far, there is no investigation on the dynamics of the stochastic two-patch prey-predator model with ratio-dependent functional responses. The purpose of this paper is to make some contribution in this direction. As in the work of Imhof and Walcher [16], assuming that the environmental noise is proportional to the variables, we obtain the following stochastic two-patch prey-predator model:

$$\begin{cases} dx_1(t) = x_1(t)\left[r_1 - a_1x_1(t) - \dfrac{\alpha_1 v y(t)}{x_1(t) + \beta_1 y(t)}\right]dt + \sigma_1 x_1(t)dw_1(t), \\ dx_2(t) = x_2(t)\left[r_2 - a_2x_2(t) - \dfrac{\alpha_2(1-v)y(t)}{x_2(t) + \beta_2 y(t)}\right]dt + \sigma_2 x_2(t)dw_2(t), \\ dy(t) = y(t)\left[-m_1 v - m_2(1-v) + \dfrac{e_1\alpha_1 v x_1(t)}{x_1(t) + \beta_1 y(t)} + \dfrac{e_2\alpha_2(1-v)x_2(t)}{x_2(t) + \beta_2 y(t)}\right]dt + \sigma_3 y(t)dw_3(t), \end{cases} \tag{4}$$

with initial value $(x_1(0\quad), x_2(0), y(0)) = (x_{10}, x_{20}, = y_0) \in \mathbb{R}_3^+ = \{(x, y, z) \in \mathbb{R}^3\colon x > 0, y > 0, z > 0\}$. All meanings of the parameters are exact as or similar to those for model (3) except the following. Here, $w = \{w_1(t), w_2(t), w_3(t)\colon t \geq 0\}$ represents the three-dimensional standard Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. σ_i^2 represents the intensity of noise $w_i(t)$ $(i = 1, 2, 3)$.

Furthermore, if the intraspecific competition of the predator is considered in model (4), then one can obtain the following stochastic two-patch predator-prey model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_1x_1(t) - \dfrac{\alpha_1 v y(t)}{x_1(t) + \beta_1 y(t)}\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[r_2 - a_2x_2(t) - \dfrac{\alpha_2(1-v)y(t)}{x_2(t) + \beta_2 y(t)}\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}y(t) = y(t)\left[-m_1 v - m_2(1-v) - by(t) + \dfrac{e_1\alpha_1 v x_1(t)}{x_1(t) + \beta_1 y(t)} + \dfrac{e_2\alpha_2(1-v)x_2(t)}{x_2(t) + \beta_2 y(t)}\right]\mathrm{d}t + \sigma_3 y(t)\mathrm{d}w_3(t), \end{cases} \tag{5}$$

with initial value $(x_1(0), x_2(0), y(0)) = (x_{10}, x_{20}, = y_0) \in \mathbb{R}_3^+$. Here, b is the interspecific competition coefficient of the predator.

In this paper, we first investigate the dynamics of the stochastic two-patch predator-prey population model (5). Then, we apply the main results in the stochastic predator-prey population model (4). The rest of this paper is organized as follows. In Section 2, we first prove that model (5) has a unique global positive solution by the stochastic comparison theorem. Then, we discuss the long-time properties of the solutions pathwise. Using the exponential martingale inequality and the Borel–Cantelli lemma, we show that the sample Lyapunov exponents of the solutions are non-positive. Moreover, we prove that, under certain conditions, the sample Lyapunov exponents of the solutions are zero. In Section 3, we establish the sufficient conditions for the extinction and persistence in mean of model (5). In Section 4, we first prove the stochastic ultimate boundedness of model (5) by using two different methods. Then, we show that model (5) is stochastically permanent. Moreover, in section 5, by constructing a suitable Lyapunov function, we establish sufficient conditions for the existence of an ergodic stationary distribution to model (5). Next, in Section 6, we apply the main results to two stochastic two-species predator-prey population models and stochastic two-patch predator-prey population model (4). Section 7 contains some numerical results, which are used to demonstrate the theoretical results in this paper. Moreover, through numerical calculation, we find other dynamic properties of the model. The paper ends with a conclusion.

For simplicity, in the coming discussion, we introduce the notations

$$\begin{aligned} \lambda_1 &\doteq r_1 - \frac{\alpha_1 v}{\beta_1}, \\ \lambda_2 &\doteq r_2 - \frac{\alpha_2(1-v)}{\beta_2}, \\ \lambda_3 &\doteq e_1\alpha_1 v + e_2\alpha_2(1-v) - m_1 v - m_2(1-v). \end{aligned} \tag{6}$$

2. Global Positive Solution and Pathwise Estimation

In this section, we first show that model (5) has a unique positive global solution by the stochastic comparison theorem. Then, we discuss the long-time properties of the solutions pathwise.

Theorem 1. *For any given initial value $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, model (5) has a unique global solution $x_1(t)$ on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. We consider the following system:

$$\begin{cases} \mathrm{d}X_1(t) = \left[r_1 - a_1 e^{X_1(t)} - \dfrac{\alpha_1 v e^{Y(t)}}{e^{X_1(t)} + \beta_1 e^{Y(t)}} - \dfrac{\sigma_1^2}{2}\right]\mathrm{d}t + \sigma_1 \mathrm{d}w_1(t), \\ \mathrm{d}X_2(t) = \left[r_2 - a_2 e^{X_2(t)} - \dfrac{\alpha_2(1-v)e^{Y(t)}}{e^{X_2(t)} + \beta_2 e^{Y(t)}} - \dfrac{\sigma_2^2}{2}\right]\mathrm{d}t + \sigma_2 \mathrm{d}w_2(t), \\ \mathrm{d}Y(t) = \left[-m_1 v - m_2(1-v) - be^{Y(t)} + \dfrac{e_1\alpha_1 v e^{X_1(t)}}{e^{X_1(t)} + \beta_1 e^{X_1(t)}} + \dfrac{e_2\alpha_2(1-v)e^{X_2(t)}}{e^{X_2(t)} + \beta_2 e^{Y(t)}} - \dfrac{\sigma_3^2}{2}\right]\mathrm{d}t + \sigma_3 \mathrm{d}w_3(t), \end{cases} \tag{7}$$

where $(X_1(0), X_2(0), Y(0)) = (\ln x_{10}, \ln x_{20}, \ln y_0)$. It is clear that the coefficients of system (7) are locally Lipschitz continuous. Hence, system (7) has a unique maximal local solution $(X_1(t), X_2(t), Y(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Let $x_i(t) = e^{X_i(t)}$ $(i = 1, 2)$ and $y(t) = e^{Y(t)}$. From Itô formula, it follows that $(x_1(t), x_2(t), y(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{Y(t)})$ is the unique positive local solution of model (5) with initial value (x_{10}, x_{20}, y_0) on $[0, \tau_e)$.

If we can verify that $\tau_e = \infty$ a.s., then $(X_1(t), X_2(t), Y(t))$ is a global solution to system (7). Now, using the stochastic comparison theorem, we show that $\tau_e = \infty$ a.s. We consider the following two stochastic differential systems:

$$\begin{cases} d\Phi_1(t) = \Phi_1(t)[r_1 - a_1\Phi_1(t)]dt + \sigma_1\Phi_1(t)dw_1(t), \\ d\Phi_2(t) = \Phi_2(t)[r_2 - a_2\Phi_2(t)]dt + \sigma_2\Phi_2(t)dw_2(t), \\ d\Psi(t) = \Psi(t)[e_1\alpha_1 v + e_2\alpha_2(1-v) - b\Psi(t)]dt + \sigma_3\Psi(t)dw_3(t), \end{cases} \tag{8}$$

with initial value $(\Phi_1(0), \Phi_2(0), \Psi(0)) = (x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$ and

$$\begin{cases} d\phi_1(t) = \phi_1(t)[\lambda_1 - a_1\phi_1(t)]dt + \sigma_1\phi_1(t)dw_1(t), \\ d\phi_2(t) = \phi_2(t)[\lambda_2 - a_2\phi_2(t)]dt + \sigma_2\phi_2(t)dw_2(t), \\ d\psi(t) = \psi(t)\left[\lambda_3 - \left(b + \dfrac{e_1\alpha_1 v\beta_1}{\phi_1(t)} + \dfrac{e_2\alpha_2(1-v)\beta_2}{\phi_2(t)}\right)\psi(t)\right]dt + \sigma_3\psi(t)dw_3(t), \end{cases} \tag{9}$$

with initial value $(\phi_1(0), \phi_2(0), \psi(0)) = (x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$.

Thanks to Lemma 4.2 in [17], systems (8) and (9) can be explicitly solved as follows:

$$\begin{cases} \Phi_1(t) = \dfrac{\exp\{(r_1 - (\sigma_1^2/2))t + \sigma_1 w_1(t)\}}{(1/x_{10}) + a_1\int_0^t \exp\{(r_1 - (\sigma_1^2/2))z + \sigma_1 w_1(z)\}dz}, \\ \Phi_2(t) = \dfrac{\exp\{(r_2 - (\sigma_2^2/2))t + \sigma_2 w_2(t)\}}{(1/x_{20}) + a_2\int_0^t \exp\{(r_2 - (\sigma_2^2/2))z + \sigma_2 w_2(z)\}dz}, \\ \Psi(t) = \dfrac{\exp\{(e_1\alpha_1 v + e_2\alpha_2(1-v) - (\sigma_3^2/2))t + \sigma_3 w_3(t)\}}{(1/y_0) + b\int_0^t \exp\{(e_1\alpha_1 v + e_2\alpha_2(1-v) - (\sigma_3^2/2))z + \sigma_3 w_3(z)\}dz}, \end{cases}$$
$$\begin{cases} \phi_1(t) = \dfrac{\exp\{(\lambda_1 - (\sigma_1^2/2))t + \sigma_1 w_1(t)\}}{(1/x_{10}) + a_1\int_0^t \exp\{(\lambda_1 - (\sigma_1^2/2))z + \sigma_1 w_1(z)\}dz}, \\ \phi_2(t) = \dfrac{\exp\{(\lambda_2 - (\sigma_2^2/2))t + \sigma_2 w_2(t)\}}{(1/x_{20}) + a_2\int_0^t \exp\{(\lambda_2 - (\sigma_2^2/2))z + \sigma_2 w_2(z)\}dz}, \\ \psi(t) = \dfrac{\exp\{(\lambda_3 - (\sigma_3^2/2))t + \sigma_3 w_3(t)\}}{(1/y_0) + \int_0^t (b + (e_1\alpha_1 v\beta_1/\phi_1(s)) + (e_2\alpha_2(1-v)\beta_2/\phi_2(s)))\exp\{(\lambda_3 - (\sigma_3^2/2))z + \sigma_3 w_3(z)\}dz}. \end{cases} \tag{10}$$

It is clear that the local solution $(x_1(t), x_2(t), y(t))$ is positive on $[0, \tau_e)$. Thus, from the stochastic comparison theorem (see Theorem 3.1 in [18]), it follows that $0 < \phi_i(t) \le x_i(t) \le \Phi_i(t)$ $(i = 1, 2)$ and $0 < \psi(t) \le y(t) \le \Psi(t)$ almost surely for $t \in [0, \tau_e)$. Thus, for $t \in [0, \tau_e)$,

$$\begin{aligned} &\ln\phi_i(t) \le X_i(t) \le \ln\Phi_i(t), \\ &\ln\psi(t) \le Y(t) \le \ln\Psi(t), a.s., \quad i = 1, 2. \end{aligned} \tag{11}$$

Note that $\ln\phi_i(t)$, $\ln\Phi_i(t)$, $\ln\psi(t)$, and $\ln\Psi(t)$ $(i = 1, 2)$ exist on $[0, \infty)$. Thus, $\tau_e = \infty$ a.s. This means that, for any $(X_1(0), X_2(0), Y(0)) = (\ln x_{10}, \ln x_{20}, \ln y_0) \in \mathbb{R}^3$, system

(7) has a unique global solution $(X_1(t), X_2(t), Y(t))$ on $[0,\infty)$ a.s. Thus, for any initial value $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, model (5) has a unique global positive solution $(x_1(t), x_2(t), y(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{Y(t)})$ on $[0,\infty)$ a.s.

Now, we discuss the long-time properties of the solutions pathwise. We denote $\langle u(t)\rangle = (1/t)\int_0^t u(s)\mathrm{d}s$.

Theorem 2. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). Then,*

$$\limsup_{t\longrightarrow\infty} \frac{\ln x_i(t)}{t} \leq 0,$$
$$\limsup_{t\longrightarrow\infty} \frac{\ln y(t)}{t} \leq 0 \text{ a.s.}, \quad i = 1, 2. \tag{12}$$

Proof. We consider the stochastic process $\Phi_1(t)$ in system (8). Applying Itô's formula to $e^t \ln\Phi_1$ leads to

$$e^t \ln \Phi_1(t) = \ln x_{10} + \int_0^t e^s \left[\ln \Phi_1(s) + r_1 - a_1\Phi_1(s) - \frac{\sigma_1^2}{2}\right]\mathrm{d}s + M_1(t), \tag{13}$$

where $M_1(t) = \int_0^t \sigma_1 e^s \mathrm{d}w_1(s)$ is a continuous local martingale with $M_1(0) = 0$ and $\langle M_1, M_1\rangle_t = \int_0^t \sigma_1^2 e^{2s}\mathrm{d}s$. Let $n = 1, 2, \ldots$, $\gamma > 0$, $\theta > 1$ and $0 < \varepsilon < 1$. We choose $T = n\gamma$, $\alpha = \varepsilon e^{-n\gamma}$, and $\beta = (\theta e^{n\gamma}\ln n)/\varepsilon$. By the exponential martingale inequality (see Theorem 1.7.4 in [7]), one can get

$$\mathbb{P}\left\{\sup_{0\leq t\leq T}\left[M_1(t) - \frac{\alpha}{2}\langle M_1, M_1\rangle_t\right] > \beta\right\} \leq e^{-\alpha\beta} = \frac{1}{n^\theta}. \tag{14}$$

Since $\sum_{n=0}^{\infty}(1/n^\theta) < \infty$ for $\theta > 1$, the Borel–Cantelli lemma (see Lemma 1.2.4 in [7]) implies that there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and an integer-valued random variable $n_0 = n_0(\omega)$ such that, for every $\omega \in \Omega_0$, $M_1(t) \leq (\theta e^{n\gamma}\ln n/\varepsilon) + (\varepsilon e^{-n\gamma}/2)\langle M_1, M_1\rangle_t$ holds for all $0 \leq t \leq n\gamma$, $n \geq n_0$. Substituting the abovementioned inequality into (13), we have

$$e^t \ln \Phi_1(t) \leq \ln x_{10} + \int_0^t e^s [\ln \Phi_1(s) + r_1 - a_1\Phi_1(s)]\mathrm{d}s - \frac{1}{2}\int_0^t \sigma_1^2 e^s \mathrm{d}s + \frac{\varepsilon e^{-n\gamma}}{2}\int_0^t \sigma_1^2 e^{2s}\mathrm{d}s + \frac{\theta e^{n\gamma}\ln n}{\varepsilon}, \tag{15}$$

which holds for all $0 \leq t \leq n\gamma$, $n \geq n_0$. Note that, for $0 \leq s \leq t \leq n\gamma$,

$$\frac{1}{2}\varepsilon e^{-n\gamma}\sigma_1^2 e^{2s} - \frac{1}{2}\sigma_1^2 e^s = \frac{1}{2}\sigma_1^2 e^s\left(\varepsilon e^{s-n\gamma} - 1\right) \leq \frac{1}{2}\sigma_1^2 e^s(\varepsilon - 1) < 0. \tag{16}$$

Thus, it follows from (15) that

$$e^t \ln \Phi_1(t) \leq \ln x_{10} + \int_0^t e^s [\ln \Phi_1(s) + r_1 - a_1\Phi_1(s)]\mathrm{d}s + \frac{\theta e^{n\gamma}\ln n}{\varepsilon} \tag{17}$$

holds for all $0 \leq t \leq n\gamma$, $n \geq n_0$. Consider function $q_1(x) = \ln x + r_1 - a_1 x$ on $(0,\infty)$. Obviously, q_1 has maximum value for $x = (1/a_1) > 0$ and $q_{1,\max} = \ln(1/a_1) + r_1 - 1$. We denote $K_1 \doteq (\ln(1/a_1) + r_1 - 1)\vee 1$. Then,

$$e^t \ln \Phi_1(t) \leq \ln x_{10} + K_1 e^t + \frac{\theta e^{n\gamma}\ln n}{\varepsilon}, \tag{18}$$

holds for all $0 \leq t \leq n\gamma$, $n \geq n_0$. Thus, for all $0 \leq (n-1)\gamma \leq t \leq n\gamma$, $n \geq n_0$, we have

$$\frac{\ln \Phi_1(t)}{\ln t} \leq \frac{\ln x_{10}}{e^t \ln t} + \frac{K_1}{\ln t} + \frac{\theta e^\gamma \ln n}{\varepsilon \ln[(n-1)\gamma]}. \tag{19}$$

Letting $n \longrightarrow \infty$ (and so $t \longrightarrow \infty$), we obtain $\limsup_{t\longrightarrow\infty}(\ln \Phi_1(t)/\ln t) \leq (\theta e^\gamma/\varepsilon)$ a.s. Moreover, letting $\theta\downarrow 1$, $\gamma\downarrow 0$ and $\varepsilon\uparrow 1$, one can get $\limsup_{t\longrightarrow\infty}(\ln \Phi_1(t)/\ln t) \leq 1$ a.s. This, together with $\lim_{t\longrightarrow\infty}(\ln t/t) = 0$, yields $\limsup_{t\longrightarrow\infty}(\ln \Phi_1(t)/t) \leq 0$ a.s. Note that $0 < x_1(t) \leq \Phi_1(t)$ a.s. for any $t \in [0,\infty)$. Then, we have

$$\limsup_{t\longrightarrow\infty} \frac{\ln x_1(t)}{t} \leq \limsup_{t\longrightarrow\infty} \frac{\ln \Phi_1(t)}{t} \leq 0, \text{ a.s.} \tag{20}$$

By a similar discussion as that mentioned above for x_1, we also have

$$\limsup_{t\longrightarrow\infty} \frac{\ln x_2(t)}{t} \leq 0,$$
$$\limsup_{t\longrightarrow\infty} \frac{\ln y(t)}{t} \leq 0, \text{ a.s.} \tag{21}$$

The proof is, therefore, complete. □

Lemma 1 (See [12]). We consider the one-dimensional stochastic differential equation

$$\mathrm{d}x(t) = x(t)[a - bx(t)]\mathrm{d}t + \sigma x(t)\mathrm{d}w(t), \tag{22}$$

where a, b, and σ are positive constants and and $w(t)$ is the standard Brownian motion. For any $x_0 > 0$, let $x(t)$ be the solution of equation (22) with $x(0) = x_0$. If $a > (\sigma^2/2)$, then

$$\lim_{t\longrightarrow\infty} \frac{\ln x(t)}{t} = 0,$$
$$\lim_{t\longrightarrow\infty} \langle x(t)\rangle = \frac{a - \left(\sigma^2/2\right)}{b}, \text{ a.s.} \tag{23}$$

Theorem 3. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). If $\lambda_i - (\sigma_i^2/2) > 0$ $(i = 1, 2, 3)$, then*

$$\lim_{t\longrightarrow\infty} \frac{\ln x_i(t)}{t} = 0,$$
$$\lim_{t\longrightarrow\infty} \frac{\ln y(t)}{t} = 0, \text{ a.s.} \quad i = 1, 2. \tag{24}$$

Moreover, the solution obeys

$$0<\frac{\lambda_1-\left(\sigma_1^2/2\right)}{a_1}\leq \liminf_{t\longrightarrow\infty}\langle x_2(t)\rangle \leq \limsup_{t\longrightarrow\infty}\langle x_1(t)\rangle \leq \frac{r_1-\left(\sigma_1^2/2\right)}{a_1}, \text{a.s},$$
$$0<\frac{\lambda_1-\left(\sigma_2^2/2\right)}{a_2}\leq \liminf_{t\longrightarrow\infty}\langle x_2(t)\rangle \leq \limsup_{t\longrightarrow\infty}\langle x_1(t)\rangle \leq \frac{r_1-\left(\sigma_2^2/2\right)}{a_2}, \text{a.s.} \tag{25}$$

Proof. From Theorem 1, it follows that, for any $t\in[0,\infty)$,

$$\begin{aligned}&0<\phi_i(t)\leq x_i(t)\leq \Phi_i(t),\\ &0<\psi(t)\leq y(t)\leq \Psi(t), \text{a.s.} \quad i=1,2.\end{aligned} \tag{26}$$

Here $\phi_1(t)$ and $\Phi_1(t)$ are the solutions of stochastic equations, respectively.

$$\begin{aligned}\mathrm{d}\phi_1(t)&=\phi_1(t)\left[\lambda_1-a_1\phi_1(t)\right]\mathrm{d}t+\sigma_1\phi_1(t)\mathrm{d}w_1(t),\\ \mathrm{d}\Phi_1(t)&=\Phi_1(t)\left[r_1-a_1\Phi_1(t)\right]\mathrm{d}t+\sigma_1\Phi_1(t)\mathrm{d}w_1(t),\end{aligned} \tag{27}$$

where $\phi_1(0)=\Phi_1(0)=x_{10}>0$. From Lemma 1, if $\lambda_1-(\sigma_1^2/2)>0$, then

$$\begin{aligned}&\lim_{t\longrightarrow\infty}\frac{\ln\phi_1(t)}{t}=0, \quad \lim_{t\longrightarrow\infty}\langle\phi_1(t)\rangle\\ &\lim_{t\longrightarrow\infty}\frac{\ln\Phi_1(t)}{t}=0, \quad \lim_{t\longrightarrow\infty}\langle\Phi_1(t)\rangle\end{aligned} \tag{28}$$

This, together with (12), yields

$$\lim_{t\longrightarrow\infty}\frac{\ln x_1(t)}{t}=0$$
$$0<\frac{\lambda_1-\left(\sigma_1^2/2\right)}{a_1}\leq \liminf_{t\longrightarrow\infty}\langle x_1(t)\rangle \leq \limsup_{t\longrightarrow\infty}\langle x_1(t)\rangle \leq \frac{r_1-\left(\sigma_1^2/2\right)}{a_1}, \text{a.s.} \tag{29}$$

Similarly, if $\lambda_2-(\sigma_2^2/2)>0$, then

$$\lim_{t\longrightarrow\infty}\frac{\ln x_2(t)}{t}=0$$
$$0<\frac{\lambda_2-\left(\sigma_2^2/2\right)}{a_2}\leq \liminf_{t\longrightarrow\infty}\langle x_2(t)\rangle \leq \limsup_{t\longrightarrow\infty}\langle x_2(t)\rangle \leq \frac{r_2-\left(\sigma_2^2/2\right)}{a_2}, \text{a.s.} \tag{30}$$

Now, we show $\lim_{t\longrightarrow\infty}(\ln y(t)/t)=0$ a.s. Note that $\Psi(t)$ is the solution of the stochastic equation

$$\begin{aligned}\mathrm{d}\Psi(t)&=\Psi(t)\left[e_1\alpha_1\nu+e_2\alpha_2(1-\nu)-b\Psi(t)\right]\mathrm{d}t\\ &\quad+\sigma_3\Psi(t)\mathrm{d}w_3(t),\end{aligned} \tag{31}$$

with $\Psi(0)=y_0>0$. Note that $\lambda_3-(\sigma_3^2/2)>0$. Thus, from Lemma 1, $\lim_{t\longrightarrow\infty}(\ln\Psi(t)/t)=0$ a.s. From $\lim_{t\longrightarrow\infty}(\ln\phi_1(t)/t)=0$ and $\lim_{t\longrightarrow\infty}(\ln\phi_2(t)/t)=0$ a.s., it follows that, for any $\varepsilon>0$, there exists $T_1>0$ such that

$$e^{-\varepsilon t}\leq\phi_i(t)\leq e^{\varepsilon t} \text{ for } t\geq T_1, \quad i=1,2. \tag{32}$$

Moreover, from the strong law of large numbers, it follows that $\lim_{t\longrightarrow\infty}(\sigma_3 w_3(t)/t)=0$ a.s. Thus, for the abovementioned $\varepsilon>0$, there exists $T_2>0$ such that

$$-\varepsilon t\leq\sigma_3 w_3(t)\leq\varepsilon t \text{ for } t\geq T_2. \tag{33}$$

Let $\kappa=\lambda_3-(\sigma_3^2/2)$. From the expression of $\psi(t)$, it follows that, for any $t>s\geq T=T_1\vee T_2$,

$$\begin{aligned}\frac{1}{\psi(t)}&=\frac{1}{y(T)}e^{\{-\kappa(t-T)-\sigma_3(w_3(t)-w_3(T))\}}+b\int_T^t e^{\{-\kappa(t-s)-\sigma_3(w_3(t)-w_3(s))\}}\mathrm{d}s\\ &\quad+\int_T^t\left[\frac{e_1\alpha_1\nu\beta_1}{\phi_1(s)}+\frac{e_2\alpha_2(1-\nu)\beta_2}{\phi_2(s)}\right]e^{\{-\kappa(t-s)-\sigma_3(w_3(t)-w_3(s))\}}\mathrm{d}s\\ &\leq\frac{1}{y(T)}e^{\{-\kappa(t-T)+\varepsilon(t+T)\}}+b\int_T^t e^{\{-\kappa(t-s)+\varepsilon(t+s)\}}\mathrm{d}s\\ &\quad+\left(e_1\alpha_1\nu\beta_1+e_2\alpha_2(1-\nu)\beta_2\right)\int_T^t e^{\varepsilon s}e^{\{-\kappa(t-s)+\varepsilon(t+s)\}}\mathrm{d}s.\end{aligned} \tag{34}$$

Hence, from $\kappa>0$, $\varepsilon>0$ and $t>T$, we have

$$\begin{aligned}\frac{e^{-3\varepsilon(t+T)}}{\psi(t)}&\leq\frac{1}{y(T)}e^{[-\kappa(t-T)-2\varepsilon(t+T)]}+\frac{b}{\kappa+\varepsilon}e^{-\varepsilon t}e^{-3\varepsilon T}\left[1-e^{-(\kappa+\varepsilon)(t-T)}\right]\\ &\quad+\frac{e_1\alpha_1\nu\beta_1+e_2\alpha_2(1-\nu)\beta_2}{\kappa+2\varepsilon}e^{-3\varepsilon T}\left[1-e^{-(\kappa+2\varepsilon)(t-T)}\right]\\ &\leq\frac{1}{y(T)}e^{[-\kappa(t-T)-2\varepsilon(t+T)]}+\frac{b}{\kappa}+\frac{e_1\alpha_1\nu\beta_1+e_2\alpha_2(1-\nu)\beta_2}{\kappa}\\ &\leq\frac{1}{y(T)}+\frac{b}{\kappa}+\frac{e_1\alpha_1\nu\beta_1+e_2\alpha_2(1-\nu)\beta_2}{\kappa}\doteq K.\end{aligned} \tag{35}$$

Thus, $(1/\psi(t))\leq Ke^{3\varepsilon(t+T)}$ a.s., which implies $-\ln\psi(t)\leq\ln K+3\varepsilon(t+T)$ a.s. Then, from the arbitrariness of ε, it follows that $\liminf_{t\longrightarrow\infty}(\ln\psi(t)/t)\geq0$ a.s. Consequently,

$$\begin{aligned}0&\leq\liminf_{t\longrightarrow\infty}\frac{\mathrm{In}\psi(t)}{t}\leq\liminf_{t\longrightarrow\infty}\frac{\mathrm{In}\, y(t)}{t}\leq\limsup_{t\longrightarrow\infty}\frac{\mathrm{In}\, y(t)}{t}\\ &\leq\lim_{t\longrightarrow\infty}\frac{\mathrm{In}\,\Psi(t)}{t}=0, \text{a.s.}\end{aligned} \tag{36}$$

This implies

$$\lim_{x\longrightarrow\infty}\frac{\ln y(t)}{t}=0, \text{a.s} \tag{37}$$

The proof is, therefore, complete. □

From proof of Theorem 3, we can get the following result with the proof being omitted.

Corollary 1. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). For $i = 1, 2$, if $\lambda_i - (\sigma_i^2/2) > 0$, then*

$$\lim_{t \longrightarrow \infty} \frac{\ln x_i(t)}{t} = 0,$$

$$0 < \frac{\lambda_i - \left(\sigma_i^2/2\right)}{a_i} \leq \liminf_{t\longrightarrow\infty} \langle x_i(t)\rangle \limsup_{t\longrightarrow\infty} \langle x_i(t)\rangle \leq \frac{r_i - \left(\sigma_i^2/2\right)}{a_i} \text{ a.s.} \tag{38}$$

3. Persistence in Mean and Extinction

In this section, we show that, under some conditions, model (5) is persistent in mean and extinct.

Lemma 2 (See [19]). Assume that $u \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$, $G \in C(\Omega \times [0, +\infty), \mathbb{R})$, and $\lim_{t\longrightarrow\infty} (G(t)/t) = 0$ a.s.

(i) If there are $\varrho \geq 0$, $\varrho_0 > 0$ and $T > 0$ satisfying

$$\ln u(t) \leq \varrho t - \varrho_0 \int_0^t u(s) ds + G(t), \text{a.s.}, \quad t \geq T, \tag{39}$$

then $\limsup_{t\longrightarrow\infty} \langle u(t)\rangle \leq (\varrho/\varrho_0)$ a.s. Furthermore, if $\varrho = 0$, then $\limsup_{t\longrightarrow\infty} \langle u(t)\rangle = 0$ a.s.

(ii) If there exist $\varrho > 0$, $\varrho_0 > 0$ and $T > 0$ satisfying

$$\ln u(t) \geq \varrho t - \varrho_0 \int_0^t u(s) ds + G(t), \text{a.s.}, \quad t \geq T, \tag{40}$$

then $\liminf_{t\longrightarrow\infty} \langle u(t)\rangle \geq (\varrho/\varrho_0)$ a.s.

Theorem 4. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). If $\lambda_i - (\sigma_i^2/2) > 0$, $(i = 1, 2, 3)$, then*

$$\liminf_{t\longrightarrow\infty} \langle x_i(t)\rangle \geq \frac{\lambda_i - \left(\sigma_i^2/2\right)}{a_i} > 0, \text{a.s.}, \quad i = 1, 2;$$

$$\liminf_{t\longrightarrow\infty} \langle y(t) + M_1 \frac{y(t)}{x_1(t)} + M_2 \frac{y(t)}{x_2(t)}\rangle \geq \frac{\lambda_3 - \left(\sigma_3^2/2\right)}{b} > 0, \text{a.s.} \tag{41}$$

Here, $M_1 = (e_1\alpha_1 v\beta_1/b)$ and $M_2 = (e_2\alpha_2(1-v)\beta_2/b)$. This means that model (5) is persistent in mean.

Proof. From Theorem 3, it follows that if $\lambda_i - (\sigma_i^2/2) > 0$, $(i = 1, 2)$, then

$$\begin{aligned} \lim\inf_{t\longrightarrow\infty} \langle x_1(t)\rangle &\geq \frac{\lambda_1 - \left(\sigma_1^2/2\right)}{a_1} > 0, \\ \lim\inf_{t\longrightarrow\infty} \langle x_2(t)\rangle &\geq \frac{\lambda_2 - \left(\sigma_2^2/2\right)}{a_2} > 0, \text{ a.s.} \end{aligned} \tag{42}$$

For the predator $y(t)$, using the Itô formula, we obtain

$$\begin{aligned} \ln y(t) = \int_0^t \Bigg[&-m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} - by(s) \\ &+ \frac{e_1\alpha_1 v x_1(s)}{x_1(s) + \beta_1 y(s)} + \frac{e_2\alpha_2(1-v)x_2(s)}{x_2(s) + \beta_2 y(s)} \Bigg] ds + \sigma_3 w_3(t) + \ln y_0, \end{aligned} \tag{43}$$

which implies

$$\ln y(t) \geq \left[\lambda_3 - \frac{\sigma_3^2}{2}\right] t - b \int_0^t \left[y(s) + M_1 \frac{y(s)}{x_1(s)} + M_2 \frac{y(s)}{x_2(s)} \right] ds + \sigma_3 w_3(t) + \ln y_0. \tag{44}$$

Hence,

$$\frac{1}{t} \int_0^t b \left[y(s) + M_1 \frac{y(s)}{x_1(s)} + M_2 \frac{y(s)}{x_2(s)} \right] ds \geq \lambda_3 - \frac{\sigma_3^2}{2} + \frac{\sigma_3 w_3(t)}{t} + \frac{\ln y_0}{t} - \frac{\ln y(t)}{t}. \tag{45}$$

Letting $t \longrightarrow \infty$ and by the strong law of numbers and Theorem 3, we have

$$\liminf_{t\longrightarrow\infty} \langle y(t) + M_1 \frac{y(t)}{x_1(t)} + M_2 \frac{y(t)}{x_2(t)}\rangle \geq \frac{\lambda_3 - \left(\sigma_3^2/2\right)}{b} > 0, \text{ a.s.} \tag{46}$$

The proof is complete. □

Theorem 5. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). If $r_1 - (\sigma_1^2/2) < 0$, $r_2 - (\sigma_2^2/2) < 0$, and $\lambda_3 - (\sigma_3^2/2) < 0$, then*

$$\begin{aligned} &\lim_{t\longrightarrow\infty} x_1(t) = 0, \\ &\lim_{t\longrightarrow\infty} x_2(t) = 0, \\ &\lim_{t\longrightarrow\infty} y(t) = 0, \text{ a.s.} \end{aligned} \tag{47}$$

That is, for any initial value $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, model (5) is extinct with probability one.

Proof. From the Itô formula, it follows that

$$\ln x_1(t) = \int_0^t \left[r_1 - a_1 x_1(s) - \frac{\alpha_1 v y(s)}{x_1(s) + \beta_1 y(s)} - \frac{\sigma_1^2}{2} \right] ds + \sigma_1 w_1(t) + \ln x_{10}$$
$$\le \left[r_1 - \frac{\sigma_1^2}{2} \right] t + \sigma_1 w_1(t) + \ln x_{10}. \tag{48}$$

This, together with $\lim_{t\longrightarrow\infty}[(\sigma_1 w_1(t)/t) + (\ln x_{10}/t)] = 0$, yields $\limsup_{t\longrightarrow\infty}(\ln x_i(t)/t) \le r_1 - (\sigma_1^2/2) < 0$ a.s. Thus,

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \text{a.s.} \tag{49}$$

Similarly, from $r_2 - (\sigma_2^2/2) < 0$, it follows that

$$\lim_{t\longrightarrow\infty} x_2(t) = 0, \text{a.s.} \tag{50}$$

Moreover, from (43), we have

$$\ln y(t) \le \int_0^t \left[e_1\alpha_1 v + e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} \right] ds + \sigma_3 w_3(t) + \ln y_0$$
$$= \left[\lambda_3 - \frac{\sigma_3^2}{2} \right] t + \sigma_3 w_3(t) + \ln y_0. \tag{51}$$

From $\lim_{t\longrightarrow\infty}[(\sigma_3 w_3(t)/t) + (\ln y_0/t)] = 0$, it follows that $\limsup_{t\longrightarrow\infty}(\ln y(t)/t) \le \lambda_3 - (\sigma_3^2/2) < 0$ a.s. This implies

$$\lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s.} \tag{52}$$

Therefore, model (5) is extinct exponentially. The proof is complete. □

Theorem 6.

(I) If the predator is absent, i.e., $y(t) = 0$ a.s. for all $t \ge 0$, then the quantities of prey $x_1(t)$ and prey $x_2(t)$ satisfy the following:

(i) If $r_1 - (\sigma_1^2/2) > 0$ and $r_2 - (\sigma_2^2/2) > 0$, then

$$\lim_{t\longrightarrow\infty} \langle x_1(t) \rangle = \frac{r_1 - (\sigma_1^2/2)}{a_1}, \quad \lim_{t\longrightarrow\infty} \langle x_2(t) \rangle = \frac{r_2 - (\sigma_2^2/2)}{a_2}, \text{a.s} \tag{53}$$

(ii) If $r_1 - (\sigma_1^2/2) > 0$ and $r_2 - (\sigma_2^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} \langle x_1(t) \rangle = \frac{r_1 - (\sigma_1^2/2)}{a_1}, \quad \lim_{t\longrightarrow\infty} x_2(t) = 0, \text{a.s} \tag{54}$$

(iii) If $r_1 - (\sigma_1^2/2) < 0$ and $r_2 - (\sigma_2^2/2) > 0$, then

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \quad \lim_{t\longrightarrow\infty} \langle x_2(t) \rangle = \frac{r_2 - (\sigma_2^2/2)}{a_2}, \text{a.s} \tag{55}$$

(iv) If $r_1 - (\sigma_1^2/2) < 0$ and $r_2 - (\sigma_2^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \quad \lim_{t\longrightarrow\infty} x_2(t) = 0, \text{a.s} \tag{56}$$

(II) If the prey in patch 2 is absent, i.e., $x_2(t) = 0$ a.s. for all $t \ge 0$, then the quantities of prey $x_1(t)$ and predator $y(t)$ satisfy the following:

(i) If $\lambda_1 - (\sigma_1^2/2) > 0$ and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) > 0$, then

$$\liminf_{t\longrightarrow\infty} \langle x_1(t) \rangle \ge \frac{\lambda_1 - (\sigma_1^2/2)}{a_1}, \text{a.s;}$$
$$\liminf_{t\longrightarrow\infty} \left\langle y(t) + M_1 \frac{y(t)}{x_1(t)} \right\rangle \ge \frac{e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2)}{b}, \text{a.s} \tag{57}$$

(ii) If $r_1 - (\sigma_1^2/2) < 0$ and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \quad \lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s} \tag{58}$$

(III) If the prey in patch 1 is absent, i.e., $x_1(t) = 0$ a.s. for all $t \ge 0$, then the quantities of prey $x_2(t)$ and predator $y(t)$ satisfy the following:

(i) If $\lambda_2 - (\sigma_2^2/2) > 0$ and $e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2) > 0$, then

$$\liminf_{t\longrightarrow\infty} \langle x_2(t) \rangle \ge \frac{\lambda_2 - (\sigma_2^2/2)}{a_1}, \text{a.s.;}$$
$$\liminf_{t\longrightarrow\infty} \left\langle y(t) + M_2 \frac{y(t)}{x_2(t)} \right\rangle \ge \frac{e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2)}{b}, \text{a.s} \tag{59}$$

(ii) If $r_2 - (\sigma_2^2/2) < 0$ and $e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} x_2(t) = 0, \quad \lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s} \tag{60}$$

(IV) If the prey is absent, i.e., $x_1(t) = x_2(t) = 0$ a.s. for all $t \ge 0$, then the predator dies with probability one

Proof. (I) In the absence of the predator, from the Itô formula, it follows that

$$\ln x_i(t) = \left[r_i - \frac{\sigma_i^2}{2}\right]t - a_i \int_0^t x_i(s)ds + \sigma_i w_i(t) + \ln x_{i0}, \text{a.s.}, \quad i = 1, 2. \tag{61}$$

Note that $\lim_{t\longrightarrow\infty}[(\sigma_i w_i(t)/t) + (\ln x_{i0}/t)] = 0$ a.s., $(i = 1, 2)$. Thus, from Lemma 2, it follows that if $r_i - (\sigma_i^2/2) > 0$, $(i = 1, 2)$, then

$$\lim_{t\longrightarrow\infty} \langle x_i(t)\rangle = \frac{r_i - (\sigma_i^2/2)}{a_i}, \text{a.s.}, \quad i = 1, 2. \tag{62}$$

Moreover, from Lemma 2, it follows that if $r_i - (\sigma_i^2/2) < 0$, $(i = 1, 2)$, then

$$\lim_{t\longrightarrow\infty} x_i(t) = 0, \text{a.s.}, \quad i = 1, 2. \tag{63}$$

Thus, (I) holds.

Next, we prove (II). From Theorem 3, it follows that if $\lambda_1 - (\sigma_1^2/2) > 0$, then

$$\lim_{t\longrightarrow\infty} \langle x_1(t)\rangle \geq \frac{\lambda_1 - (\sigma_1^2/2)}{a_1}, \text{a.s.} \tag{64}$$

Moreover, in the absence of the prey in patch 2, from the Itô formula, it follows that

$$\begin{aligned}\ln y(t) &= \int_0^t \left[-m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} - by(s) + \frac{e_1\alpha_1 v x_1(s)}{x_1(s) + \beta_1 y(s)}\right]ds + \sigma_3 w_3(t) + \ln y_0 \\ &\geq \left[e_1\alpha_1 v - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2}\right]t - b\int_0^t \left[y(s) + M_1\frac{y(s)}{x_1(s)}\right]ds + \sigma_3 w_3(t) + \ln y_0.\end{aligned} \tag{65}$$

Hence,

$$\frac{b}{t}\int_0^t \left[y(s) + M_1\frac{y(s)}{x_1(s)}\right]ds \geq e_1\alpha_1 v - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} + \frac{\sigma_3 w_3(t)}{t} + \frac{\ln y_0}{t} - \frac{\ln y(t)}{t}. \tag{66}$$

Letting $t \longrightarrow \infty$ and by the strong law of numbers and Theorem 2, we have

$$\begin{aligned} b\liminf_{t\longrightarrow\infty}\left\langle y(t) + M_1\frac{y(t)}{x_1(t)}\right\rangle &\geq e_1\alpha_1 v - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} - \limsup_{t\longrightarrow\infty}\frac{\ln y(t)}{t} \\ &\geq e_1\alpha_1 v - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} \text{ a.s.}\end{aligned} \tag{67}$$

Thus, we have

$$\liminf_{t\longrightarrow\infty}\left\langle y(t) + M_1\frac{y(t)}{x_1(t)}\right\rangle \geq \frac{e_1\alpha_1 v - m_1 v - m_2(1-v) - \sigma_3^2/2}{b}, \text{a.s.} \tag{68}$$

Furthermore, from the proof of (I), if $r_1 - (\sigma_1^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \text{a.s.} \tag{69}$$

Moreover, in the absence of the prey in patch 2, from the Itô formula, it follows that

$$\ln y(t) \leq \left[e_1\alpha_1 v - m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2}\right]t - b\int_0^t y(s)ds + \sigma_3 w_3(t) + \ln y_0. \tag{70}$$

Note that $\lim_{t\longrightarrow\infty}[(\sigma_3 w_3(t)/t) + (\ln y_0/t)] = 0$ a.s. Thus, from Lemma 2, if $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) < 0$, then

$$\lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s.} \tag{71}$$

Hence, (II) holds. The proof of (III) is similar to (II) and, hence, is omitted.

At last, we prove (IV). In the absence of the prey, from the Itô formula, it follows that

$$\begin{aligned}\ln y(t) &= \int_0^t \left[-m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2} - by(s)\right]ds + \sigma_3 w_3(t) + \ln y_0 \\ &= \left[-m_1 v - m_2(1-v) - \frac{\sigma_3^2}{2}\right]t - b\int_0^t y(s)ds + \sigma_3 w_3(t) + \ln y_0.\end{aligned} \tag{72}$$

Note that $\lim_{t\longrightarrow\infty}[(\sigma_3 w_3(t)/t) + (\ln y_0/t)] = 0$ a.s. Thus, from Lemma 2, it follows that

$$\lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s.} \tag{73}$$

The proof is, therefore, complete. □

4. Stochastic Permanence

In this section, we investigate the stochastic permanence of model (5).

4.1. Stochastically Ultimate Boundedness. In this subsection, we first use two different ways to prove the boundedness of model (5) and then show that model (5) is stochastically ultimately bounded by Chebyshev's inequality. The definition of stochastically ultimate boundedness of model (5) was introduced in the literature [20, 21] as follows.

Definition 1 (See [20, 21]). Model (5) is called stochastically ultimately bounded if, for any $\varepsilon \in (0,1)$, there exist positive constants $H_i = H_i(\varepsilon)$, $(i = 1,2,3)$ such that the solution $(x_1(t), x_2(t), y(t))$ of model (5) with any initial value $(x_{10}, x_{20}, y_0) \in \mathbb{R}^3_+$ has the property that

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \mathbb{P}\{x_i(t) > H_i\} < \varepsilon, \\ &\limsup_{t\longrightarrow\infty} \mathbb{P}\{y(t) > H_3\} < \varepsilon, \quad i = 1,2. \end{aligned} \tag{74}$$

Theorem 7. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}^3_+$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). Then, for any $p > 0$, the solution $(x_1(t), x_2(t), y(t))$ obeys*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[x_i^p(t)\right] \le \left[\frac{p}{p+1}\right]^{p+1} \frac{\left((1/p) + r_i + (p/2)\sigma_i^2\right)^{p+1}}{a_i^p} \doteq K_i(p), \quad i = 1,2, \limsup_{t\longrightarrow\infty} \mathbb{E}\left[y^p(t)\right] \le \left[\frac{p}{p+1}\right]^{p+1} \frac{\left((1/p) + e_1\alpha_1\nu + e_2\alpha_2(1-\nu) + (p/2)\sigma_3^2\right)^{p+1}}{b^p} \doteq K_3(p). \tag{75}$$

Proof. Applying the Itô formula to $e^t\Phi_i^p$ leads to

$$\begin{aligned} \mathbb{E}\left[e^t\Phi_1^p(t)\right] &= x_{10}^p + p\mathbb{E}\int_0^t e^s\Phi_1^p(s)\left[\frac{1}{p} + r_1 + \frac{p-1}{2}\sigma_1^2 - a_1\Phi_1(s)\right]ds \\ &\le x_{10}^p + p\mathbb{E}\int_0^t e^s\Phi_1^p(s)\left[\frac{1}{p} + r_1 + \frac{p}{2}\sigma_1^2 - a_1\Phi_1(s)\right]ds. \end{aligned} \tag{76}$$

Clearly, function $f(x) = x^p(1/p)((1/p) + r_1 + (p/2)\sigma_1^2 - a_1x)$ reaches its maximum value at $x = (p((1/p) + r_1 + (p/2)\sigma_1^2)/a_1(p+1)) > 0$ and $f_{\max} = (p/a_1)^p((1/p) + r_1 + (p/2)\sigma_1^2/p+1)^{p+1}$. Thus,

$$\mathbb{E}\left[e^t\Phi_1^p(t)\right] \le x_{10}^p + p\left(\frac{p}{a_1}\right)^p\left(\frac{(1/p) + r_1 + (p/2)\sigma_1^2}{p+1}\right)^{p+1}\left[e^t - 1\right], \tag{77}$$

which implies

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\Phi_1^p(t)\right] \le \left[\frac{p}{p+1}\right]^{p+1} \frac{\left((1/p) + r_1 + (p/2)\sigma_1^2\right)^{p+1}}{a_1^p}. \tag{78}$$

By a similar discussion as in $\Phi_1(t)$, we also have

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\Phi_2^p(t)\right] \le \left[\frac{p}{p+1}\right]^{p+1} \frac{\left((1/p) + r_2 + (p/2)\sigma_2^2\right)^{p+1}}{a_2^p}, \\ &\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\Psi^p(t)\right] \le \left[\frac{p}{p+1}\right]^{p+1} \frac{\left((1/p) + e_1\alpha_1\nu + e_2\alpha_2(1-\nu) + (p/2)\sigma_3^2\right)^{p+1}}{b^p}. \end{aligned} \tag{79}$$

From (12), it follows that $0 < x_i(t) \le \Phi_i(t)$ and $0 < y(t) \le \Psi(t)$ a.s. on $t \in [0, +\infty)$, $i = 1,2$. Then, for any $p > 0$, we have

$$\begin{aligned} &0 < \mathbb{E}\left[x_i^p(t)\right] \le \mathbb{E}\left[\Phi_i^p(t)\right], \\ &0 < \mathbb{E}\left[y^p(t)\right] \le \mathbb{E}\left[\Psi^p(t)\right], \quad i = 1,2. \end{aligned} \tag{80}$$

Thus, the desired results can be obtained immediately. The proof is complete. □

Theorem 8. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}^3_+$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). Then,*

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \mathbb{E}\left[x_i(t)\right] \le \frac{K_2}{e_i\lambda}, \\ &\limsup_{t\longrightarrow\infty} \mathbb{E}\left[y(t)\right] \le \frac{K_2}{\lambda}, \quad i = 1,2. \end{aligned} \tag{81}$$

where $K_2 = (e_1(r_1+\lambda)^2/4a_1) + (e_2(r_2+\lambda)^2/4a_2)$ and $\lambda = m_1\nu + m_2(1-\nu)$.

Proof. We denote $\lambda = m_1\nu + m_2(1-\nu)$ and $N(t) = e_1x_1(t) + e_2x_2(t) + y(t)$. From model (5) and the Itô formula, it follows that

$$\begin{aligned} \mathbb{E}[N(t)] = N(0) + \mathbb{E}\int_0^t &\left[-\lambda N(s) + e_1(r_1+\lambda)x_1(s)\right. \\ &- e_1a_1x_1^2(s) + e_2(r_2+\lambda)x_2(s) - e_2a_2x_2^2(s) \\ &\left.- by^2(s)\right]ds. \end{aligned} \tag{82}$$

This, together with $e_i(r_i+\lambda)x_i(t)-e_ia_ix_i^2(t)\le (e_i(r_i+\lambda)^2/4a_i)$, $(i=1,2)$, yields

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}[N(t)]}{\mathrm{d}t} &= \mathbb{E}\left[-\lambda N(t)+e_1(r_1+\lambda)x_1(t)-e_1a_1x_1^2(t)++e_2(r_2+\lambda)x_2(t)-e_2a_2x_2^2(t)-by^2(t)\right]\\ &\le -\lambda\mathbb{E}[N(t)]+\mathbb{E}\left[e_1(r_1+\lambda)x_1(t)-e_1a_1x_1^2(t)\right]+\mathbb{E}\left[e_2(r_2+\lambda)x_2(t)-e_2a_2x_2^2(t)\right]\\ &\le -\lambda\mathbb{E}[N(t)]+\frac{e_1(r_1+\lambda)^2}{4a_1}+\frac{e_2(r_2+\lambda)^2}{4a_2}\doteq K_2-\lambda\mathbb{E}[N(t)],\end{aligned} \tag{83}$$

where $K_2=(e_1(r_1+\lambda)^2/4a_1)+(e_2(r_2+\lambda)^2/4a_2)$. Thus, by the comparison theorem, we have

$$0\le \limsup_{t\longrightarrow\infty}\mathbb{E}[N(t)]\le\frac{K_2}{\lambda}. \tag{84}$$

From the solution of model (5) which is positive, it follows that

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\mathbb{E}[x_i(t)]\le\frac{K_2}{e_i\lambda},\\ &\limsup_{t\longrightarrow\infty}\mathbb{E}[y(t)]\le\frac{K_2}{\lambda},\quad i=1,2.\end{aligned} \tag{85}$$

The proof is complete. □

Theorem 9. *Model (5) is stochastically ultimately bounded.*

Proof. For any $(x_{10},x_{20},y_0)\in\mathbb{R}_+^3$, let $(x_1(t),x_2(t),y(t))$ be the solution of model (5) with initial value (x_{10},x_{20},y_0). From Theorem 7, it follows that

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\mathbb{E}[x_i(t)]\le K_i(1),\\ &\limsup_{t\longrightarrow\infty}\mathbb{E}[y(t)]\le K_3(1),\quad i=1,2.\end{aligned} \tag{86}$$

For any $\varepsilon\in(0,1)$, let $H_i=(K_i(1)/\varepsilon)+1$, $i=1,2,3$. From Chebyshev's inequality, it follows that

$$\begin{aligned}&\mathbb{P}\{x_i(t)>H_i\}\le\frac{\mathbb{E}[x_i(t)]}{H_i},\\ &\mathbb{P}\{y(t)>H_3\}\le\frac{\mathbb{E}[y(t)]}{H_3},\quad i=1,2.\end{aligned} \tag{87}$$

Hence,

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)>H_i\}\le\limsup_{t\longrightarrow\infty}\frac{\mathbb{E}[x_i(t)]}{H_i}<\varepsilon,\quad i=1,2,\\ &\limsup_{t\longrightarrow\infty}\mathbb{P}\{y(t)>H_3\}\le\limsup_{t\longrightarrow\infty}\frac{\mathbb{E}[y(t)]}{H_3}<\varepsilon.\end{aligned} \tag{88}$$

The proof is, therefore, complete.

Similarly, from Theorem 8, together with Chebyshev's inequality, one can say that model (5) is also stochastically ultimately bounded. □

4.2. Stochastic Permanence. In this section, we show that the model (5) is stochastically permanent. The definition of stochastic permanence of model (5) is introduced as follows.

Definition 2 (See [20, 21]). Model (5) is called stochastically permanent if, for any $\varepsilon\in(0,1)$, there exist positive constants $\delta_i=\delta_i(\varepsilon)$, $H_i=H_i(\varepsilon)$, and $\delta_i<H_i$, $i=1,2,3$, such that solution $(x_1(t),x_2(t),y(t))$ of model (5) with any initial value $(x_{10},x_{20},y_0)\in\mathbb{R}_+^3$ has the property that

$$\begin{aligned}&\liminf_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)\le H_i\}\ge 1-\varepsilon\\ &\limsup_{t\longrightarrow\infty}\mathbb{P}\{y(t)\le H_3\}\\ &\ge 1-\varepsilon,\ i=1,2.\\ &\liminf_{t\longrightarrow\infty}\mathbb{P}\{x_i(t)\ge\delta_i\}\\ &\ge 1-\varepsilon\ \text{and}\ \limsup_{t\longrightarrow\infty}\mathbb{P}\{y(t)\ge\delta_3\}\ge 1-\varepsilon,\quad i=1,2.\end{aligned} \tag{89}$$

For simplicity, we denote $\kappa_i = \lambda_i - \sigma_i^2$, $(i = 1, 2, 3)$. To prove that model (5) is stochastically permanent, we define

$$\begin{aligned} u_1 &= \frac{1}{x_1}, \\ u_2 &= \frac{1}{x_2}, \\ u_3 &= \frac{1}{y}. \end{aligned} \tag{90}$$

From the Itô formula, it follows that

$$\begin{cases} \mathrm{d}u_1(t) = \left[a_1 - r_1 u_1(t) + \sigma_1^2 u_1(t) + \dfrac{\alpha_1 \nu}{u_3(t) + \beta_1 u_1(t)} u_1^2(t)\right]\mathrm{d}t - \sigma_1 u_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}u_2(t) = \left[a_2 - r_2 u_2(t) + \sigma_2^2 u_2(t) + \dfrac{\alpha_2(1-\nu)}{u_3(t) + \beta_2 u_2(t)} u_2^2(t)\right]\mathrm{d}t - \sigma_2 u_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}u_3(t) = u_3(t)\left[\dfrac{b}{u_3(t)} + m_1\nu + m_2(1-\nu) + \sigma_3^2 - \dfrac{e_1\alpha_1\nu u_3(t)}{u_3(t) + \beta_1 u_1(t)} - \dfrac{e_2\alpha_2(1-\nu)u_3(t)}{u_3(t) + \beta_2 u_2(t)}\right]\mathrm{d}t \\ -\sigma_3 u_3(t)\mathrm{d}w_3(t), \end{cases} \tag{91}$$

with initial value $(u_1(0), u_2(0), u_3(0)) = ((1/x_{10}), (1/x_{20}), (1/y_0))$. Furthermore, we consider the following auxiliary system:

$$\begin{cases} \mathrm{d}\theta_1(t) = [a_1 - \kappa_1\theta_1(t)]\mathrm{d}t - \sigma_1\theta_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\theta_2(t) = [a_2 - \kappa_2\theta_2(t)]\mathrm{d}t - \sigma_2\theta_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}\theta_3(t) = [b - \kappa_3\theta_3(t) + e_1\alpha_1\nu\beta_1\theta_1(t) + e_2\alpha_2(1-\nu)\beta_2\theta_2(t)]\mathrm{d}t - \sigma_3\theta_3(t)\mathrm{d}w_3(t), \end{cases} \tag{92}$$

with initial value $(\theta_1(0), \theta_2(0), \theta_3(0)) = ((1/x_{10}), (1/x_{20}), (1/y_0))$.

Thanks to Lemma 4.2 in [17], system (92) has the exact solution. Moreover, from the stochastic comparison theorem, it follows that, for $t \in [0, \infty)$,

$$0 < u_i(t) \le \theta_i(t), \text{a.s.}, \quad i = 1, 2, 3. \tag{93}$$

Theorem 10. *For any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, let $(x_1(t), x_2(t), y(t))$ be the solution of model (5) with initial value (x_{10}, x_{20}, y_0). If $\kappa_i > 0$, $(i = 1, 2, 3)$, then*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\frac{1}{x_i(t)}\right] \le \frac{a_i}{\kappa_i} \doteq M_i, \quad i = 1, 2, \limsup_{t\longrightarrow\infty} \mathbb{E}\left[\frac{1}{y(t)}\right] \le \frac{b}{\kappa_3} + \frac{a_1 e_1\alpha_1\nu\beta_1}{\kappa_1\kappa_3} + \frac{a_2 e_2\alpha_2(1-\nu)\beta_2}{\kappa_2\kappa_3} \doteq M_3. \tag{94}$$

Proof. Integrating the both sides of the first equation of (92) and taking the expectation yields

$$\mathbb{E}[\theta_1(t)] = \frac{1}{x_{10}} + \mathbb{E}\int_0^t [a_1 - \kappa_1\theta_1(s)]\mathrm{d}s. \tag{95}$$

Then, we have the differential equation $(\mathrm{d}\mathbb{E}[\theta_1(t)]/\mathrm{d}t) = a_1 - \kappa_1\mathbb{E}[\theta_1(t)]$ with $\mathbb{E}[\theta_1(0)] = (1/x_{10})$. Thus, one can get

$$\mathbb{E}[\theta_1(t)] = \frac{1}{x_{10}} e^{-\kappa_1 t} + \frac{a_1}{\kappa_1}\left[1 - e^{-\kappa_1 t}\right]. \tag{96}$$

This, together with $\kappa_1 > 0$, it yields $\lim_{t\longrightarrow\infty}\mathbb{E}[\theta_1(t)] = (a_1/\kappa_1)$. From $(1/x_1(t)) = u_1(t)$ and (93), it follows that

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{x_1(t)}\right]=\limsup_{t\longrightarrow\infty}\mathbb{E}\left[u_1(t)\right]\leq\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\theta_1(t)\right]=\frac{a_1}{\kappa_1}\doteq M_1. \tag{97}$$

Similarly, one can get

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{x_2(t)}\right]=\limsup_{t\longrightarrow\infty}\mathbb{E}\left[u_2(t)\right]\leq\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\theta_2(t)\right]=\frac{a_2}{\kappa_2}\doteq M_2. \tag{98}$$

Integrating the both sides of the third equation of system (92) and taking the expectation yields

$$\begin{aligned}\mathbb{E}\left[\theta_3(t)\right]&=\frac{1}{y_0}+\mathbb{E}\int_0^t\left[b-\kappa_3\theta_3(s)+e_1\alpha_1\nu\beta_1\theta_1(s)\right.\\&\quad\left.+e_2\alpha_2(1-\nu)\beta_2\theta_2(s)\right]\mathrm{d}s.\end{aligned} \tag{99}$$

Thus, we can get $(\mathrm{d}\mathbb{E}[\theta_3(t)]/\mathrm{d}t)=b-\kappa_3\mathbb{E}[\theta_3(t)]+e_1\alpha_1\nu\beta_1\mathbb{E}[\theta_1(t)]+e_2\alpha_2(1-\nu)\beta_2\mathbb{E}[\theta_2(t)]$ with $\mathbb{E}[\theta_3(0)]=(1/y_0)$. Therefore, we have

$$\begin{aligned}\mathbb{E}\left[\theta_3(t)\right]&=\frac{1}{y_0}e^{-\kappa_3 t}+\frac{b}{\kappa_3}\left[1-e^{-\kappa_3 t}\right]\\&\quad+e_1\alpha_1\nu\beta_1\int_0^t e^{-\kappa_3(t-s)}\mathbb{E}\left[\theta_1(s)\right]\mathrm{d}s\\&\quad+e_2\alpha_2(1-\nu)\beta_2\int_0^t e^{-\kappa_3(t-s)}\mathbb{E}\left[\theta_2(s)\right]\mathrm{d}s.\end{aligned} \tag{100}$$

From L'Hospital's rule and (97), it follows that

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_0^t e^{-\kappa_3(t-s)}\mathbb{E}\left[\theta_1(s)\right]\mathrm{d}s&=\lim_{t\longrightarrow\infty}\frac{\int_0^t e^{\kappa_3 s}\mathbb{E}\left[\theta_1(s)\right]\mathrm{d}s}{e^{\kappa_3 t}}\\&=\lim_{t\longrightarrow\infty}\frac{\mathbb{E}\left[\theta_1(t)\right]}{\kappa_3}=\frac{a_1}{\kappa_1\kappa_3}.\end{aligned} \tag{101}$$

Similarly, we also have

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_0^t e^{-\kappa_3(t-s)}\mathbb{E}\left[\theta_2(s)\right]\mathrm{d}s&=\lim_{t\longrightarrow\infty}\frac{\int_0^t e^{\kappa_3 s}\mathbb{E}\left[\theta_2(s)\right]\mathrm{d}s}{e^{\kappa_3 t}}\\&=\lim_{t\longrightarrow\infty}\frac{\mathbb{E}\left[\theta_2(t)\right]}{\kappa_3}=\frac{a_1}{\kappa_2\kappa_3}.\end{aligned} \tag{102}$$

This, together with (100) and $\kappa_3>0$, yields

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\theta_3(t)\right]\leq\frac{b}{\kappa_3}+\frac{a_1e_1\alpha_1\nu\beta_1}{\kappa_1\kappa_3}+\frac{a_2e_2\alpha_2(1-\nu)\beta_2}{\kappa_2\kappa_3}\doteq M_3. \tag{103}$$

Thus, from $(1/y(t))=u_3(t)$ and (93), it follows that

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{y(t)}\right]=\limsup_{t\longrightarrow\infty}\mathbb{E}\left[u_3(t)\right]\leq\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\theta_3(t)\right]\leq M_3. \tag{104}$$

The proof is, therefore, complete. □

Theorem 11. *If $\kappa_i>0$, $(i=1,2,3)$, then model (5) is stochastically permanent.*

Proof. For any $(x_{10},x_{20},y_0)\in\mathbb{R}_+^3$, let $(x_1(t),x_2(t),y(t))$ be the solution of model (5) with initial value (x_{10},x_{20},y_0). From Theorem 10, it follows that

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{x_i(t)}\right]\leq M_i,\\&\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{y(t)}\right]\leq M_3,\quad 1=1,2.\end{aligned} \tag{105}$$

For any $\varepsilon\in(0,1)$, let $\delta_i=(\varepsilon/M_i)$, $(i=1,2,3)$; then,

$$\begin{aligned}&\mathbb{P}\left\{x_i(t)<\delta_i\right\}=\mathbb{P}\left\{\frac{1}{x_i(t)}>\frac{1}{\delta_i}\right\}\leq\frac{\mathbb{E}\left[(1/x_i(t))\right]}{(1/\delta_i)}=\delta_i\mathbb{E}\left[\frac{1}{x_i(t)}\right],\quad i=1,2,\\&\mathbb{P}\left\{y(t)<\delta_3\right\}=\mathbb{P}\left\{\frac{1}{y(t)}>\frac{1}{\delta_3}\right\}\leq\frac{\mathbb{E}\left[(1/y(t))\right]}{(1/\delta_3)}=\delta_3\mathbb{E}\left[\frac{1}{y(t)}\right].\end{aligned} \tag{106}$$

Thus,

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\mathbb{P}\left\{x_i(t)<\delta_i\right\}\leq\limsup_{t\longrightarrow\infty}\delta_i\mathbb{E}\left[\frac{1}{x_i(t)}\right]=\varepsilon,\quad i=1,2,\\&\limsup_{t\longrightarrow\infty}\mathbb{P}\left\{y(t)<\delta_3\right\}\leq\limsup_{t\longrightarrow\infty}\delta_3\mathbb{E}\left[\frac{1}{y(t)}\right]=\varepsilon.\end{aligned} \tag{107}$$

This implies

$$\begin{aligned}&\liminf_{t\longrightarrow\infty}\mathbb{P}\left\{x_i(t)\geq\delta_i\right\}\geq 1-\varepsilon,\\&\liminf_{t\longrightarrow\infty}\mathbb{P}\left\{y(t)\geq\delta_3\right\}\geq 1-\varepsilon,\quad i=1,2.\end{aligned} \tag{108}$$

Let $\varepsilon\in(0,1)$ be sufficiently small such that $\delta_i<H_i$, $(i=1,2,3)$. Then, from Theorem 9 and Definition 2, one can say that model (5) is stochastically permanent. The proof is, therefore, complete.

5. Stationary Distribution and Ergodicity

This section will show that there is an ergodic stationary distribution for the solution of model (5). Let $\mathscr{B}(\mathbb{R}_+^3)$ be the Borel σ-algebra on $\mathbb{R}_+^3$ and $(\mathbb{R}_+^3,\mathscr{B}(\mathbb{R}_+^3),\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered complete probability space satisfying usual hypotheses. Consider a Markov process $X(t)$ in the state space $(\mathbb{R}_+^3,\mathscr{B}(\mathbb{R}_+^3),\mathbb{P})$ that satisfies the following stochastic differential equation:

$$\mathrm{d}X(t)=b(X(t))\mathrm{d}t+g(X(t))\mathrm{d}W(t),\ X(0)=X_0. \tag{109}$$

Here, $W(t)$ is a standard 3-dimensional standard Brownian motion, b: $\mathbb{R}_+^3\longrightarrow\mathbb{R}^3$, and g: $\mathbb{R}_+^3\longrightarrow\mathbb{R}^{3\times 3}$ are all locally Lipschitz functions. The diffusion matrix of $X(t)$ is defined as $J(X)=g(X)g^{\mathbb{T}}(X)=(a_{ij}(X))$. Let $X=(x_1,x_2,y)$,

$$b(X)=\begin{pmatrix} x_1\left[r_1-a_1x_1-\dfrac{\alpha_1 \nu y}{x_1+\beta_1 y}\right] \\ x_2\left[r_2-a_2x_2-\dfrac{\alpha_2(1-\nu)y}{x_2+\beta_2 y}\right] \\ y\left[-m_1\nu-m_2(1-\nu)-by+\dfrac{e_1\alpha_1\nu y}{x_1+\beta_1 y}+\dfrac{e_2\alpha_2(1-\nu)y}{x_2+\beta_2 y}\right]\end{pmatrix}, \tag{110}$$

$g(X)=\operatorname{diag}(\sigma_1x_1,\sigma_2x_2,\sigma_3y)$, and $W=(w_1,w_2,w_3)^{\mathbb{T}}$. Then, model (5) reduces to the abstract form (110) with diffusion matrix $J(X)=\operatorname{diag}(\sigma_1^2x_1^2,\sigma_2^2x_2^2,\sigma_3^2y^2)$. The norm $|X|$ is given by $|X|=\sqrt{x_1^2+x_2^2+y^2}$. Moreover, we denote $P_t(X_0,A)$ the transition probability

$$P_t(X_0,A)=\mathbb{P}(X(t)\in A|X(0)=X_0), \cdot\forall t\in\mathbb{R}_+,\ \forall X_0\in\mathbb{R}_+^3,\ \forall X_0\in\mathscr{B}\left(\mathbb{R}_+^3\right). \tag{111}$$

Definition 3 (See [22]). Let $\mathbb{P}(t,X,\cdot)$ be the probability measure induced by $X(t)$ in (110) with $X(0)=X_0$. That is, $\mathbb{P}(t,X_0,A)=\mathbb{P}(X(t)\in A|Xt(0)n=qX_0)$, for any Borel set $A\in\mathscr{B}(\mathbb{R}_+^3)$. If there exists a probability measure $\mu(\cdot)$ such that $\lim_{t\longrightarrow\infty}\mathbb{P}(t,X_0,A)=\mu(A)$ for all $X_0\in\mathbb{R}_+^3$ and $A\in\mathscr{B}(\mathbb{R}_+^3)$, then equation (110) has a stationary distribution $\mu(\cdot)$.

Lemma 3 (See [23, 24]). The Markov process $X(t)$ in (110) has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded open domain $D\subset E_3$ (E_3 denotes 3-dimensional Euclidean space) with regular boundary Γ, and

(A1): there is a positive number M such that $\sum_{i,j=1}^3 a_{ij}(X)\xi_i\xi_j\ge M|\xi|^2$, $X\in D$, $\xi\in\mathbb{R}^3$

(A2): there exists a nonnegative C^2 function V such that LV is negative for any $X\in E_3/D$

Then,

$$\mathbb{P}\left\{\lim_{T\longrightarrow\infty}\frac{1}{T}\int_0^T f(X(t))\mathrm{d}t=\int_{E_d} f(x)\mu(\mathrm{d}x)\right\}=1, \tag{112}$$

for all $x\in E_3$, where The caption of Figure 5 is unclear. Please rephrase the caption for clarity and correctness. is a function integrable with respect to the measure μ.

Theorem 12. *If $\kappa_1-e_1\alpha_1\nu\beta_1>0$, $\kappa_2-e_2\alpha_2(1-\nu)\beta_2>0$, and $\kappa_3>0$, then for any initial value $(x_{10},x_{20},y_0)\in\mathbb{R}_+^3$, model (5) has a stationary distribution $\mu(\cdot)$ and the solutions have ergodic property.*

Proof. In what follows, for the simplification, we denote $x_1(t)$, $x_2(t)$, and $y(t)$ as x_1, x_2, and y, respectively. We define C^2 function V_1: $\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ by

$$V_1(X)=x_1+x_2+y, \tag{113}$$

for $X=(x_1,x_2,y)\in\mathbb{R}_+^3$. From the Itô formula, it follows that

$$\begin{aligned} LV_1(X)&=x_1\left[r_1-a_1x_1-\frac{\alpha_1\nu y}{x_1+\beta_1y}\right]+x_2\left[r_2-a_2x_2-\frac{\alpha_2(1-\nu)y}{x_2+\beta_2y}\right]\\ &\quad+y\left[-m_1\nu-m_2(1-\nu)-by+\frac{e_1\alpha_1\nu x_1}{x_1+\beta_1y}+\frac{e_2\alpha_2(1-\nu)x_2}{x_2+\beta_2y}\right]\\ &\le x_1(r_1-a_1x_1)+x_2(r_2-a_2x_2)+y(e_1\alpha_1\nu+e_2\alpha_2(1-\nu)-by),\\ &=-a_1x_1^2+r_1x_1-a_2x_2^2+r_2x_2-by^2+(e_1\alpha_1\nu+e_2\alpha_2(1-\nu))y. \end{aligned} \tag{114}$$

We define V_2: $\mathbb{R}_+^3\longrightarrow\mathbb{R}_+$ by

$$V_2(X)=x_1^{-1}+x_2^{-1}+y^{-1}, \tag{115}$$

for $X=(x_1,x_2,y)\in\mathbb{R}_+^3$. Applying the Itô formula, we have

$$\begin{aligned} LV_2(X)&=-x_1^{-1}\left[r_1-a_1x_1-\frac{\alpha_1\nu y}{x_1+\beta_1y}-\sigma_1^2\right]-x_2^{-1}\left[r_2-a_2x_2-\frac{\alpha_2(1-\nu)y}{x_2+\beta_2y}-\sigma_2^2\right]\\ &\quad-y^{-1}\left[-m_1\nu-m_2(1-\nu)-by+\frac{e_1\alpha_1\nu x_1}{x_1+\beta_1y}+\frac{e_2\alpha_2(1-\nu)x_2}{x_2+\beta_2y}-\sigma_3^2\right]\\ &\le -x_1^{-1}[\kappa_1-a_1x_1]-x_2^{-1}[\kappa_2-a_2x_2]-y^{-1}\left[\kappa_3-by-\frac{e_1\alpha_1\nu\beta_1y}{x_1+\beta_1y}-\frac{e_2\alpha_2(1-\nu)\beta_2y}{x_2+\beta_2y}\right]\\ &\le -x_1^{-1}[\kappa_1-a_1x_1]-x_2^{-1}[\kappa_2-a_2x_2]-y^{-1}\left[\kappa_3-by-\frac{e_1\alpha_1\nu\beta_1y}{x_1}-\frac{e_2\alpha_2(1-\nu)\beta_2y}{x_2}\right],\\ &=-[\kappa_1-e_1\alpha_1\nu\beta_1]x_1^{-1}+a_1-[\kappa_2--e_2\alpha_2(1-\nu)\beta_2]x_2^{-1}+a_2-\kappa_3y^{-1}+b. \end{aligned} \tag{116}$$

Let $V(X) = V_1(X) + V_2(X) = x_1 + x_2 + y + x_1^{-1} + x_2^{-1} + y^{-1}$. Then,

$$\begin{aligned} LV(X) &= LV_1(X) + LV_2(X) \\ &\leq -a_1x_1^2 + r_1x_1 - [\kappa_1 - e_1\alpha_1\nu\beta_1]x_1^{-1} + a_1 \\ &\quad - a_2x_2^2 + r_2x_2 - [\kappa_2 - e_2\alpha_2(1-\nu)\beta_2]x_2^{-1} + a_2 \\ &\quad - by^2 + (e_1\alpha_1\nu + e_2\alpha_2(1-\nu))y - \kappa_3 y^{-1} + b, \\ &\doteq f(x_1) + g(x_2) + h(y), \end{aligned} \tag{117}$$

where

$$\begin{aligned} f(x_1) &= -a_1x_1^2 + r_1x_1 - [\kappa_1 - e_1\alpha_1\nu\beta_1]x_1^{-1} + a_1, \\ g(x_2) &= -a_2x_2^2 + r_2x_2 - [\kappa_2 - e_2\alpha_2(1-\nu)\beta_2]x_2^{-1} + a_2, \\ h(y) &= -by^2 + (e_1\alpha_1\nu + e_2\alpha_2(1-\nu))y - \kappa_3 y^{-1} + b. \end{aligned} \tag{118}$$

Obviously, $f(x_1)$, $g(x_2)$, and $h(y)$ are all functions with an upper bound on $\mathbb{R}_+$. Thus, we denote

$$\begin{aligned} f^u &= \sup_{x_1\in\mathbb{R}_+}\{f(x_1)\}, \\ g^u &= \sup_{x_2\in\mathbb{R}_+}\{g(x_2)\}, \\ h^u &= \sup_{y\in\mathbb{R}_+}\{h(y)\}. \end{aligned} \tag{119}$$

Let ρ be a sufficiently small positive number. We define a bounded open set as follows:

$$D = \left\{(x_1, x_2, y) \in \mathbb{R}_+^3 | \rho < x_1 < \frac{1}{\rho}, \rho < x_2 < \frac{1}{\rho}, \rho < y < \frac{1}{\rho}\right\} \subset \mathbb{R}_+^3. \tag{120}$$

Now, we prove that $LV(X) \leq -1$ on $(\mathbb{R}_+^3/D)$. From $\kappa_1 - e_1\alpha_1\nu\beta_1 > 0$, it follows that

$$\begin{aligned} LV(X) &\leq f(x_1) + g(x_2) + h(y) \leq f(x_1) + g^u \\ &\quad + h^u \longrightarrow -\infty, \text{a.s.}, \quad x_1 \longrightarrow 0^+ \text{ or } x_1 \longrightarrow +\infty. \end{aligned} \tag{121}$$

Similarly, from $\kappa_2 - e_2\alpha_2(1-\nu)\beta_2 > 0$ and $\kappa_3 > 0$, we also have

$$\begin{aligned} LV(X) &\leq f(x_1) + g(x_2) + h(y) \leq f^u + g(x_2) \\ &\quad + h^u \longrightarrow -\infty, \text{a.s.}, \quad x_2 \longrightarrow 0^+ \text{ or } x_2 \longrightarrow +\infty. \\ LV(X) &\leq f(x_1) + g(x_2) + h(y) \leq f^u + g^u \\ &\quad + h(y) \longrightarrow -\infty, \text{a.s.}, \quad y \longrightarrow 0^+ \text{ or } y \longrightarrow +\infty. \end{aligned} \tag{122}$$

Consequently, for sufficiently small ρ, one can see

$$LV(X) \leq -1, \quad \text{for all } (x_1, x_2, y) \in \left(\frac{\mathbb{R}_+^3}{D}\right). \tag{123}$$

Hence, condition (A2) of Lemma 3 holds.

Let $\sigma^2 = \sigma_1^2 \wedge \sigma_2^2 \wedge \sigma_3^2$ and $M = \rho^2\sigma^2$. Then, for any $X = (x_1, x_2, y) \in D$ and $\xi = (\xi_1, \xi_2, \xi_2) \in \mathbb{R}^3$, we have

$$\begin{aligned} \sum_{i,j=1}^{3} a_{ij}(X)\xi_i\xi_j &= \sigma_1^2x_1^2\xi_1^2 + \sigma_2^2x_2^2\xi_2^2 + \sigma_3^2y^2\xi_3^2 \\ &\geq \rho^2\sigma^2\left(\xi_1^2 + \xi_2^2 + \xi_3^2\right) \doteq M|\xi|^2. \end{aligned} \tag{124}$$

Thus, (A1) in Lemma 3 is satisfied. From Lemma 3, we can say that model (5) has a stationary distribution $\mu(\cdot)$ and the solutions of model (5) have ergodic property. □

6. Application of Main Results

In this section, we first apply the main results to two stochastic two-species predator-prey models. Then, we present the application of the main results to stochastic two-patch predator-prey model (4).

6.1. Two-Species Predator-Prey Model. If the predator only stays in one patch, then one can obtain the following stochastic predator-prey model (obtained by taking $\nu = 0$ or $\nu = 1$ in model (5)).

$$\begin{cases} \mathrm{d}x(t) = x(t)\left[r - ax(t) - \dfrac{\alpha y(t)}{x(t) + \beta y(t)}\right]\mathrm{d}t + \sigma_1 x(t)\mathrm{d}w_1(t), \\ \mathrm{d}y(t) = y(t)\left[-m - by(t) + \dfrac{e\alpha x(t)}{x(t) + \beta y(t)}\right]\mathrm{d}t + \sigma_2 y(t)\mathrm{d}w_2(t), \end{cases} \tag{125}$$

where $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}_+^2$. Model (125) was discussed in [25]. Furthermore, Linh and Ton [26] considered the corresponding nonautonomous model of (125). By a similar discussion as in Theorem 1, for any $(x_0, y_0) \in \mathbb{R}_+^2$, model (21) has a unique global positive solution $(x(t), y(t))$. Moreover, for model (125), we have the following results. For simplicity, we denote $\delta_1 = r - (\alpha/\beta)$ and $\delta_2 = e\alpha - m$.

Corollary 2. *For any $(x_0, y_0) \in \mathbb{R}_+^2$, let $(x(t), y(t))$ be the solution of model (125) with initial value (x_0, y_0).*

(I) By a similar discussion as in Theorems 2–6, we can obtain the following results:

(i) The solution $(x(t), y(t))$ of model (126) obeys

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \frac{\ln x(t)}{t} \leq 0, \\ &\limsup_{t\longrightarrow\infty} \frac{\ln y(t)}{t} \leq 0, \text{a.s.} \end{aligned} \tag{126}$$

Moreover, if $\delta_1 - (\sigma_1^2/2) > 0$ and $\delta_2 - (\sigma_2^2/2) > 0$; then,

$$\begin{aligned} &\lim_{t\longrightarrow\infty} \frac{\ln x(t)}{t} = 0, \\ &\lim_{t\longrightarrow\infty} \frac{\ln y(t)}{t} = 0, \text{a.s.} \end{aligned} \tag{127}$$

(ii) If $\delta_1 - (\sigma_1^2/2) > 0$ and $\delta_2 - (\sigma_2^2/2) > 0$, then

$$\liminf_{t\longrightarrow\infty}\langle x(t)\rangle \geq \frac{\delta_1-\left(\sigma_1^2/2\right)}{a}>0,$$

$$\liminf_{t\longrightarrow\infty}\left\langle y(t)+M\frac{y(t)}{x(t)}\right\rangle \geq \frac{\delta_2-\left(\sigma_2^2/2\right)}{b}>0, \text{a.s.} \tag{128}$$

Here, $M=(e\alpha\beta/b)$. *This means that model (125) is persistent in mean.*

(iii) *If* $r-(\sigma_1^2/2)<0$ *and* $\delta_2-(\sigma_2^2/2)<0$, *then*

$$\lim_{t\longrightarrow\infty} x(t)=0,$$
$$\lim_{t\longrightarrow\infty} y(t)=0, \text{a.s.} \tag{129}$$

(iv) *If the predator is absent, i.e.,* $y(t)=0$ *a.s. for all* $t\geq 0$, *then the quantity of prey in model (125) satisfies the following:*

$$\lim_{t\longrightarrow\infty}\langle x(t)\rangle=\frac{r-\left(\sigma_1^2/2\right)}{a_1}, \text{a.s.}, \quad \text{if } r-\left(\sigma_1^2/2\right)>0;$$

$$\lim_{t\longrightarrow\infty} x(t)=0, \text{a.s.}, \quad \text{if } r-\left(\sigma_1^2/2\right)<0. \tag{130}$$

(v) *If the prey is absent, i.e.,* $x(t)=0$ *a.s. for all* $t\geq 0$, *then the predator in model (125) dies with probability one.*

(II) *By a similar discussion as in Theorems 7, 10 and 11, one can get the following results:*

(i) *The solution* $(x(t),y(t))$ *of model (125) obeys*

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[x^p(t)\right]\leq\left[\frac{p}{p+1}\right]^{p+1}\frac{\left((1/p)+r+(p/2)\sigma_1^2\right)^{p+1}}{a^p}$$

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[y^p(t)\right]$$

$$\leq\left[\frac{p}{p+1}\right]^{p+1}\frac{\left((1/p)+e\alpha+(p/2)\sigma_2^2\right)^{p+1}}{b^p}. \tag{131}$$

Moreover, from Chebyshev's inequality, model (125) is stochastically ultimately bounded.

(ii) *If* $\varsigma_i \doteq \delta_i-\sigma_i^2>0$ $(i=1,2)$, *then the solution* $(x(t),y(t))$ *of model (125) obeys*

$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{x(t)}\right]\leq\frac{a}{\varsigma_1},$$
$$\limsup_{t\longrightarrow\infty}\mathbb{E}\left[\frac{1}{y(t)}\right]\leq\frac{b}{\varsigma_2}+\frac{ae\alpha\beta}{\varsigma_1\varsigma_2}. \tag{132}$$

Furthermore, if $\varsigma_i>0$, $(i=1,2)$, *then model (125) is stochastically permanent.*

By a similar discussion as in Theorem 12, if $\varsigma_1-e\alpha\beta>0$ *and* $\varsigma_2>0$, *then any* $(x_0,y_0)\in\mathbb{R}_+^2$, *model (125) has a stationary distribution* $\mu(\cdot)$ *and the solutions have an ergodic property.*

Remark 1. In [25], the authors show that model (125) has a unique global positive solution by using stopping times and contradiction. In this paper, the stochastic comparison theorem is used to prove that the model has a unique global positive solution. Reference [25] only shows that if $\delta_1-(\sigma_1^2/2)>0$ and $\delta_2-(\sigma_2^2/2)>0$, then $\lim_{t\longrightarrow\infty}(\ln x(t)/t)=0$ and $\lim_{t\longrightarrow\infty}(\ln y(t)/t)=0$ a.s. However, we also show that the sample Lyapunov exponents of the solutions are nonpositive in the absence of conditions. In [25], the authors only show that the solutions are uniformly bounded in the p-th moment. However, we give the concrete upper bound for the p-th moment. It is clear that the results of (ii) and (iii) in Corollary 2 (I) are consistent with Theorems 7 and 8 in [25]. However, the result of (III) in Corollary 2 is not reflected in [25]. Thus, our work can be seen as the extension of [25].

Remark 2. For the deterministic version of model (125), from [26], we can see that $\lim_{t\longrightarrow\infty}y(t)=0$ holds under some special conditions, i.e., the predator dies out, but it never gets $\lim_{t\longrightarrow\infty}x(t)=0$ (if $\lim_{t\longrightarrow\infty}y(t)=0$, then $\liminf_{t\longrightarrow\infty}x(t)\geq(r/a)>0$). However, the result of (iii) in Corollary 2 (I) shows that great noise intensities σ_1^2 and σ_2^2 can make both the prey and predator in model (125) die out. This means that a relatively large stochastic perturbation can cause the extinction of the population.

However, Linh and Ton [26] only consider the asymptotic estimations of moments, the upper-growth rates, and exponential death rates of species in the corresponding nonautonomous stochastic model of model (125). Moreover, the results of (iv) and (v) in Corollary 2 (I) are consistent with Theorems 4.3 and 4.4 in [26]. Thus, our paper can be regarded as the extension and supplement of [26].

Furthermore, if we do not consider the intraspecific competition of the predator, i.e., $b=0$ in model (125), then one can obtain the following stochastic model:

$$\begin{cases} \mathrm{d}x(t)=x(t)\left[r-ax(t)-\dfrac{\alpha y(t)}{x(t)+\beta y(t)}\right]\mathrm{d}t+\sigma_1 x(t)\mathrm{d}w_1(t), \\ \mathrm{d}y(t)=y(t)\left[-m+\dfrac{e\alpha x(t)}{x(t)+\beta y(t)}\right]\mathrm{d}t+\sigma_2 y(t)\mathrm{d}w_2(t), \end{cases} \tag{133}$$

with initial value $(x_0,y_0)\in\mathbb{R}_+^2$. This is a stochastic predator-prey model discussed in [27]. Wu et al. [10] considered the corresponding nonautonomous model of stochastic model (133). By a similar discussion as in Theorem 1, for any $(x_0,y_0)\in\mathbb{R}_+^2$, model (133) has a unique global positive solution $(x(t),y(t))$.

Corollary 3. *For any* $(x_0,y_0)\in\mathbb{R}_+^2$, *let* $(x(t),y(t))$ *be the solution of model (133) with initial value* (x_0,y_0).

(I) *By a similar discussion as in Theorems 5 and 6, one can get the following results:*

(i) *If* $r - (\sigma_1^2/2) > 0$ *and* $\delta_2 - (\sigma_2^2/2) < 0$, *then*

$$\lim_{t\longrightarrow\infty} x(t) = 0, \quad \lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s.} \tag{134}$$

(ii) *If the predator is absent, i.e.,* $y(t) = 0$ *a.s. for all* $t \geq 0$, *then the quantity of prey in model (133) satisfies the following:*

$$\lim_{t\longrightarrow\infty} \langle x(t)\rangle = \frac{r - (\sigma_1^2/2)}{a_1}, \text{a.s.}, \quad \text{if } r - (\sigma_1^2/2) > 0;$$
$$\lim_{t\longrightarrow\infty} x(t) = 0, \text{a.s.}, \quad \text{if } r - (\sigma_1^2/2) < 0 \tag{135}$$

(iii) *If the prey is absent, i.e.,* $x(t) = 0$ *a.s. for all* $t \geq 0$, *then the predator in model (133) dies with probability one*

(II) *By a similar discussion as in Theorems 8 and 10, one can get the following results:*

(i) *The solution* $(x(t), y(t))$ *of model (133) obeys*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}[x(t)] \leq \frac{K_3}{em}, \quad \limsup_{t\longrightarrow\infty} \mathbb{E}[y(t)] \leq \frac{K_3}{m}. \tag{136}$$

where $K_3 = (e(r+m)^2/4a)$. *Then, from Chebyshev's inequality, model (133) is stochastically ultimately bounded.*

(ii) *If* $\varsigma_i \doteq \delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$, *then the solution* $(x(t), y(t))$ *of model (125) obeys*

Furthermore, by a similar discussion as in Theorem 11, if $\varsigma_i > 0$, $(i = 1, 2)$, *then model (125) is stochastically permanent.*

Remark 3. If $\varsigma_i \doteq \delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$, then by Theorem 3.3 in [27], model (133) is persistent in mean. However, from (II) in Corollary 3, model (133) is stochastically permanent. This means that Theorem 11 generalizes and improves Theorem 3.3 in [27].

Remark 4. From Theorem 4.11 in [14], it ca be seen that if $\delta_i - (3/2)\sigma_i^2 > 0$, $(i = 1, 2)$, then model (133) is stochastically permanent. However, the results in Corollary 3 show that if $\delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$, then model (133) is stochastically permanent. Obviously, if $\delta_i - (3/2)\sigma_i^2 > 0$, $(i = 1, 2)$ holds, then $\delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$ holds. On the contrary, it is not set up. Thus, we can say that Corollary 3 generalizes and improves Theorem 4.11 in [14].

6.2. Two-Patch Predator-Prey Model (4). By a similar discussion as in Theorem 1, for any $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, model (4) has a unique global positive solution $(x_1(t), x_2(t), y(t))$. Moreover, for model (4), we have the following results.

Corollary 4.

(I) *By a similar discussion as in Theorems 5 and 6, we can obtain the following results:*

(i) *If* $r_1 - (\sigma_1^2/2) < 0$, $r_2 - (\sigma_2^2/2) < 0$ *and* $\lambda_3 - (\sigma_2^3/2) < 0$, *then*

$$\lim_{t\longrightarrow\infty} x_1(t) = 0, \quad \lim_{t\longrightarrow\infty} x_2(t) = 0, \quad \lim_{t\longrightarrow\infty} y(t) = 0, \text{a.s.} \tag{138}$$

(ii) *If the predator is absent, i.e.,* $y(t) = 0$ *a.s. for all* $t \geq 0$, *then the prey* x_i $(i = 1, 2)$ *in model (4) satisfies the following:*

$$\lim_{t\longrightarrow\infty} \langle x_i(t)\rangle = \frac{r_i - (\sigma_i^2/2)}{a_i}, \text{a.s.}, \quad \text{if } r_i - (\sigma_i^2/2) > 0;$$
$$\lim_{t\longrightarrow\infty} x_i(t) = 0, \text{a.s.}, \quad \text{if } r_i - (\sigma_i^2/2) < 0 \tag{139}$$

(iii) *If the prey is absent, i.e.,* $x_1(t) = x_2(t) = 0$ *a.s. for all* $t \geq 0$, *then the predator in model (4) dies with probability one*

(II) *By a similar discussion as in Theorems 8 and 10, one can get the following results:*

(i) *The solution* $(x_1(t), x_2(t), y(t))$ *of model (4) obeys*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}[x_i(t)] \quad \limsup_{t\longrightarrow\infty} \mathbb{E}[y(t)] \leq \frac{K_2}{\lambda}, \quad i = 1, 2, \tag{140}$$

where $K_2 = (e_1(r_1+\lambda)^2/4a_1) + (e_2(r_2+\lambda)^2/4a_2)$. *Then, from Chebyshev's inequality, model (4) is stochastically ultimately bounded.*

(ii) *If* $\kappa_i > 0$, $(i = 1, 2, 3)$, *then the solution* $(x_1(t), x_2(t), y(t))$ *of model (4) obeys*

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\frac{1}{x_i(t)}\right] \leq \frac{a_i}{\kappa_i}, \quad i = 1, 2,$$
$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[\frac{1}{y(t)}\right] \leq \frac{a_1 e_1 \alpha_1 \nu \beta_1}{\kappa_1 \kappa_3} + \frac{a_2 e_2 \alpha_2 (1-\nu)\beta_2}{\kappa_2 \kappa_3}. \tag{141}$$

Moreover, if $\kappa_i > 0$, $(i = 1, 2, 3)$, then model (4) is stochastically permanent.

7. Numerical Simulations

In this section, we use the Milstein method (see [28]) to substantiate the main results. The parameters are given in the following table.

Example 1. denote $k_1(v) = r_1 - (\alpha_1 v/\beta_1) - (\sigma_1^2/2)$, $k_2(v) = r_2 - \alpha_2((1-v)/\beta_2) - (\sigma_2^2/2)$, and $k_3(v) = (e_1\alpha_1 - m_1)v + (e_2\alpha_2 - m_2)(1-v) - (\sigma_3^2/2)$. By Theorem 4, if $k_i(v) > 0$, $(i = 1, 2, 3)$, then model (5) is persistent in mean. If we take the parameter values as in Table 1, $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$, then one can get Figure 1. It can be seen from Figure 1 that when $0.042 < v < 0.44875$, model (5) will be persistent in mean.

Example 2. We denote $k_1(v, \sigma_1^2) = r_1 - (\alpha_1 v/\beta_1) - (\sigma_1^2/2)$, $k_2(v, \sigma_2^2) = r_2 - \alpha_2((1-v)/\beta_2) - (\sigma_2^2/2)$ and $k_3(v, \sigma_3^2) = (e_1\alpha_1 - m_1)v + (e_2\alpha_2 - m_2)(1-v) - (\sigma_3^2/2)$. If we take the parameter values as in Table 1, then we can obtain figures about $k_i(v, \sigma_i^2)$ (see Figure 2).

To illustrate the results, we take the parameter values as in Table 1 and $v = 0.4$. Here we give numerical simulations of model (5) with $(x_{10}, x_{20}, y_0) = (1200, 1000, 500)$ and different noise intensities. In Figure 3, we choose $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0$ and get the solution of deterministic model (3).

Example 3. We assume that $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.01$. By a simple computation, $\lambda_1 - (\sigma_1^2/2) = 0.03 > 0$, $\lambda_2 - (\sigma_2^2/2) = 0.205 > 0$, and $\lambda_3 - (\sigma_3^2/2) = 0.0664 > 0$. Thus, the conditions of Theorem 4 hold. In view of Theorem 4, all the populations in model (5) will be persistent in mean (see Figure 4).

Example 4. We assume that $\sigma_1^2 = 1$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.2$. Thus, $r_1 - (\sigma_1^2/2) = -0.05 < 0$, $r_2 - (\sigma_2^2/2) = -0.02 < 0$, and $\lambda_3 - (\sigma_3^2/2) = -0.0286 < 0$. Then, from Theorem 5, it follows that $\lim_{t\longrightarrow\infty} x_1(t) = 0$, $\lim_{t\longrightarrow\infty} x_2(t) = 0$, and $\lim_{t\longrightarrow\infty} y(t) = 0$ a.s. This means that all the population in model (5) will go to extinction (see Figure 5).

Example 5. We assume that $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$. By a simple computation, $\kappa_1 = 0.0475 > 0$, $\kappa_2 = 0.2225 > 0$, and $\kappa_3 = 0.071 > 0$. Thus, from Theorem 11, it follows that model (5) is stochastically permanent (see Figure 6).

Example 6. We assume that $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$. By a simple computation, $\kappa_1 - e_1\alpha_1 v\beta_1 = 0.0317 > 0$, $\kappa_2 - e_2\alpha_2(1-v)\beta_2 = 0.1721 > 0$, and $\kappa_3 = 0.071 > 0$. Thus, the conditions of Theorem 12 hold. According to Theorem 12, model (5) has a stationary distribution (see Figure 7–11).

Example 7. The predator is absent, i.e., $y(t) = 0$ a.s. for all $t \geq 0$.

(i) We assume that $\sigma_1^2 = 0.04$ and $\sigma_2^2 = 0.04$. By a simple computation, $r_1 - (\sigma_1^2/2) = 0.43 > 0$ and $r_2 - (\sigma_2^2/2) = 0.58 > 0$. Thus, from Theorem 6, $\lim_{t\longrightarrow\infty}\langle x_1(t)\rangle = 2150$ and $\lim_{t\longrightarrow\infty}\langle x_2(t)\rangle = 1450$ a.s. This means that, in the absence of the predator, the prey x_i, $(i = 1, 2)$ will be persistent in mean in the absence of the predator y (see Figure 8).

(ii) We assume that $\sigma_1^2 = 0.04$ and $\sigma_2^2 = 1.24$. Thus, $r_1 - (\sigma_1^2/2) = 0.43 > 0$ and $r_2 - (\sigma_2^2/2) = -0.02 < 0$. From Theorem 6, it follows that $\lim_{t\longrightarrow\infty}\langle x_1(t)\rangle = 2150$ and $\lim_{t\longrightarrow\infty} x_2(t) = 0$ a.s. This means that, in the absence of the predator y, the prey x_1 will be persistent in mean, while the prey x_2 will go to extinction (see Figure 9).

(iii) We assume that $\sigma_1^2 = 1$ and $\sigma_2^2 = 0.04$. By a simple computation, $r_1 - (\sigma_1^2/2) = -0.05 < 0$ and $r_2 - (\sigma_2^2/2) = 0.58 > 0$. Thus, from Theorem 6, $\lim_{t\longrightarrow\infty} x_1(t) = 0$ and $\lim_{t\longrightarrow\infty}\langle x_2(t)\rangle = 1450$ a.s. This means that, in the absence of the predator y, the prey x_2 will be persistent in mean, while the prey x_1 will go to extinction (see Figure 10).

(iv) We assume that $\sigma_1^2 = 1$ and $\sigma_2^2 = 1.24$. Thus, $r_1 - (\sigma_1^2/2) = -0.05 < 0$ and $r_2 - (\sigma_2^2/2) = -0.02 < 0$. According to Theorem 6, we have $\lim_{t\longrightarrow\infty} x_1(t) = 0$ and $\lim_{t\longrightarrow\infty} x_2(t) = 0$ a.s. Thus, the prey x_1 and x_2 will go to extinction in the absence of the predator y (see Figure 11).

Example 8. The prey in patch 2 is absent, i.e., $x_2(t) = 0$ a.s. for all $t \geq 0$ (Figures 12–15).

(i) We assume that $\sigma_1^2 = 0.04$ and $\sigma_3^2 = 0.002$. By a simple computation, $\lambda_1 - (\sigma_1^2/2) = 0.03 > 0$ and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) = 0.0074 > 0$. Thus, from Theorem 6, it follows that

$$\liminf_{t\longrightarrow\infty}\langle x_1(t)\rangle \geq 150,$$
$$\liminf_{t\longrightarrow\infty}\left\langle y(t) + \frac{e_1\alpha_1 v\beta_1}{b}\frac{y(t)}{x_1(t)}\right\rangle \geq 24.67, \text{a.s.} \tag{142}$$

This means that the prey x_1 and the predator y will be persistent in mean in the absence of the prey x_2 (see Figure 12).

(ii) We assume that $\sigma_1^2 = 1$ and $\sigma_3^2 = 0.04$. By a simple computation, $r_1 - (\sigma_1^2/2) = -0.05 < 0$ and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) = -0.0116 < 0$. Thus, from Theorem 6, $\lim_{t\longrightarrow\infty} x_1(t) = 0$ and $\lim_{t\longrightarrow\infty} y(t) = 0$ a.s. This means that, in the absence of the prey x_2, the prey x_1 and predator population y will go to extinction (see Figure 13).

Example 9. The prey in patch 1 is absent, i.e., $x_1(t) = 0$ a.s. for all $t \geq 0$.

(i) We assume that $\sigma_2^2 = 0.04$ and $\sigma_3^2 = 0.02$. By a simple computation, $\lambda_2 - (\sigma_2^2/2) = 0.205 > 0$ and

Table 1: Physical interpretation of the parameters.

Parameters	Description	Values	
r_i	Intrinsic growth rate of the prey in patch i	$r_1 = 0.45$,	$r_2 = 0.6$
e_i	Expected biomass of the prey converted to predators in patch i	$e_1 = 0.11$,	$e_2 = 0.21$
m_i	Per capita mortality rate of predators in patch i	$m_1 = 0.03$,	$m_2 = 0.01$
a_i	Intraspecific competition coefficient of the prey in patch i	$a_1 = 0.0002$,	$a_2 = 0.0004$
b	Intraspecific competition coefficient of the predator	$b = 0.0003$	
α_i	Encounter rate with the prey in patch i	$\alpha_1 = 0.6$,	$\alpha_2 = 0.5$
β_i	Half saturation constant for the prey in patch i	$\beta_1 = 0.6$,	$\beta_2 = 0.8$

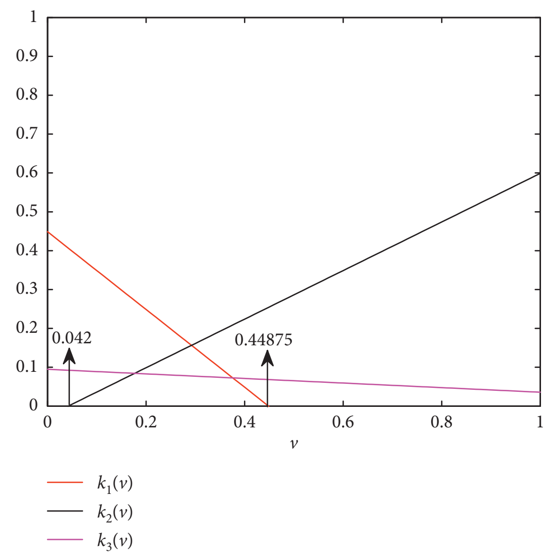


Figure 1: The trajectories of $k_1(v)$, $k_2(v)$, and $k_3(v)$ with $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$ (color figure online).

$e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2) = 0.035 > 0$. From Theorem 6, it follows that

$$\liminf_{t\longrightarrow\infty}\langle x_2(t)\rangle \geq 512.5,$$

$$\liminf_{t\longrightarrow\infty}\langle y(t) + \frac{e_2\alpha_2(1-v)\beta_2}{b}\frac{y(t)}{x_1(t)}\rangle \geq 116.67, \text{a.s.} \tag{143}$$

This means that, in the absence of the prey x_1, the prey x_2 and the predator y will be persistent in mean (see Figure 14).

(ii) We assume that $\sigma_2^2 = 1.24$ and $\sigma_3^2 = 0.2$. Thus, $r_2 - (\sigma_2^2/2) = -0.02 < 0$ and $e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2) = -0.055 < 0$. From Theorem 6, $\lim_{t\longrightarrow\infty} x_2(t) = 0$ and $\lim_{t\longrightarrow\infty} y(t) = 0$ a.s. This means that the prey x_2 and the predator $y(t)$ will go to extinction in the absence of the prey x_1 (see Figure 15).

Example 10. The prey is absent, i.e., $x_1(t) = x_2(t) = 0$ a.s. for all $t \geq 0$. From Theorem 6, it follows that the predator $y(t)$ will go to extinction (see Figure 16–22).

Example 11.

(i) We assume that $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.2$. Thus, $r_1 - (\sigma_1^2/2) = 0.43 > 0$, $r_2 - (\sigma_2^2/2) = 0.58 > 0$, and $\lambda_3 - (\sigma_3^2/2) = -0.0286 < 0$. From Theorem 5, we have $\lim_{t\longrightarrow\infty} y(t) = 0$ a.s. Moreover, from Figure 17, we can see that $\lim_{t\longrightarrow\infty}\langle x_1(t)\rangle = 2150$ and $\lim_{t\longrightarrow\infty}\langle x_2(t)\rangle = 1450$ a.s.

(ii) We assume that $\sigma_1^2 = 0.04$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.03$. By a simple computation, $r_1 - (\sigma_1^2/2) = 0.43 > 0$, $r_2 - (\sigma_2^2/2) = -0.02 < 0$, and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) = -0.0066 < 0$. From Theorem 5, it follows that $\lim_{t\longrightarrow\infty} x_2(t) = 0$ a.s. Moreover, from Figure 18, we can see that $\lim_{t\longrightarrow\infty}\langle x_1(t)\rangle = 2150$ and $\lim_{t\longrightarrow\infty} y(t) = 0$ a.s.

(iii) We assume that $\sigma_1^2 = 0.04$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.002$. Thus, $\lambda_1 - (\sigma_1^2/2) = 0.03 > 0$, $r_2 - (\sigma_2^2/2) = -0.02 < 0$, and $e_1\alpha_1 v - m_1 v - m_2(1-v) - (\sigma_3^2/2) = 0.0074 > 0$. From Theorem 5, it follows that $\lim_{t\longrightarrow\infty} x_2(t) = 0$ a.s. Moreover, from Figure 19, we can see that prey x_1 and predator y will be persistent in mean.

(iv) We assume that $\sigma_1^2 = 1$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.1$. By a simple computation, $r_1 - (\sigma_1^2/2) = -0.05 < 0$, $r_2 - (\sigma_2^2/2) = 0.58 > 0$, and $e_2\alpha_2(1-v) - m_1 v - m_2(1-v) - (\sigma_3^2/2) = -0.005 < 0$. Thus, from

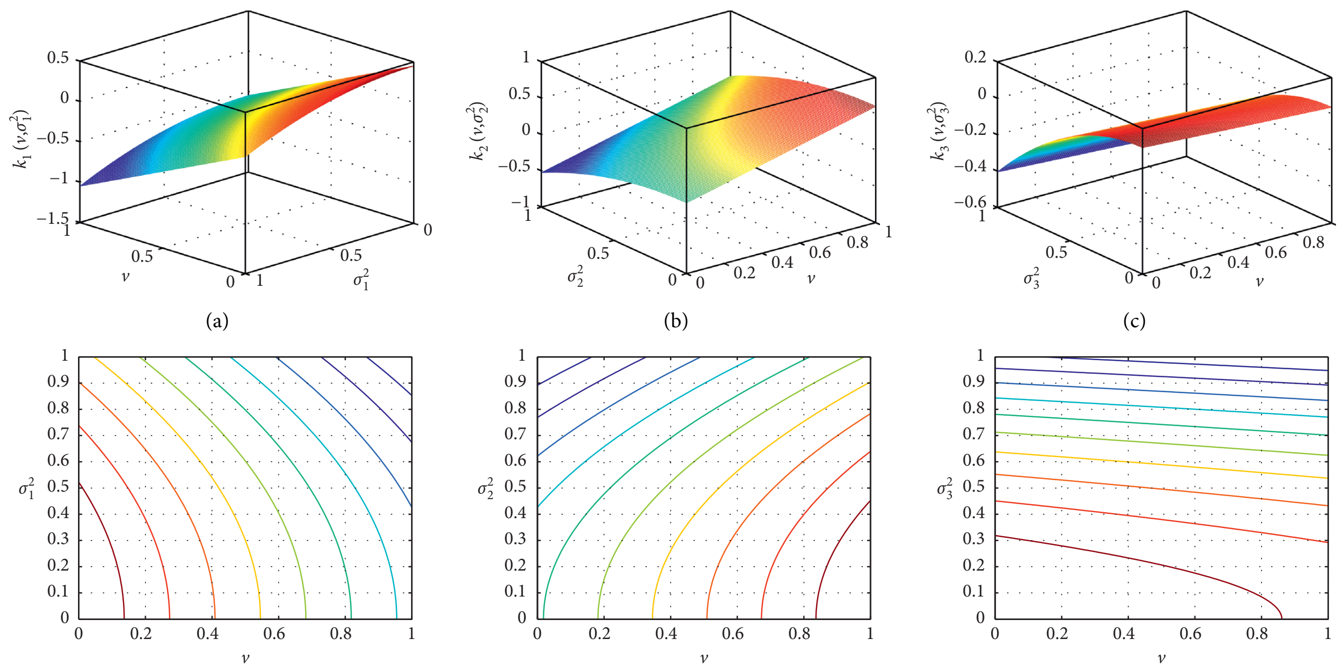


FIGURE 2: $k_i(\nu, \sigma_i^2)$ and its phase plan $(i = 1, 2, 3)$ (color figure online): (a) $k_1(\nu, \sigma_1^2)$, (b) $k_2(\nu, \sigma_2^2)$, (c) $k_3(\nu, \sigma_3^2)$, (d) phase plan of $k_1(\nu, \sigma_1^2)$, (e) phase plan of $k_2(\nu, \sigma_2^2)$, and (f) phase plan of $k_3(\nu, \sigma_3^2)$.

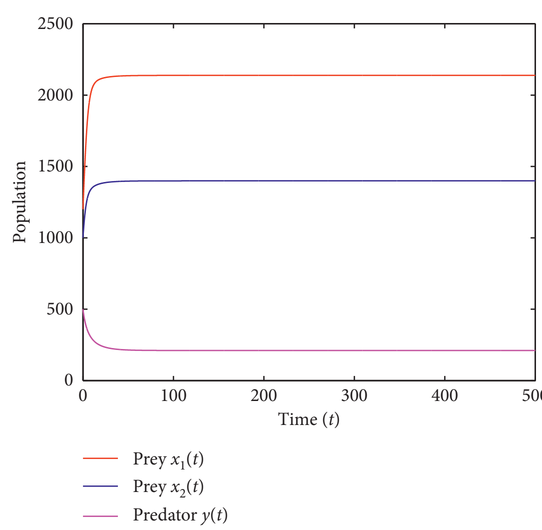


FIGURE 3: The trajectories of model (5) with $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0$ (color figure online).

Theorem 5, it follows that $\lim_{t\to\infty} x_1(t) = 0$ a.s. Moreover, from Figure 20, we can see that $\lim_{t\to\infty}\langle x_2(t)\rangle = 1450$ and $\lim_{t\to\infty} y(t) = 0$ a.s.

(v) We assume that $\sigma_1^2 = 1$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.02$. Thus, $r_1 - (\sigma_1^2/2) = -0.05 < 0$, $\lambda_2 - (\sigma_2^2/2) = 0.205 > 0$, and $e_2\alpha_2(1-\nu) - m_1\nu - m_2(1-\nu) - (\sigma_3^2/2) = 0.035 > 0$. From Theorem 5, it follows that $\lim_{t\to\infty} x_1(t) = 0$ a.s. Moreover, from Figure 21, we can see that prey x_2 and predator y will be persistent in mean.

(vi) We assume that $\sigma_1^2 = 1$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.002$. Thus, $r_1 - (\sigma_1^2/2) = -0.05 < 0$, $r_2 - (\sigma_2^2/2) = -0.02 < 0$, and $\lambda_3 - (\sigma_3^2/2) = 0.0704 > 0$. Then, from Theorem 5, it follows that $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_2(t) = 0$ a.s. Moreover, from Figure 22, we can see that predator y will go to extinction.

Example 12. We assume that $r_1 = 0.9$, $e_2 = 0.1$, $m_1 = 0.03$, $m_2 = 0.03$, $\alpha_2 = 0.2$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.02$, and $\sigma_3^2 = 0.002$, and the values of other parameters are shown in Table 1. Moreover, if we take $\nu = 0.8$, then $\kappa_1 = \lambda_1 - \sigma_1^2 = 0.08 > 0$, $\kappa_2 = \lambda_2 - \sigma_2^2 = 0.53 > 0$, and $\kappa_3 = \lambda_3 - \sigma_3^2 = 0.0248 > 0$. Thus, from Theorem 11, model (5) is stochastically permanent (see Figures 23(a)–23(c)). However, if we take $\nu = 0$, then $r_1 - (\sigma_1^2/2) = 0.89 > 0$, $r_2 - (\sigma_2^2/2) = 0.59 > 0$, and $\lambda_3 - (\sigma_3^2/2) = -0.011 < 0$. From Theorem 5, it follows that $\lim_{t\to\infty} y(t) = 0$ a.s. Moreover, from Figures 23(d) and 23(e), one can see that $\lim_{t\to\infty}\langle x_1(t)\rangle = 4450$ and $\lim_{t\to\infty}\langle x_2(t)\rangle = 1475$ a.s. This means that the prey population $x_i(t)$ $(i = 1, 2)$ will be persistent in mean and the predator population $y(t)$ will go to extinction. Furthermore, comparing Figures 23(c) and 23(f), we conclude that the patch structure is conducive to the survival of the predator population.

Example 13. We assume that $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$. If we take $\nu = 0.1$ ($\nu = 0.2$, $\nu = 0.3$, or $\nu = 0.4$); then, all the conditions of Theorem 11 hold. Thus, from Theorem 11, it follows that model (5) is stochastically permanent. Here, we give the numerical simulations of

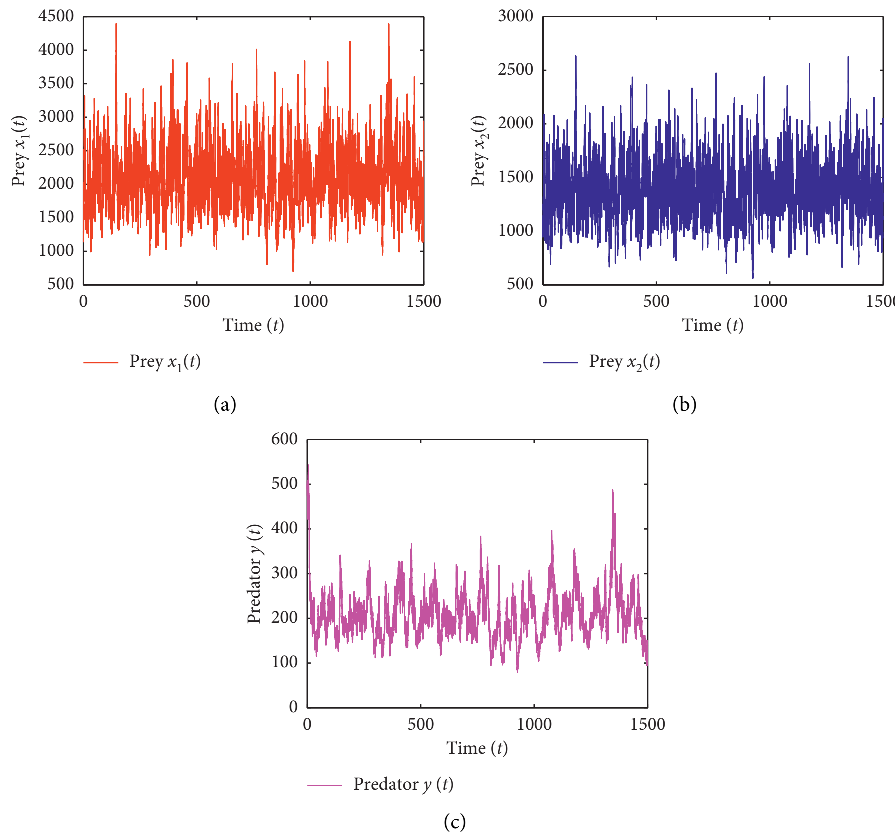


Figure 4: The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.01$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

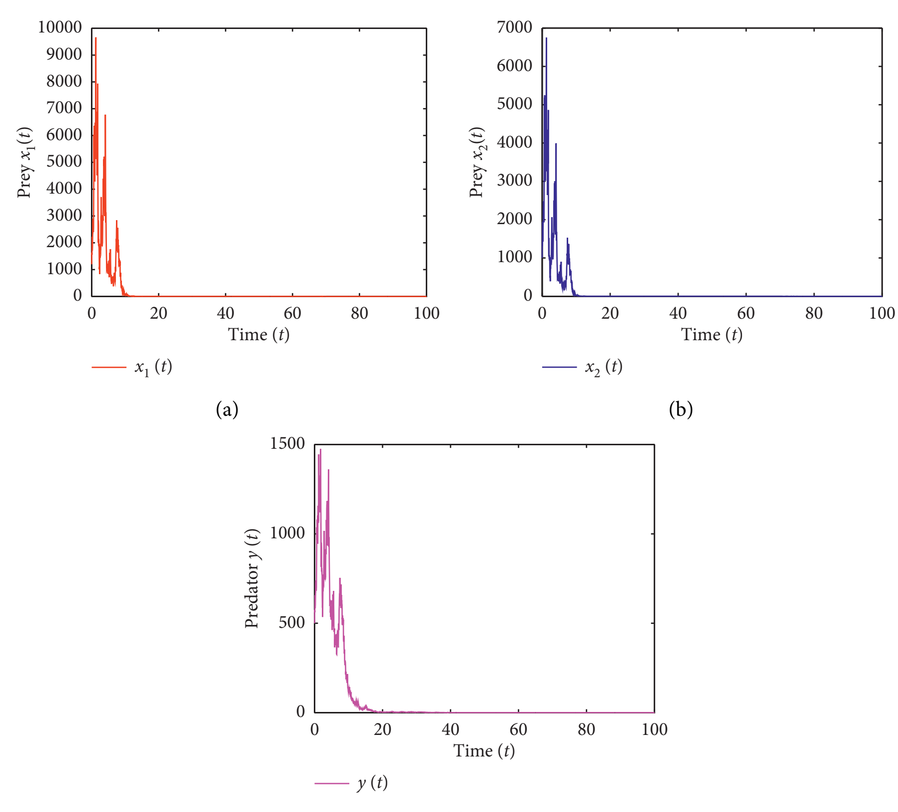


Figure 5: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.2$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

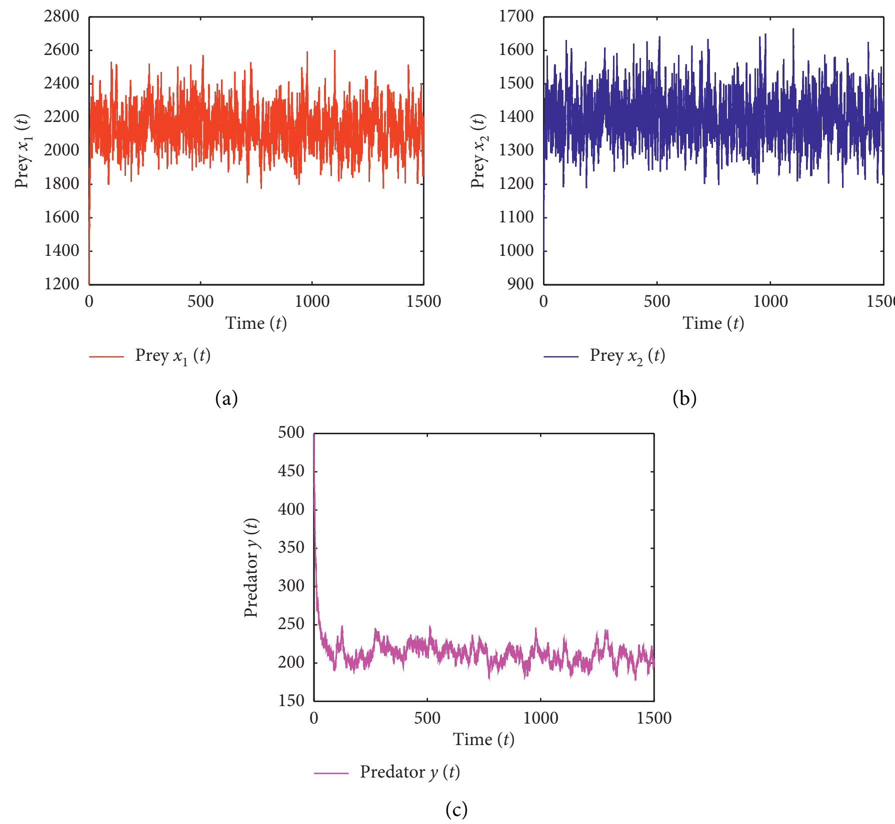


Figure 6: The trajectories of stochastic model (5) with $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, and $\sigma_3 = 0.02$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

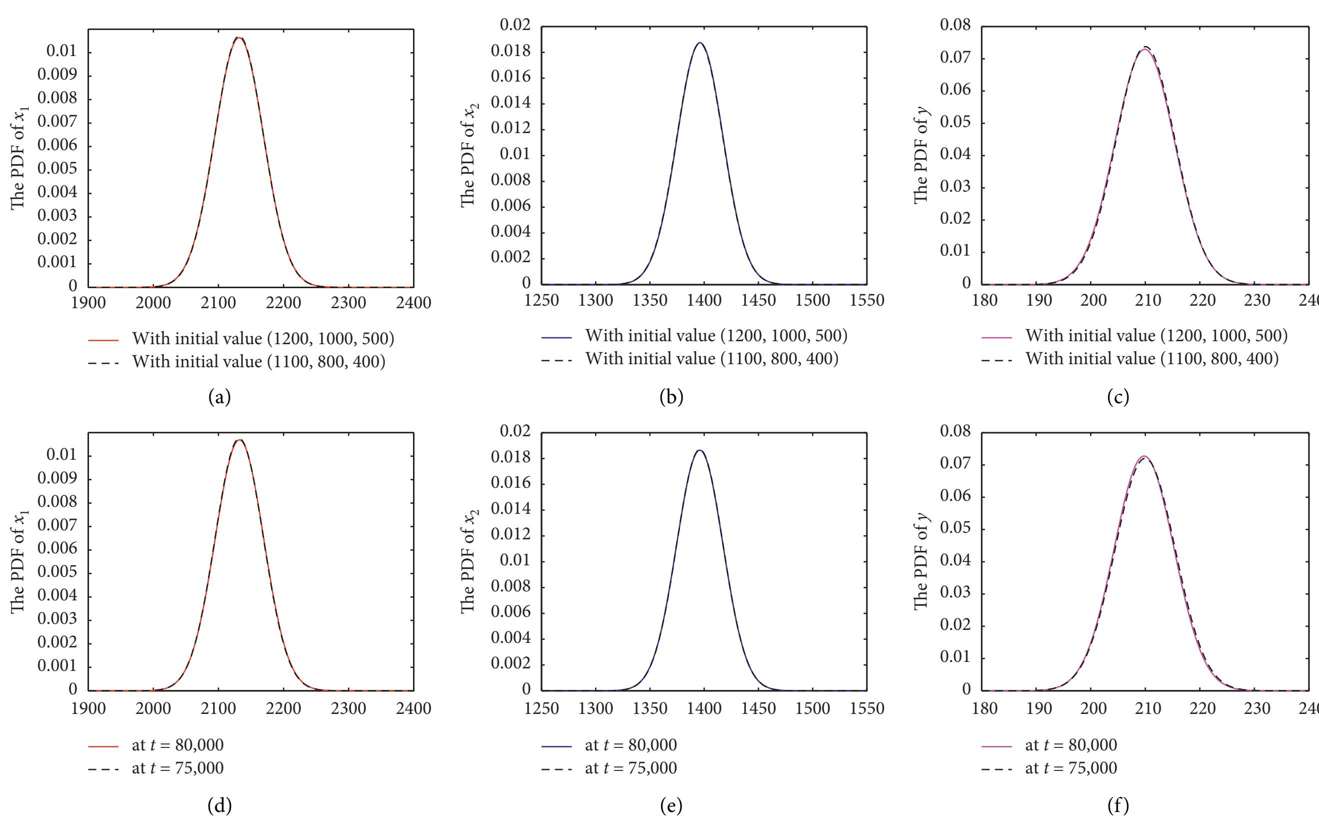


Figure 7: The density based on 10 stochastic simulations for each population. (a)–(c) The density at $t = 80,000$ with different initial value; (d)–(f) the density with initial value $(1500, 100, 500)$ at different time periods (color figure online). (a) PDF of prey $x_1(t)$, (b) PDF of prey $x_2(t)$, (c) PDF of prey $y(t)$, (d) PDF of prey $x_1(t)$, (e) PDF of prey $x_2(t)$, and (f) PDF of prey $y(t)$.

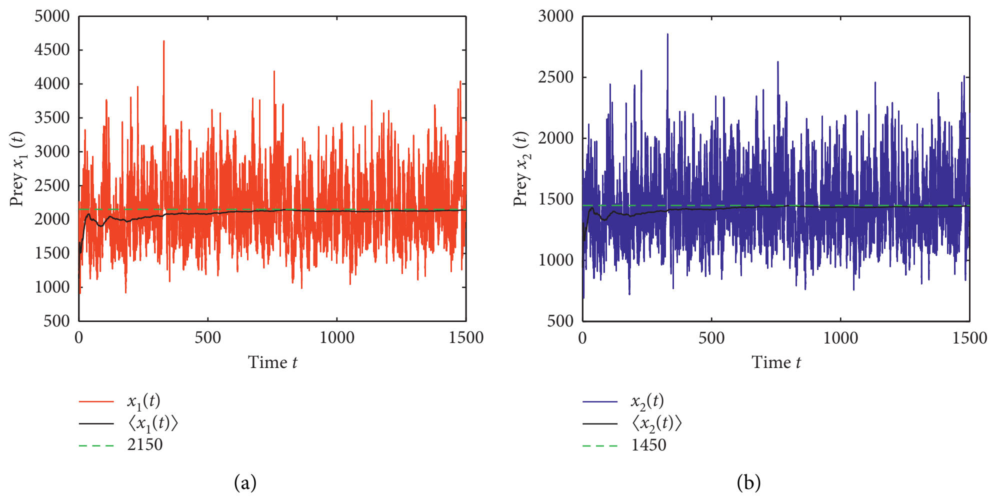


FIGURE 8: The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$ and $\sigma_2^2 = 0.04$, in the absence of the predator (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.

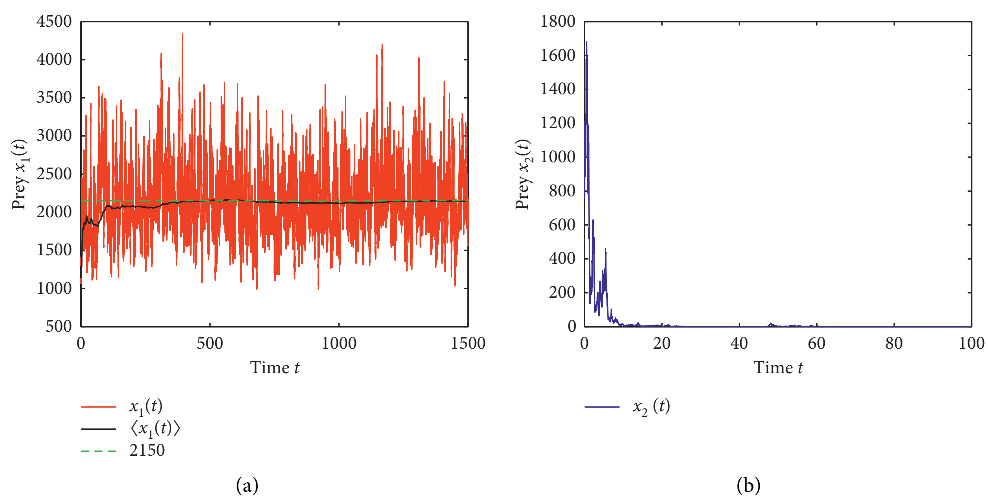


FIGURE 9: t The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$ and $\sigma_2^2 = 1.24$, in the absence of the predator (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.

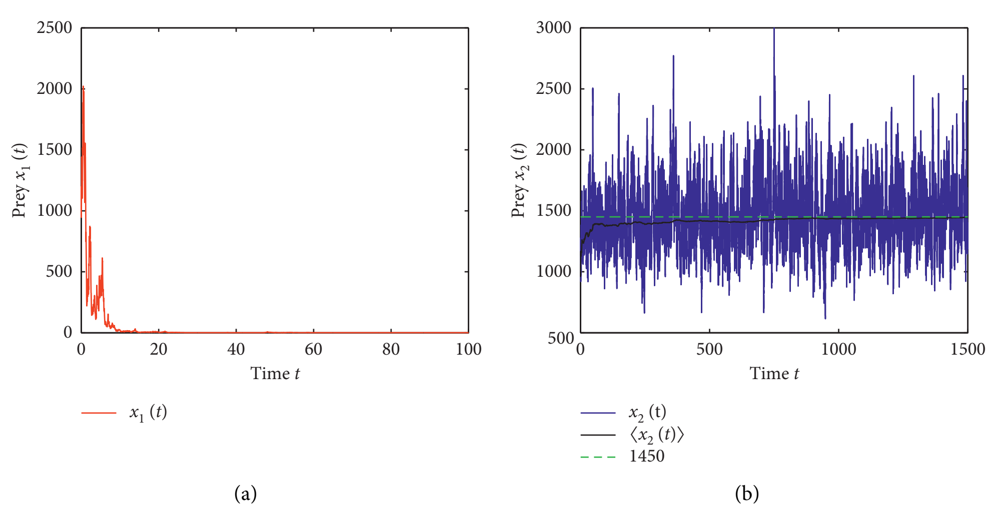
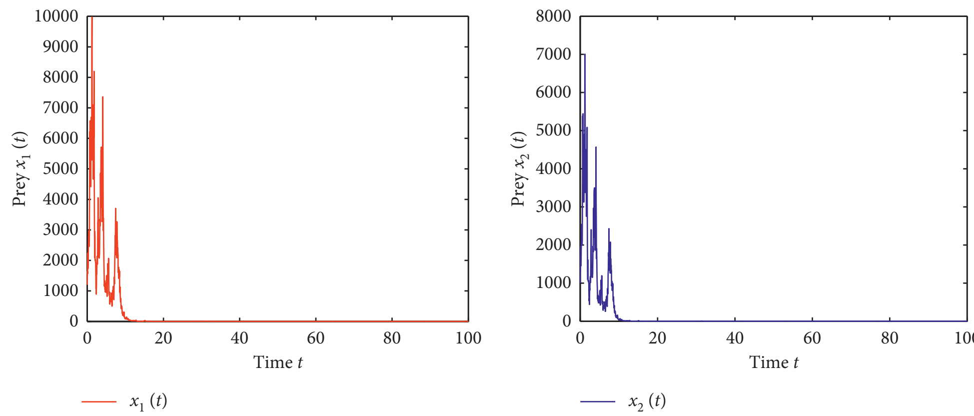
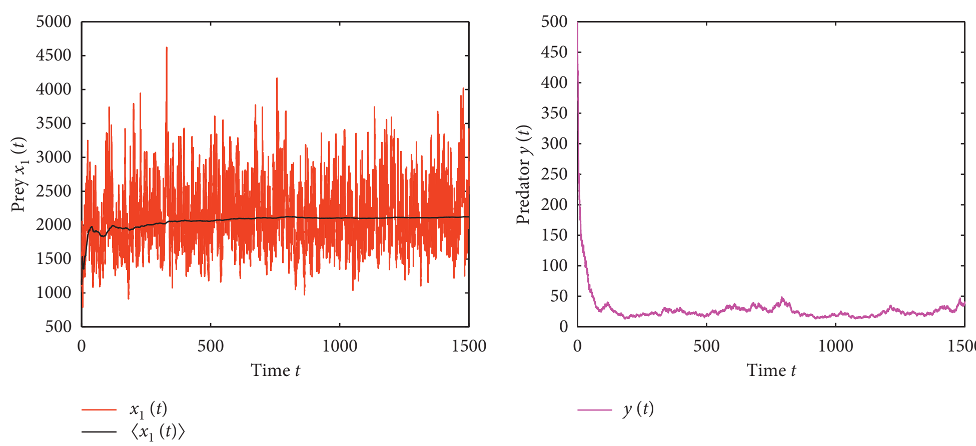


FIGURE 10: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$ and $\sigma_2^2 = 0.04$, in the absence of the predator (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.



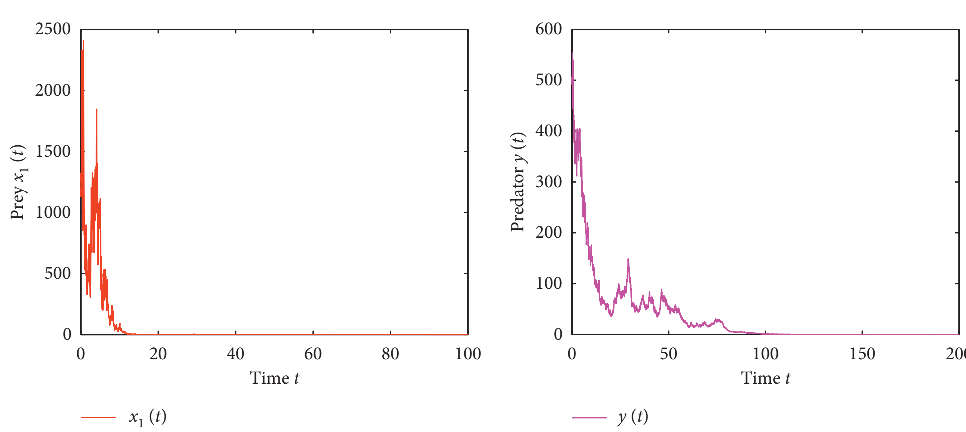
(a) (b)

Figure 11: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$ and $\sigma_2^2 = 1.24$, in the absence of the predator (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.



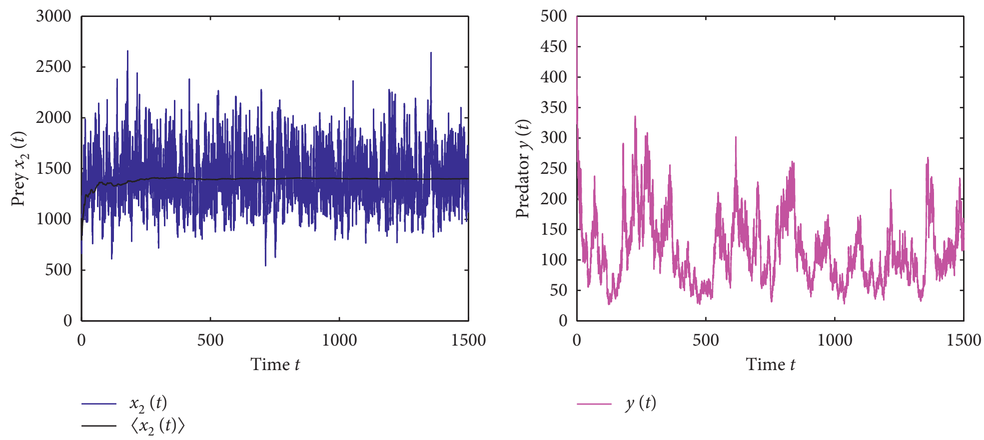
(a) (b)

Figure 12: The trajectories of (5) with $\sigma_1^2 = 0.04$ and $\sigma_3^2 = 0.002$, in the absence of the prey in patch 2 (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.



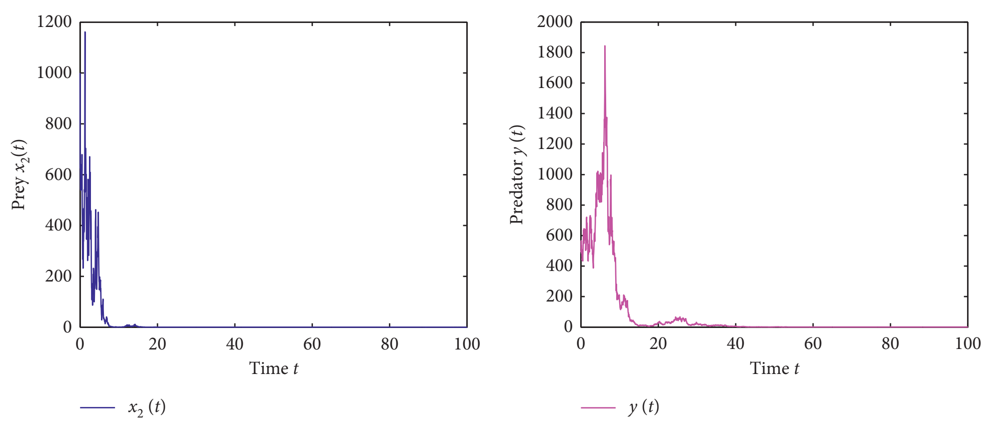
(a) (b)

Figure 13: The trajectories of (5) with $\sigma_1^2 = 1$ and $\sigma_3^2 = 0.04$, in the absence of the prey in patch 2 (color figure online): (a) prey $x_1(t)$ and (b) prey $x_2(t)$.



(a) (b)

Figure 14: The trajectories of (5) with $\sigma_2^2 = 0.04$ and $\sigma_3^2 = 0.02$, in the absence of the prey in patch 1 (color figure online): (a) prey $x_2(t)$ and (b) predator $y(t)$.



(a) (b)

Figure 15: The trajectories of (5) with $\sigma_2^2 = 1.24$ and $\sigma_3^2 = 0.2$, in the absence of the prey in patch 1 (color figure online): (a) prey $x_2(t)$ and (b) predator $y(t)$.

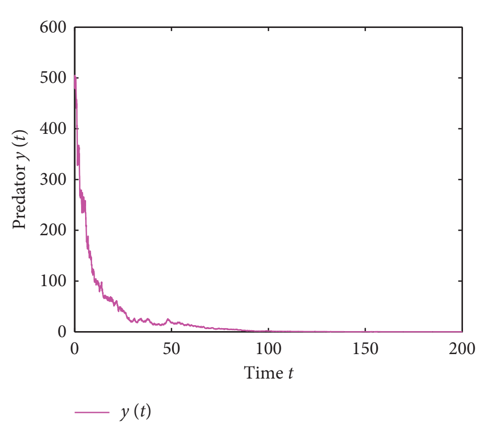


Figure 16: The trajectories of (5) with $\sigma_3^2 = 0.02$, in the absence of the prey (color figure online).

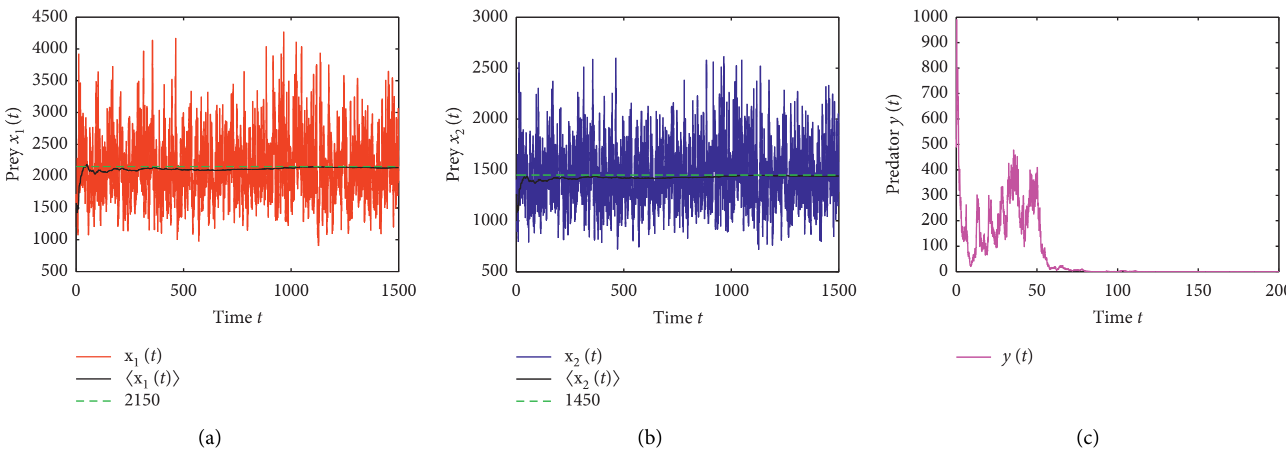


FIGURE 17: The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.2$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

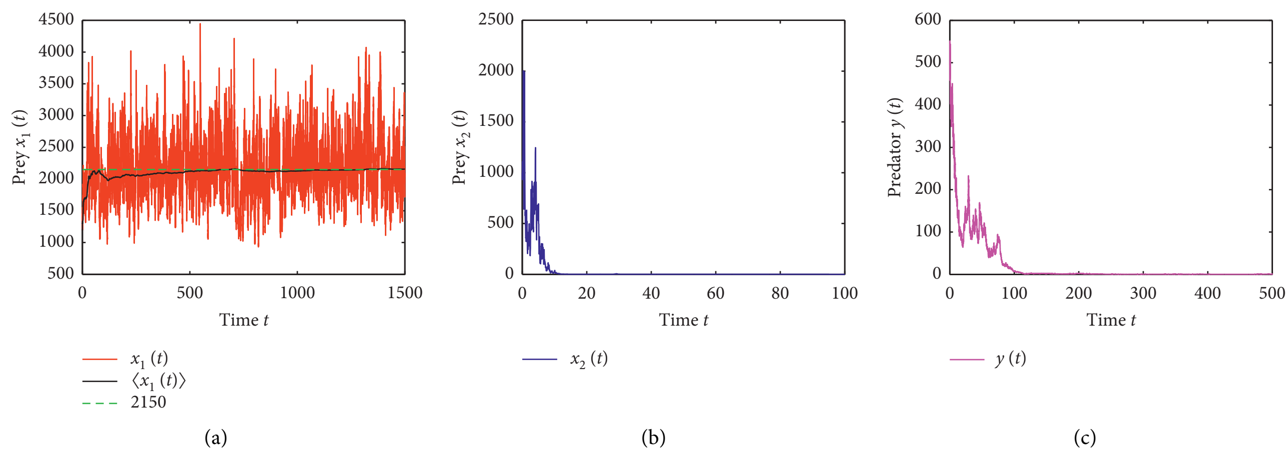


FIGURE 18: The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.03$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

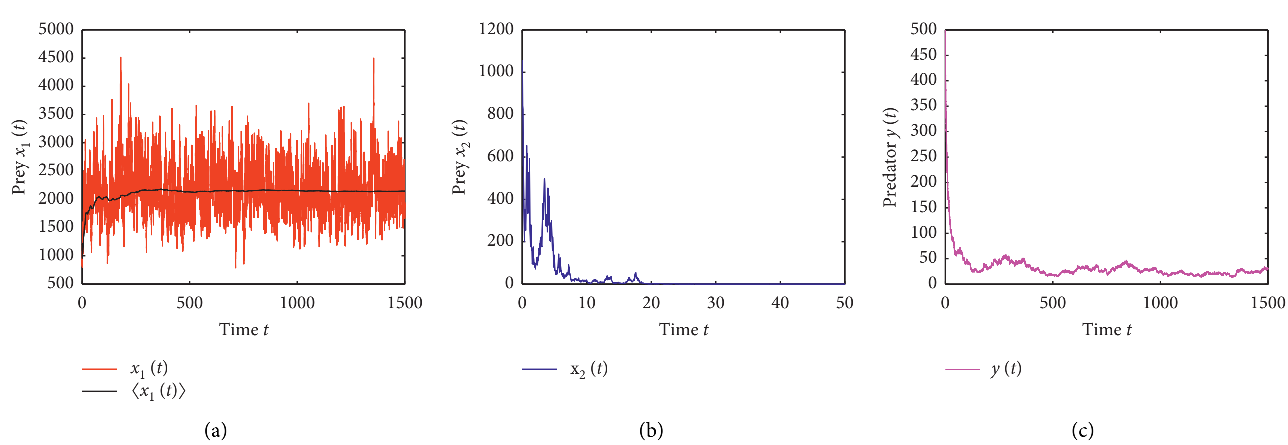
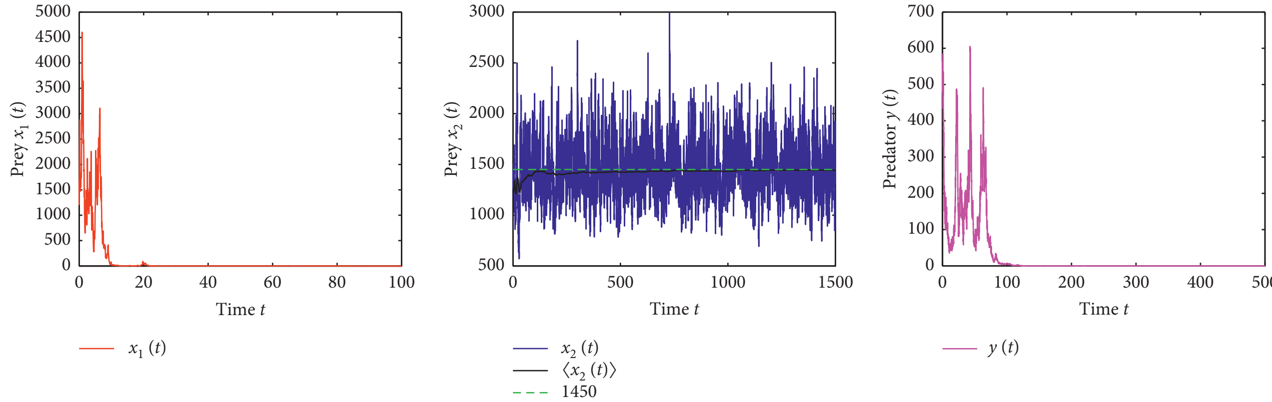
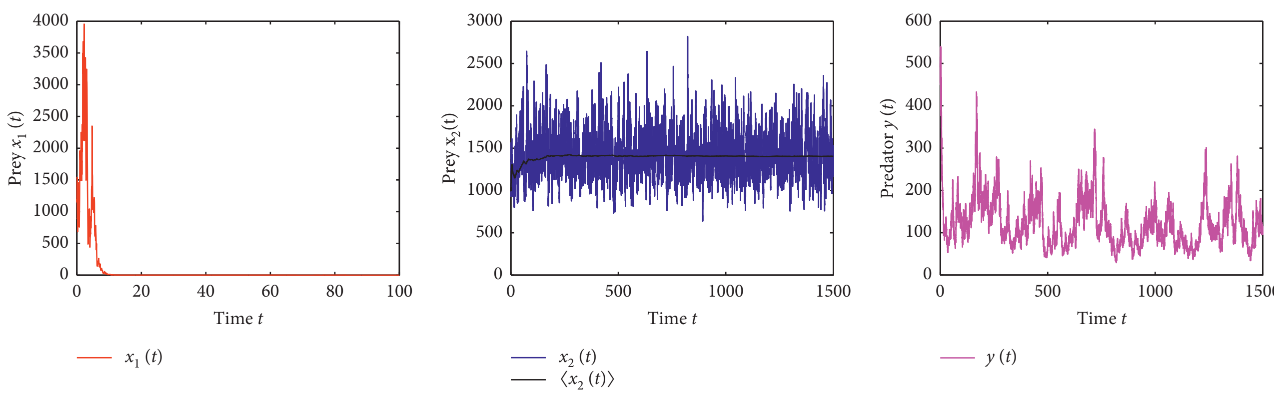


FIGURE 19: The trajectories of stochastic model (5) with $\sigma_1^2 = 0.04$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.002$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.



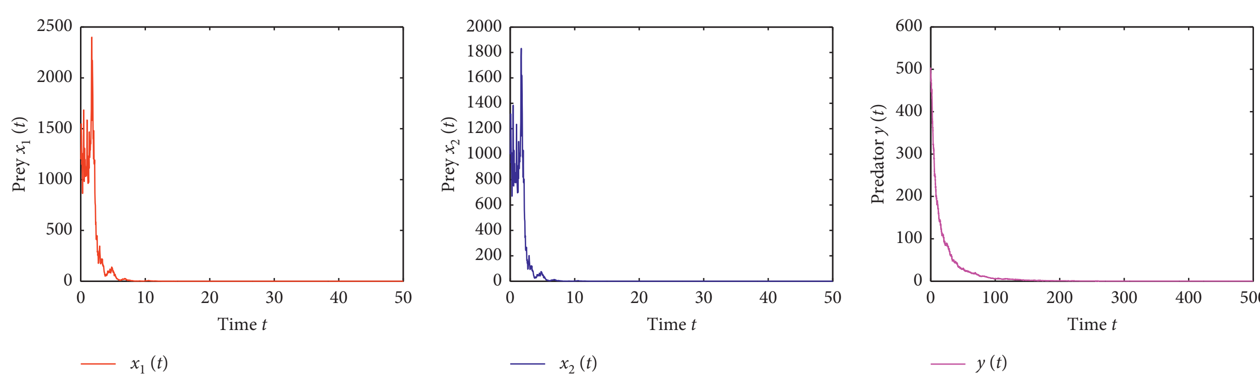
(a) (b) (c)

FIGURE 20: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.1$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.



(a) (b) (c)

FIGURE 21: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$, $\sigma_2^2 = 0.04$, and $\sigma_3^2 = 0.02$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.



(a) (b) (c)

FIGURE 22: The trajectories of stochastic model (5) with $\sigma_1^2 = 1$, $\sigma_2^2 = 1.24$, and $\sigma_3^2 = 0.002$ (color figure online): (a) prey $x_1(t)$, (b) prey $x_2(t)$, and (c) predator $y(t)$.

model (5) with different v (see Figure 24). As can be seen from Figure 24, with the increase of v, that is, the proportion of time that predators stay in patch 1 increases, the number of prey in patch 1 decreases, while the number of prey in patch 2 increases. This has a reasonable biological significance. However, the mortality, the encounter rate with the

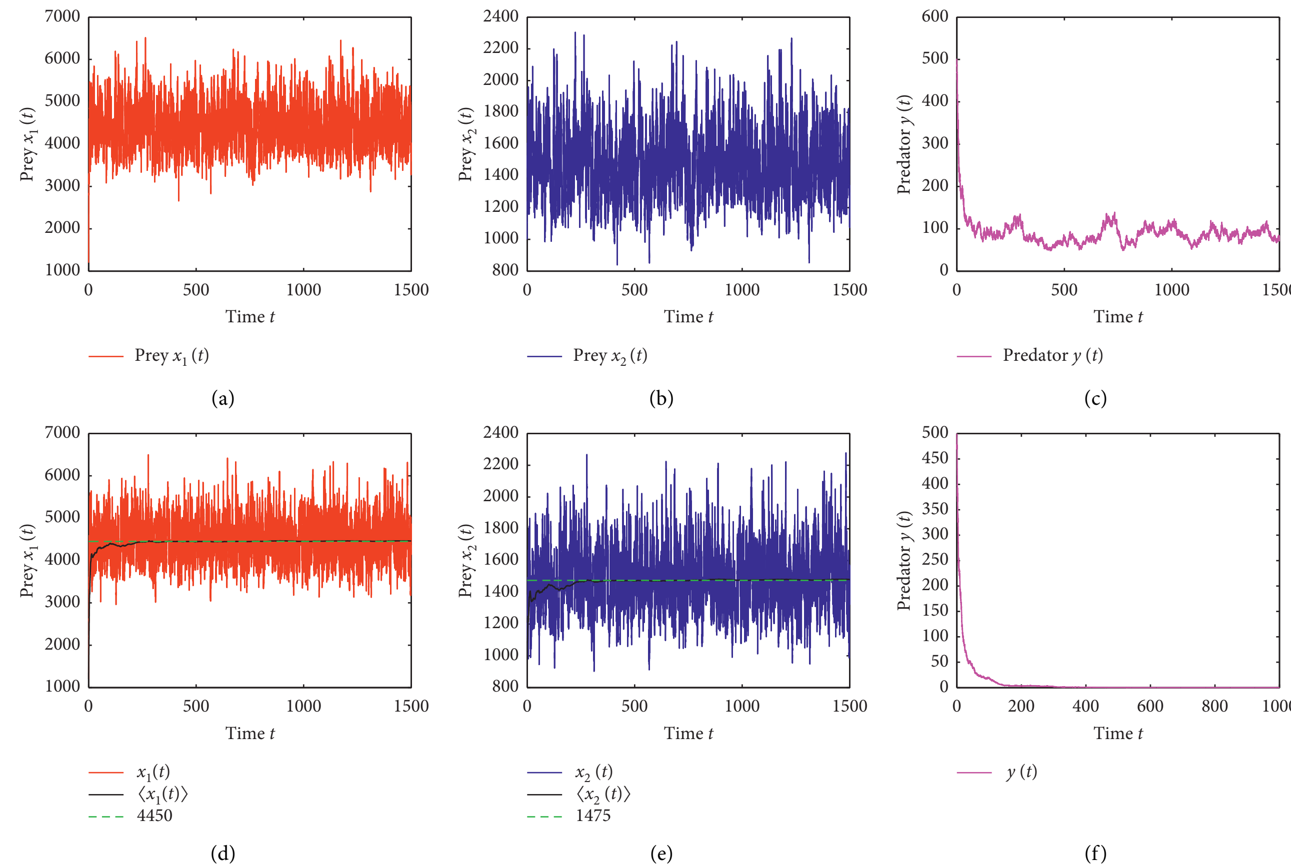


FIGURE 23: The trajectories of stochastic model (5) with different v (color figure online): (a) prey $x_1(t)$with $v = 0.8$, (b) prey $x_2(t)$with $v = 0.8$, (c) predator $y(t)$with $v = 0.8$, (d) prey $x_1(t)$with $v = 0$, (e) prey $x_2(t)$with $v = 0$, and (f) predator $y(t)$with $v = 0$.

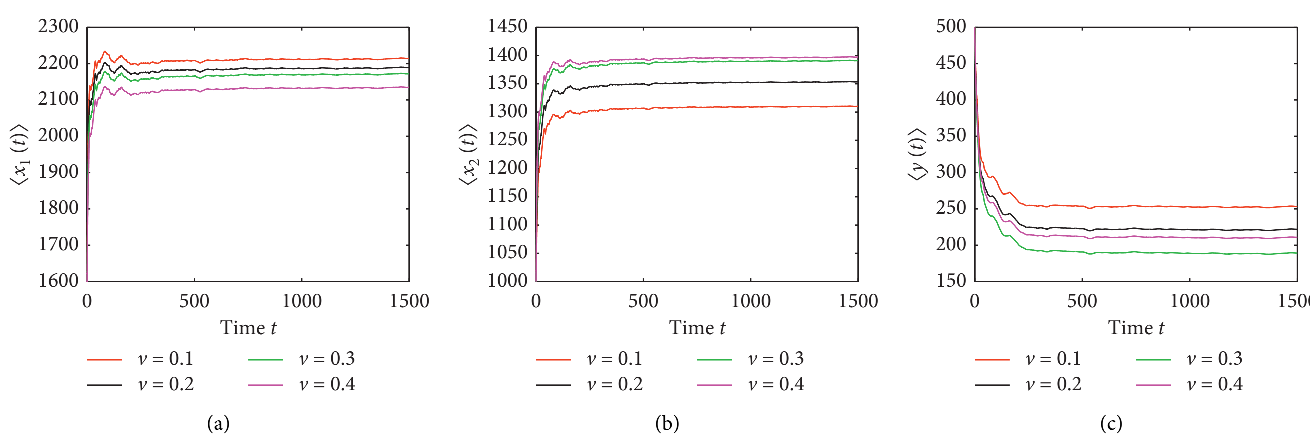


FIGURE 24: The trajectories of stochastic model (5) with different v (color figure online): (a) prey $\langle x_1(t)\rangle$, (b) prey $\langle x_2(t)\rangle$, and (c) predator $\langle y(t)\rangle$.

prey, the half-saturation constant, and the conversion rate of the predators are different in different patches, and it is impossible to determine the number of predators with the change of v.

8. Conclusions and Discussion

This paper is concerned with a stochastic two-patch predator-prey model with ratio-dependent functional responses. First, by using the comparison theorem of stochastic differential equations, we show that the model has a unique global positive solution. Then, the long-time properties of the solutions are discussed pathwise. Using the exponential martingale inequality and Borel–Cantelli lemma, we show that the sample Lyapunov exponents are nonpositive. Moreover, under certain conditions, we show that the sample Lyapunov exponents are zero. Next, the sufficient conditions for the extinction and persistence in mean of the model are given. Then, we investigate the stochastically ultimate boundedness and stochastic persistence of the model. Moreover, by constructing a suitable Lyapunov function, we show that the model has an ergodic stationary distribution. Next, we apply the main results to two special stochastic population models. Finally, some numerical simulations are introduced to support the main results. Furthermore, other dynamic properties of the model are found through numerical simulations.

In Section 2, by using the stochastic comparison theorem, we show that the model has a unique global positive solution. Then, the long-time properties of the solutions are discussed pathwise. Using the exponential martingale inequality and Borel–Cantelli lemma, we show that the sample Lyapunov exponents are nonpositive. Moreover, we show that if the noise intensities σ_i^2 $(i = 1, 2, 3)$ are small compared to the other parameters, then the sample Lyapunov exponents are zero.

Section 3 reveals the effects of stochastic perturbations on the persistence and extinction of prey x_1, prey x_2, and predator y. From Theorem 4, if the noise intensities σ_i^2 $(i = 1, 2, 3)$ are small such that $\lambda_i - (\sigma_i^2/2) > 0$, $(i = 1, 2, 3)$, then all populations in model (5) will be persistent in mean. Furthermore, from Theorem 5, if $r_1 - (\sigma_1^2/2) < 0$, $r_2 - (\sigma_2^2/2) < 0$, and $\lambda_3 - (\sigma_3^2/2) < 0$, then the solution of model (5) tends to zero almost surely. This means that great noise intensities σ_i^2 $(i = 1, 2, 3)$ can make all populations in model (5) will become extinct.

Furthermore, Theorem 6 discusses the effects of noise on the dynamics of other species in the absence of the predator y, prey in patch 1, and prey in patch 2, respectively. In the absence of the predator, if $r_i - (\sigma_i^2/2) > 0$, $(i = 1, 2)$, then $\lim_{t\longrightarrow\infty}\langle x_i(t)\rangle = (r_i - (\sigma_i^2/2))/a_i$ a.s., $(i = 1, 2)$; if $r_i - (\sigma_i^2/2) > 0$ and $r_j - (\sigma_j^2/2) < 0$, $(i \neq j, i, j = 1, 2)$, then $\lim_{t\longrightarrow\infty}\langle x_i(t)\rangle = (r_i - (\sigma_i^2/2))/a_i$ and $\lim_{t\longrightarrow\infty}x_j(t) = 0$ a.s. $(i \neq j, i, j = 1, 2)$; moreover, if $r_i - (\sigma_i^2/2) < 0$, $(i = 1, 2)$, then $\lim_{t\longrightarrow\infty}x_i(t) = 0$ a.s., $(i = 1, 2)$. Hence, in the absence of the predator, with the increase of noise intensity σ_i^2, the prey in patch i will go to extinction, while the environment noise $\dot{w}_i(t)$ has no effect on the extinction of the prey in patch j $(i \neq j, i, j = 1, 2)$. Moreover, in the absence of the predator, with the decrease of noise intensity σ_i^2, the prey in patch i can be persistent better, while the environment noise $\dot{w}_i(t)$ has no effect on the persistent level of the prey in patch j, $(i \neq j, i, j = 1, 2)$. In the absence of the prey in patch 2, if $\lambda_1 - (\sigma_1^2/2) > 0$ and $e_1\alpha_1 v - m_1 v - m_2(1 - v) - (\sigma_3^2/2) > 0$, then the prey in patch 1 and the predator y will be persistent in mean, while if $r_1 - (\sigma_1^2/2) < 0$ and $e_1\alpha_1 v - m_1 v - m_2(1 - v) - (\sigma_3^2/2) < 0$, then the prey in patch 1 and the predator y will go to extinction exponentially. This means that, in the absence of the prey in patch 2, with the increase of noise intensity σ_i^2, $(i = 1, 3)$, the prey in patch 1 and the predator y will become extinct. In the absence of the prey in patch 1, if $\lambda_2 - (\sigma_2^2/2) > 0$ and $e_2\alpha_2(1 - v) - m_1 v - m_2(1 - v) - (\sigma_3^2/2) > 0$, then the prey in patch 2 and the predator y will be persistent in mean, while if $r_2 - (\sigma_2^2/2) < 0$ and $e_2\alpha_2(1 - v) - m_1 v - m_2(1 - v) - (\sigma_3^2/2) < 0$, then the prey in patch 2 and the predator y will go to extinction exponentially. Hence, in the absence of the prey in patch 1, with the increase of noise intensity σ_i^2, $(i = 2, 3)$, the prey in patch 2 and the predator y will become extinct. Moreover, from Theorem 6, in the absence of the prey, the predator dies with probability one.

Theorem 6 shows the effects of noise on the dynamics of other species in the absence of the predator y, prey in patch 1, and prey in patch 2, respectively. However, through the numerical simulation of Example 11, we can conclude the following results: (i) In the case of the predator extinction, the prey has the same dynamic behavior same as that in the absence of the predator. (ii) In the case of the prey x_1 extinction, the dynamic behaviors of the prey in patch 2 and the predator y are the same as those in the absence of the prey x_1. (iii) In the case of the prey x_2 extinction, the dynamic behaviors of the prey in patch 1 and the predator y are the same as those in the absence of the prey x_2. (iv) If the prey extinction, then the predator will go to extinction. This is consistent with the results in the absence of the prey.

In Section 4, we investigate the stochastically ultimate boundedness and stochastic persistence of the model. First, we use two different ways to prove the boundedness of the model and then show that the model is stochastically ultimately bounded by Chebyshev's inequality. Next, we investigate the stochastic persistence of the model. The results show that if the noise intensities σ_i^2 $(i = 1, 2, 3)$ are small such that $\kappa_i > 0$, $(i = 1, 2, 3)$, then model (5) is stochastically permanent. This means that the species in model (5) will survive forever at low noise levels.

In Section 5, by constructing a suitable Lyapunov function, we show that the solution of model (5) has an ergodic stationary distribution. From Theorem 12, if $\kappa_1 - e_1\alpha_1 v\beta_1 > 0$, $\kappa_2 - e_2\alpha_2(1 - v)\beta_2 > 0$, and $\kappa_3 > 0$, then for any initial value $(x_{10}, x_{20}, y_0) \in \mathbb{R}_+^3$, model (5) has a stationary distribution $\mu(\cdot)$ and the solutions have an ergodic property. Hence, the small noise intensities σ_i^2 $(i = 1, 2, 3)$ can ensure that the solution of the model has an ergodic stationary distribution.

In Section 6, we first apply the main results to two stochastic two-species predator-prey models. Then, we present the application of the main results to stochastic two-patch predator-prey model (4). Moreover, we compare the results with the known closely related models.

In [25], the authors discussed the stochastic model (125). It is clear that the results of (ii) and (iii) in Corollary 2 (I) are consistent with Theorems 7 and 8 in [25]. Moreover, [25] only shows that if $\delta_1 - (\sigma_1^2/2) > 0$ and $\delta_2 - (\sigma_2^2/2) > 0$, then $\lim_{t\longrightarrow\infty}(\ln x(t)/t) = 0$ and $\lim_{t\longrightarrow\infty}(\ln y(t)/t) = 0$ a.s. However, we also show that the sample Lyapunov exponents of the solutions are nonpositive in the absence of conditions. Furthermore, the ergodic stationary distribution of model (125) is not reflected in [25]. Thus, our work can be seen as the extension of [25].

Linh and Ton [26] considered the corresponding nonautonomous model of (125). Moreover, the results of (iv) and (v) in Corollary 2 (I) are consistent with Theorems 4.3 and 4.4 in [26]. Furthermore, for the deterministic version of model (125), from [26], if $\lim_{t\longrightarrow\infty}y(t) = 0$, then $\liminf_{t\longrightarrow\infty}x(t) \geq (r/a) > 0$. This means that when the predator dies out, the prey must survive forever. However, the result of (iii) in Corollary 2 (I) shows that great noise intensities σ_1^2 and σ_2^2 can make both the prey and predator in model (125) go to extinction.

Ji et al. [27] considered the stochastic model (133), while Wu, Huang, and Wang [14] discussed the corresponding nonautonomous model of (133). On the one hand, from Theorem 3.3 in [27], if $\varsigma_i \doteq \delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$, then model (133) is persistent in mean. However, from (II) in Corollary 3, model (133) is stochastically permanent. Thus, Theorem 11 generalizes and improves Theorem 3.3 in [27]. On the other hand, from Theorem 4.11 in [14], if $\delta_i - (3/2)\sigma_i^2 > 0$, $(i = 1, 2)$, then model (133) is stochastically permanent. However, the results in Corollary 3 show that if $\delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$, then model (133) is stochastically permanent. Obviously, if $\delta_i - (3/2)\sigma_i^2 > 0$, $(i = 1, 2)$ holds, then $\delta_i - \sigma_i^2 > 0$, $(i = 1, 2)$ holds. On the contrary, it is not set up. Thus, we can say that Corollary 3 generalizes and improves Theorem 4.11 in [14].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Nos. 11971279 and 12001341) and the Youth Natural Science Foundation of Shanxi Province (No. 201901D211410).

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