Hindawi
Complexity
Volume 2021, Article ID 6684784, 23 pages
https://doi.org/10.1155/2021/6684784

Research Article

Hamilton Connectivity of Convex Polytopes with Applications to Their Detour Index

Sakander Hayat[1], **Asad Khan**[2], **Suliman Khan**[1], and **Jia-Bao Liu**[3]

[1]*Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, Khyber Pakhtunkhwa 23460, Pakistan*
[2]*School of Computer Science and Cyber Engineering, Guangzhou University, Guangzhou 510006, China*
[3]*School of Mathematics and Physics, Anhui Jianzhu University, Hefei, Anhui 230000, China*

Correspondence should be addressed to Asad Khan; asad@gzhu.edu.cn and Jia-Bao Liu; liujiabaoad@163.com

Received 1 November 2020; Revised 21 November 2020; Accepted 30 November 2020; Published 25 January 2021

Academic Editor: M. Irfan Uddin

A connected graph is called Hamilton-connected if there exists a Hamiltonian path between any pair of its vertices. Determining whether a graph is Hamilton-connected is an NP-complete problem. Hamiltonian and Hamilton-connected graphs have diverse applications in computer science and electrical engineering. The detour index of a graph is defined to be the sum of lengths of detours between all the unordered pairs of vertices. The detour index has diverse applications in chemistry. Computing the detour index for a graph is also an NP-complete problem. In this paper, we study the Hamilton-connectivity of convex polytopes. We construct three infinite families of convex polytopes and show that they are Hamilton-connected. An infinite family of non-Hamilton-connected convex polytopes is also constructed, which, in turn, shows that not all convex polytopes are Hamilton-connected. By using Hamilton connectivity of these families of graphs, we compute exact analytical formulas of their detour index.

1. Introduction and Preliminaries

All graphs in this paper are simple, loopless, finite, and connected.

A cycle in a graph G is called Hamiltonian if it travels all the vertices of G. Moreover, a path in G is called Hamiltonian path if it traverses all the vertices of G. Not every graph contains a Hamiltonian cycle. For instance, any tree is an acyclic graph, so it cannot contain a Hamiltonian cycle. However, a tress may still contain a Hamiltonian path. A graph G is called Hamiltonian if there exist a Hamiltonian cycle in it. By definition, any cycle graph or a clique graph are Hamiltonian. Furthermore, G is called traceable if it contains a Hamiltonian path. Of course all Hamiltonian graphs are traceable. However, there exist graphs which are traceable but not Hamiltonian. The first nontrivial example which comes in mind immediately is the so-called Petersen graph which is traceable but not Hamiltonian.

A graph which contains a Hamiltonian path between every two vertices of G is called Hamilton connected. They were introduced by Ore [1] in 1963. Frucht [2] studied a canonical representation of trivalent Hamiltonian graphs. A bipartite can never be a Hamilton-connected, since there cannot exist a Hamiltonian path between vertices of the same partite set. In that case, a bipartite graph is called Hamilton-laceable if there exist Hamiltonian paths between vertices of different partite sets. There is an extensive literature available on Hamiltoniancity and Hamilton-connectivity of graphs, see, for instance, [3–7].

Chartrand et al. [8] showed that the square of a block graph is Hamilton-connected. Thomasson [9] studied Hamilton-connected tournaments in graphs. Chang et al. [10] studied panconnectivity, fault-tolerant Hamiltonicity, and Hamiltonian-connectivity in alternating group graphs by considering them as interconnection networks. Kewen et al. [11] derived a sufficient condition for a graph to be Hamilton-connected. Zhou and Wang [12] proved certain sufficient conditions for a graph to be Hamilton-connected in terms of the edge number, the spectral radius, and the signless Laplacian spectral radius of the graph. Zhou et al.

[13] calculated the Wiener and Harary indices of Hamilton-connected graphs with large diameter. Wei et al. [14] derived some spectral analogues of Erdös' theorems for Hamilton-connected graphs. Hung et al. [15] studied Hamilton-connectivity of alphabet grid graphs. Zhou et al. [16] extended a result of Fiedler and Nikiforov and derived signless Laplacian spectral conditions for Hamilton-connected graphs with large minimum degree. Recently, Shabbir et al. [17] studied Hamilton-connectivity in Teoplitz graphs.

By preserving the vertex-edge incidence relation in convex polytopes, their graphs are constructed. Bača [18–20] was the first researcher to consider these families of geometric graphs. In [20] (resp. [19]), Bača studied the problem of magic (resp. graceful and antigraceful) labeling of convex polytopes, whereas, in [18], the problem of face antimagic labeling of convex polytopes was studied. Miller et al. [21] studied the vertex-magic total labeling of convex polytopes. Imran et al. [22–24] computed the minimum metric dimension of various infinite families of convex polytopes. In particular, they showed that these infinite families of convex polytopes have constant metric dimension. Malik et al. [25] also constructed two infinite families of convex polytopes having constant metric dimension. Other closely related infinite families of graphs with constant metric dimension are studied in [26]. Kratica et al. [27] studied the strong metric dimension of certain infinite families of convex polytopes by constructing their doubly resolving sets. The fault-tolerant metric dimension (resp. mixed metric dimension) of convex polytopes was studied by Raza et al. [28] (resp. Raza et al. [29]). Metric dimension of other related families of graphs such as Cayley graph is studied by Vetrik and Abas [30, 31]. The binary locating-dominating number of convex polytopes is studied by Simić et al. [32] and Raza et al. [33]. The open-locating-dominating number of certain convex polytopes has recently been studied by Savic et al. [34]. Liu et al. [35–39] studied application of various graph parameters in electrical networks and related systems.

For a graph G, let $\ell(x, y)$ be the length of a longest path (i.e., detour) between vertices x and y of G. The detour index [40] is defined to be the sum of lengths of all detours between unordered pairs of vertices in G. The detour index of a graph G is usually denoted by $\omega(G)$:

$$\omega(G) = \sum_{\{x,y\}\subset V(G)} \ell(x, y). \tag{1}$$

The detour index has important applications in chemistry. Applications of this parameter in quantitative structure activity and property relationship models were put forward by Lukovits [41]. Besides giving further applications of the detour index, Trinajstić et al. [42] conducted a comparative analysis with the Wiener index in terms of applicability in correlating structure boiling point of organic compounds. Moreover, its further applications in predicting the normal boiling points of cyclic and acyclic alkanes were studied by Rücker and Rücker [43].

An algorithm for tracing the detour between two vertices in a graph was proposed by Lukovits and Razinger [44]. They also applied their algorithm in detecting detours and computing the detour index of graphs corresponding to fused bicyclic skeletons. Rücker and Rücker [43] and Trinajstić et al. [45] proposed certain computer methods to trace out detours and thus calculation of the detour index of graphs. It has already been shown in [46] that the problem of finding the detour index of a given graph is computationally NP-complete. Trinajstić et al. [45] also proposed a method of calculating the detour matrix of reasonably small sizes. Note that the detour index is equal to the sum of all the entries of the detour matrix dividing by two.

Mahmiani et al. [47] proposed the edge versions of the detour index and studied their mathematical properties. Zhou and Cai [48] proved some upper and lower bounds on the detour index of graphs. Qi and Zhou [49] studied minimum uncyclic graphs with respect to the detour index. Du [50] studied the minimum detour index of bicyclic graphs. Fang et al. [51] characterized the minimum detour index of some families of tricyclic graphs. Karbasioun et al. [52] studied the applications of the detour index in infinite families of nanostar dendrimers. Wu and Deng [53] computed the detour index for a chain of C20 fullerenes. Kaladevi [54] studied spectral properties of the detour index in relation with the Laplacian energy of graphs. Recently, Abdullah and Omar [55] introduced the restricted edge version of the detour index and studied it for some families of graphs.

We end this section with an important and well-known result bounding the detour index in terms of graph parameters.

Theorem 1 (see [56]). *Let G be a connected graph with $n \geq 3$ vertices. Then,*

$$(n-1)^2 \leq \omega(G) \leq \frac{n(n-1)^2}{2}, \tag{2}$$

with left equality if and only if $G \cong S_n$ and with right equality if and only if G is Hamilton-connected.

In this paper, we study the Hamilton connectivity of certain infinite families of convex polytopes. More precisely, we construct three infinite families of Hamilton-connected convex polytopes. Moreover, we construct an infinite family non-Hamilton-connected convex polytope. By using Hamilton-connectivity of these families of graphs, we compute exact formulas of their detour index.

2. Hamilton Connectivity and the Detour Index of R_n

In this section, we show that the n-dimensional infinite family of convex polytopes R_n is Hamilton-connected. Afterwards, we use its Hamiltonconnectivity to find analytical exact expression of the detour index of the graph R_n.

For $n \geq 5$, the graph of convex polytope R_n is defined in [20]. It has the vertex set

$$V(R_n) = \{x_j; y_j; z_j: 1 \leq j \leq n\}, \tag{3}$$

and edge set

$$E(R_n) = \{(x_j, x_{j+1}); (x_j, y_j); (x_{j+1}, y_j); (y_j, y_{j+1}); \cdot (y_j, z_j); (z_j, z_{j+1}): 1 \le j \le n\}. \quad (4)$$

By convention, we suppose that $x_{n+1} = x_n$, $y_{n+1} = y_n$, and $z_{n+1} = z_n$. See Figure 1 to view the n-dimensional convex polytope R_n.

For a positive integer $v \in \mathbb{Z}^+$, we write $v|2$ (resp. $v\nmid 2$) if v is even (resp. odd).

Theorem 2. *The n-dimensional convex polytope R_n, with $n \ge 4$, is Hamilton-connected.*

Proof. We prove this result by definition. For this, we have to show that there exists a Hamiltonian path between any pair of vertices of R_n.

Let $P_H(u, v)$ be a Hamiltonian path between vertices u and v in R_n. Let $V(R_n) = Z \cup Y \cup X$ such that $Z = \{z_1, z_2, \ldots, z_n\}$, $Y = \{y_1, y_2 \ldots y_n\}$, and $X = \{x_1, x_2, \ldots, x_n\}$, see, Figure 1.

Case 1: $u = z_1$ and $v = z_i$, $2 \le i \le n$.

Case 1.1: $2 \le i \le n-1$:

$$P_H(u, v)\colon\ u = z_1 \sim \{z_{n-j}\colon\ 0 \le j \le n-i-1\} \sim \{y_{i-j+1}\colon\ 0 \le j \le i-2\} \sim \{x_j\colon\ 3 \le j \le i+1\} \sim \{x_j y_j\colon\ i+2 \le j \le n\} \sim x_1 y_1 x_2 y_2 \sim \{z_j\colon\ 2 \le j \le i\} = v. \quad (5)$$

Case 1.2: $i = n$:

$$P_H(u, v)\colon\ u = z_1 y_1 \sim \{x_j\colon\ 1 \le j \le n\} \sim \{y_{n-j}\colon\ 0 \le j \le n-2\} \sim \{z_j\colon\ 2 \le j \le n\} = v. \quad (6)$$

Case 2: $u = z_1$ and $v = y_i$, $1 \le i \le n$.

Case 2.1: $1 \le i \le n-1$:

$$P_H(u, v)\colon\ u = z_1 \sim \{z_j\colon\ 2 \le j \le n\} \sim \{y_{n-j}\colon\ 0 \le j \le n-i-1\} \sim \{x_j\colon\ i+1 \le j \le n\} \sim \{x_j y_j\colon\ 1 \le j \le i\} = v. \quad (7)$$

Case 2.2: $i = n$:

$$P_H(u, v)\colon u = z_1 y_1 \sim \{x_j\colon 1 \le j \le n\} \sim \{y_{n-j}\colon 1 \le j \le n-2\} \sim \{z_j\colon 2 \le j \le n\} \sim y_n = v. \quad (8)$$

Case 3: $u = z_1$ and $v = x_i$, $1 \le i \le n$.

Case 3.1: $i = 1$:

$$P_H(u, v)\colon u = z_1 \sim \{z_j\colon 2 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-1\} \sim \{x_j\colon 2 \le j \le n\} \sim x_1 = v. \quad (9)$$

Case 3.2: $2 \le i \le n$:

$$P_H(u, v)\colon\ u = z_1 \sim \{z_j\colon\ 2 \le j \le n\} \sim \{y_{n-j}\colon\ 0 \le j \le n-i\} \sim \{y_{i-j} x_{i-j}\colon\ 1 \le j \le i-1\} \sim \{x_{n-j}\colon\ 0 \le j \le n-i\} = v. \quad (10)$$

Case 4: $u = y_1$ and $v = z_i$, $1 \le i \le n$.

Case 4.1: $i = 1$:

$$P_H(u, v)\colon u = y_1 \sim \{x_j\colon 1 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-2\} \sim \{z_j\colon 2 \le j \le n\} \sim z_1 = v. \quad (11)$$

Case 4.2: $2 \le i \le n-1$:

$$P_H(u, v)\colon\ u = y_1 x_1 \sim \{x_j y_j\colon\ 2 \le j \le i\} \sim \{x_j\colon\ i+1 \le j \le n\} \sim \{y_{n-j}\colon\ 0 \le j \le n-i-1\} \sim \{z_j\colon\ i+1 \le j \le n\} \sim \{z_j\colon\ 1 \le j \le i\} = v. \quad (12)$$

Case 4.3: $i = n$:

$$P_H(u, v)\colon u = y_1 \sim \{z_j\colon 1 \le j \le n-1\} \sim \{y_{n-j}\colon 1 \le j \le n-2\} \sim \{x_j\colon 2 \le j \le n\} \sim x_1 y_n z_n = v. \quad (13)$$

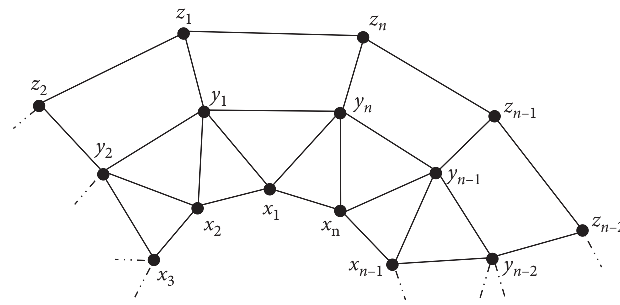

Figure 1: The n-dimensional convex polytope R_n.

Case 5: $u = y_1$ and $v = y_i$, $2 \le i \le n$.

Case 5.1: $i = 2$:

$$P_H(u,v): u = y_1 \sim \{x_j: 1 \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-3\} \sim \{z_j: 3 \le j \le n\} \sim z_1 z_2 y_2 = v. \quad (14)$$

Case 5.2: $3 \le i \le n-1$:

$$P_H(u,v): u = y_1 x_1 \sim \{x_j y_j: 2 \le j \le i-1\} \sim \{x_j: i \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-i-1\} \sim \{z_j: i+1 \le j \le n\} \sim \{z_j: 1 \le j \le i\} = v. \quad (15)$$

Case 5.3: $i = n$:

$$P_H(u,v): u = y_1 z_1 \sim \{z_{n-j}: 0 \le j \le n-2\} \sim \{y_j: 2 \le j \le n-1\} \sim \{x_{n-j}: 0 \le j \le n-1\} \sim y_n = v. \quad (16)$$

Case 6: $u = y_1$ and $v = x_i$, $1 \le i \le n$.

Case 6.1: $i = 1, 2$:

$$P_H(u,v): u = y_1 \sim \{z_j: 1 \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-2\} \sim \{x_j: i+1 \le j \le n\} \sim \{x_j: 1 \le j \le i\} = v. \quad (17)$$

Case 6.2: $3 \le i \le n-1$:

$$P_H(u,v): u = y_1 \sim \{z_j: 1 \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-i\} \sim \{x_j: i+1 \le j \le n\} \sim x_1 \sim \{x_j y_j: 2 \le j \le i-1\} \sim x_i = v. \quad (18)$$

Case 6.3: $i = n$:

$$P_H(u,v): u = y_1 x_1 y_n z_n \sim \{z_j: 1 \le j \le n-1\} \sim \{y_{n-j}: 1 \le j \le n-2\} \sim \{x_j: 2 \le j \le n\} = v. \quad (19)$$

Case 7: $u = x_1$ and $v = z_i$, $1 \le i \le n$.

Case 7.1: $i = 1$:

$$P_H(u,v): u = x_1 \sim \{x_{n-j}: 0 \le j \le n-2\} \sim \{y_j: 1 \le j \le n\} \sim \{z_{n-j}: 0 \le j \le n-2\} \sim z_1 = v. \quad (20)$$

Case 7.2: $i = 2$:

$$P_H(u,v): u = x_1 \sim \{x_j: 2 \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-1\} \sim z_1 \sim \{z_{n-j}: 0 \le j \le n-3\} \sim z_2 = v. \quad (21)$$

Case 7.3: $3 \le i \le n$:

$$P_H(u,v): u = \{x_j y_j: 1 \le j \le i-2\} \sim \{x_j: i-1 \le j \le n\} \sim \{y_{n-j}: 0 \le j \le n-i+1\} \sim \{z_{i-j}: 1 \le j \le i-1\} \sim \{z_{n-j}: 0 \le j \le n-i\} = v. \quad (22)$$

Case 8: $u = x_1$ and $v = y_i$, $1 \le i \le n$.

Case 8.1: $i = 1$:

$$P_H(u, v)\colon u = x_1 \sim \{x_j\colon 2 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-2\} \sim \{z_j\colon 2 \le j \le n\} \sim z_1 y_1 = v. \tag{23}$$

Case 8.2: $2 \le i \le n-1$:

$$P_H(u, v)\colon u = \{x_j y_j\colon 1 \le j \le i-1\} \sim \{x_j\colon i \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-i-1\} \sim \{z_j\colon i+1 \le j \le n\} \sim \{z_j\colon 1 \le j \le i\} \sim y_i = v. \tag{24}$$

Case 8.3: $i = n$:

$$P_H(u, v)\colon u = x_1 \sim \{x_j\colon 2 \le j \le n\} \sim \{y_{n-j}\colon 1 \le j \le n-1\} \sim \{z_j\colon 1 \le j \le n\} \sim z_n = v. \tag{25}$$

Case 9: $u = x_1$ and $v = x_i$, $2 \le i \le n$.

Case 9.1: $2 \le i \le n-1$:

$$P_H(u, v)\colon u = x_1 \sim \{x_{n-j}\colon 0 \le j \le n-i-1\} \sim \{y_j\colon i \le j \le n\} \sim \{z_{n-j}\colon 0 \le j \le n-1\} \sim \{y_{j-1} x_j\colon 2 \le j \le i\} = v. \tag{26}$$

Case 9.2: $i = n$:

$$P_H(u, v)\colon u = x_1 \sim \{x_j\colon 2 \le j \le n-1\} \sim \{y_{n-j}\colon 1 \le j \le n-1\} \sim \{z_j\colon 1 \le j \le n\} \sim y_n x_n = v. \tag{27}$$

Existence of Hamiltonian path between any two vertices of the n-dimensional convex polytope R_n completes the proof.

Using Theorems 1 and 2, the following proposition computes the detour index of of R_n. □

Proposition 1. *Let $G = R_n$, where $n \ge 4$. Then, the detour index of G is*

$$\omega(G) = \frac{3n(3n-1)^2}{2}. \tag{28}$$

Proof. The number of vertices in the graph G is $3n$. Replacing $3n$ with n in Theorem 1 shows the proposition. □

3. Hamilton Connectivity and the Detour Index of S_n

In this section, we show that the n-dimensional infinite family of convex polytopes S_n is Hamilton-connected. Then, we use its Hamilton connectivity to find analytical exact expression of the detour index of the graph S_n.

The graph of convex polytope S_n (Figure 2) consists of $2n$ 3-sided faces, $2n$ 4-sided faces, and a pair of n-sided faces [57]. We have

$$\begin{aligned} V(S_n) &= \{w_j; x_j; y_j; z_j\colon 1 \le j \le n\}, \\ E(S_n) &= \{w_j w_{j+1}; x_j x_{j+1}; y_j y_{j+1}; z_j z_{j+1}\colon 1 \le j \le n\} \cup \{w_{j+1} x_j; w_j x_j; x_j y_j; y_j z_j\colon 1 \le j \le n\}. \end{aligned} \tag{29}$$

See Figure 2 to view the n-dimensional convex polytope S_n.

Next, we show the main result of this section.

Theorem 3. *The n-dimensional convex polytope S_n, with $n \ge 4$, is Hamilton-connected.*

Proof. We prove it by definition. This implies that we need to show the existence of Hamiltonian paths between any pair of vertices of G.

Let $P_H(u, v)$ be a Hamiltonian path between vertices u and v in S_n. By following the labeling of vertices exhibited in Figure 2, we show the existence of Hamiltonian paths between vertices of S_n in a number of cases.

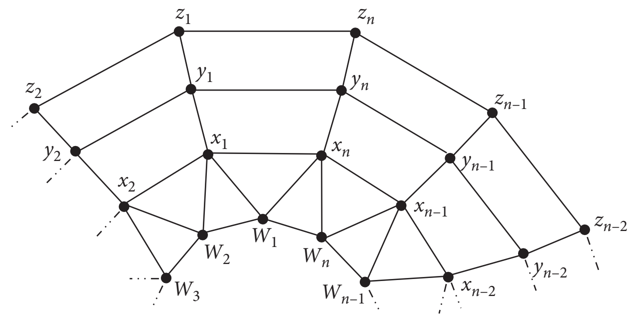

FIGURE 2: The n-dimensional convex polytope S_n.

Case 1: $u = z_1$ and $v = z_i$, $2 \le i \le n$.

Case 1.1: $i|2$:

$$P_H(u,v)\colon u = z_1 y_1 x_1 w_1 \sim \left\{ w_{j-n} x_{j-n} y_{j-n} z_{j-n} z_{j-n+1} y_{j-n+1} x_{j-n+1} w_{j-n+1}\colon n+2 \le j \le \left(\frac{n+i}{2}\right) \right\} \sim \left\{ w_j\colon i \le j \le n \right\} \\ \sim \left\{ x_{n-j}\colon 0 \le j \le n-i \right\} \sim \left\{ y_j\colon i \le j \le n \right\} \sim \left\{ z_{n-j}\colon 0 \le j \le n-i \right\} = v. \tag{30}$$

Case 1.2: $i \nmid 2$:

$$P_H(u,v)\colon u = z_1 \sim \left\{ z_j\colon 2 \le j \le i-1 \right\} \sim \left\{ y_j\colon i-1 \le j \le n-1 \right\} \sim \left\{ x_{n-j}\colon 1 \le j \le n-i+1 \right\} \sim \left\{ w_j\colon i-1 \le j \le n \right\} \\ \sim x_n \sim \left\{ w_j\colon 1 \le j \le i-2 \right\} \sim \left\{ x_{i-j} y_{i-j} y_{i-j-1} x_{i-j-1}\colon 2 \le j \le \frac{(i-1)}{2} \right\} \sim x_1 y_1 y_n \sim \left\{ z_{n-j}\colon 0 \le j \le n-i \right\} = v. \tag{31}$$

Case 2: $u z_1$ and $v y_i$, $1 \le i \le n$:

Case 2.1: $1 \le i \le n-1$:

$$P_H(u,v)\colon u = z_1 \sim \left\{ z_j\colon 2 \le j \le n \right\} \sim \left\{ y_{n-j}\colon 0 \le j \le n-i-1 \right\} \sim \left\{ x_j\colon i+1 \le j \le n \right\} \sim w_1 \sim \left\{ w_{n-j}\colon 0 \le j \le n-i-1 \right\} \\ \sim \left\{ x_{i-j} w_{i-j}\colon 0 \le j \le i-2 \right\} \sim x_1 \sim \left\{ y_j\colon 1 \le j \le i \right\} = v. \tag{32}$$

Case 2.2: $i = n$:

$$P_H(u,v)\colon u = z_1 y_1 x_1 x_n w_1 \sim \left\{ w_{n-j}\colon 0 \le j \le n-2 \right\} \sim \left\{ x_j\colon 2 \le j \le n-1 \right\} \sim \left\{ y_{n-j}\colon 1 \le j \le n-2 \right\} \sim \left\{ z_j\colon 2 \le j \le n \right\} \sim y_n = v. \tag{33}$$

Case 3: $u = z_1$ and $v = x_i$, $1 \le i \le n$.

Case 3.1: $i = 1$:

$$P_H(u,v):\ u = z_1 \sim \{z_{n-j}\colon 0 \le j \le n-2\} \sim y_1 \sim \{y_{n-j}\colon 0 \le j \le n-3\} \sim \{x_j\colon 3 \le j \le n\} \sim w_1 \sim \{w_{n-j}\colon 0 \le j \le n-2\} \sim x_2 x_1 = v. \tag{34}$$

Case 3.2: $2 \le i \le n-1$:

$$P_H(u,v)\colon u = z_1 \sim \{z_{n-j}\colon 0 \le j \le n-2\} \sim \{y_j\colon 2 \le j \le n\} \sim y_1 x_1 \sim \{x_{n-j}\colon 0 \le j \le n-i-1\} \sim \{w_j\colon i+1 \le j \le n\} \sim w_1 \sim \{w_j x_j\colon 2 \le j \le i\} = v. \tag{35}$$

Case 3.3: $i = n$:

$$P_H(u,v)\colon u = \{z_j\colon 1 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-1\} \sim \{x_j\colon 1 \le j \le n-1\} \sim \{w_{n-j}\colon 0 \le j \le n-1\} \sim x_n = v. \tag{36}$$

Case 4: $u = z_1$ and $v = w_i$, $1 \le i \le n$.

Case 4.1: $i = 1$:

$$P_H(u,v)\colon u = \{z_j\colon 1 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-1\} \sim \{x_j\colon 1 \le j \le n\} \sim \{w_{n-j}\colon 0 \le j \le n-1\} = v. \tag{37}$$

Case 4.2: $2 \le i \le n-1$:

$$P_H(u,v)\colon u = \{z_j\colon 1 \le j \le n\} \sim y_n \sim \{y_j\colon 1 \le j \le n-1\} \sim \{x_{n-j}\colon 1 \le j \le n-i\} \sim \{w_j\colon i+1 \le j \le n\} \sim w_1 x_n \sim \{x_{j-1} w_j\colon 2 \le j \le i\} = v. \tag{38}$$

Case 4.3: $i = n$:

$$P_H(u,v)\colon u = \{z_j\colon 1 \le j \le n\} \sim \{y_{n-j}\colon 0 \le j \le n-1\} \sim \{x_j\colon 1 \le j \le n\} \sim \{w_j\colon 1 \le j \le n\} = v. \tag{39}$$

Case 5: $u = y_1$ and $v = z_i$, $1 \le i \le n$.

Case 5.1: $i = 1$:

$$P_H(u,v)\colon\ u = \{y_j\colon\ 1 \le j \le n-1\} \sim \{x_{n-j}\colon\ 1 \le j \le n-1\} \sim \{w_j\colon\ 1 \le j \le n\} \sim x_n y_n \sim \{z_{n-j}\colon\ 0 \le j \le n-1\} = v. \tag{40}$$

Case 5.2: $2 \le i \le n-1$:

$$P_H(u,v)\colon\ u = \left\{y_j\colon\ 1\le j\le i\right\} \sim \left\{x_{i-j}w_{i-j}\colon\ 0\le j\le i-1\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{x_j\colon\ i+1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{z_j\colon\ i+1\le j\le n\right\} \left\{z_j\colon\ 1\le j\le i\right\} = v. \tag{41}$$

Case 5.3: $i=n$:

$$P_H(u,v)\colon\ u = y_1 \sim \left\{z_j\colon\ 1\le j\le n-1\right\} \sim \left\{y_{n-j}\colon\ 1\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n-1\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-1\right\} \sim x_1x_ny_nz_n = v. \tag{42}$$

Case 6: $u=y_1$ and $v=y_i$, $2\le i\le n$.
Case 6.1: $2\le i\le n-1$:

$$P_H(u,v)\colon\ u = y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{x_{i-j+1}\colon\ 0\le j\le i-2\right\} \sim \left\{w_j\colon\ 3\le j\le i+1\right\} \sim \left\{w_jx_j\colon\ i+2\le j\le n\right\} \sim w_1x_1w_2x_2 \sim \left\{y_j\colon\ 2\le j\le i\right\} = v. \tag{43}$$

Case 6.2: $i=n$:

$$P_H(u,v)\colon\ u = y_1z_1 \sim \left\{z_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{y_j\colon\ 2\le j\le n-1\right\} \sim \left\{x_{n-j}\colon\ 1\le j\le n-1\right\} \sim \left\{w_j\colon\ 1\le j\le n\right\} \sim x_ny_n = v. \tag{44}$$

Case 7: $u=y_1$ and $v=x_i$, $1\le i\le n$.
Case 7.1: $1\le i\le n-1$:

$$P_H(u,v)\colon\ u = y_1z_1 \sim \left\{z_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{y_j\colon\ 2\le j\le n\right\} \sim \left\{x_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{w_j\colon\ i+1\le j\le n\right\} \sim \left\{w_jx_j\colon\ 1\le j\le i\right\} = v. \tag{45}$$

Case 7.2: $i=n$:

$$P_H(u,v)\colon\ u = y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n-1\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-1\right\} \sim x_1x_n = v. \tag{46}$$

Case 8: $u=y_1$ and $v=w_i$, $1\le i\le n$.
Case 8.1: $i=1$:

$$P_H(u,v)\colon\ u = y_1z_1 \sim \left\{z_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{y_j\colon\ 2\le j\le n\right\} \sim \left\{x_{n-j}\colon\ 0\le j\le n-1\right\} \sim \left\{w_j\colon\ 2\le j\le n\right\} \sim w_1 = v. \tag{47}$$

Case 8.2: $2\le i\le n$:

$$P_H(u,v):\ u = y_1 z_1 \sim \left\{z_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{y_j:\ 2 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-i\right\} \sim \left\{x_{i-j} w_{i-j}:\ 1 \le j \le i-1\right\} \sim \left\{w_{n-j}:\ 0 \le j \le n-i\right\} = v. \tag{48}$$

Case 9: $u = x_1$ and $v = z_i$, $1 \le i \le n$.

Case 9.1: $i = 1$:

$$P_H(u,v):\ u = x_1 \sim \left\{y_j:\ 1 \le j \le n-1\right\} \sim \left\{x_{n-j}:\ 1 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n\right\} \sim x_1 x_n y_n \sim \left\{z_{n-j}:\ 0 \le j \le n-1\right\} = v. \tag{49}$$

Case 9.2: $2 \le i \le n-1$:

$$P_H(u,v):\ u = x_1 y_1 z_1 \sim \left\{z_{n-j}:\ 0 \le j \le n-i-1\right\} \sim \left\{y_j:\ i+1 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-i-1\right\} \sim \left\{w_j:\ i+1 \le j \le n\right\} \sim w_1 \sim \left\{w_j x_j:\ 2 \le j \le i\right\} \sim \left\{y_{i-j}:\ 0 \le j \le i-2\right\} \sim \left\{z_j:\ 2 \le j \le i\right\} = v. \tag{50}$$

Case 9.3: $i = n$:

$$P_H(u,v):\ u = x_1 w_1 \sim \left\{w_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{x_j:\ 2 \le j \le n\right\} \sim \left\{y_{n-j}:\ 0 \le j \le n-1\right\} \sim \left\{z_j:\ 1 \le j \le n\right\} = v. \tag{51}$$

Case 10: $u = x_1$ and $v = y_i$, $1 \le i \le n$.

Case 10.1: $i = 1$:

$$P_H(u,v):\ u = x_1 \sim \left\{w_j:\ 1 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{y_j:\ 2 \le j \le n\right\} \sim \left\{z_{n-j}:\ 0 \le j \le n-1\right\} \sim y_1 = v. \tag{52}$$

Case 10.2: $2 \le i \le n-1$:

$$P_H(u,v):\ u = x_1 y_1 \sim \left\{z_j:\ 1 \le j \le n\right\} \sim \left\{y_{n-j}:\ 0 \le j \le n-i-1\right\} \sim \left\{x_j w_j:\ i+1 \le j \le n\right\} \sim \left\{w_j:\ 1 \le j \le i\right\} \sim \left\{x_{i-j}:\ 0 \le j \le i-2\right\} \sim \left\{y_j:\ 2 \le j \le i\right\} = v. \tag{53}$$

Case 10.3: $i = n$:

$$P_H(u,v):\ u = x_1 y_1 z_1 \sim \left\{z_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{y_j:\ 2 \le j \le n-1\right\} \sim \left\{x_{n-j}:\ 1 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n\right\} \sim w_1 x_n y_n = v. \tag{54}$$

Case 11: $u = x_1$ and $v = x_i$, $2 \le i \le n$.

Case 11.1: $2 \le i \le n-1$:

$$P_H(u,v)\colon\ u = x_1y_1z_1 \sim \{z_{n-j}\colon\ 0\le j\le n-2\} \sim \{y_j\colon\ 2\le j\le n\} \sim \{x_{n-j}\colon\ 0\le j\le n-i-1\} \sim \{w_j\colon\ i+1\le j\le n\} \sim w_1 \sim \{w_jx_j\colon\ 2\le j\le i\} = v. \tag{55}$$

Case 11.2: $i = n$:

$$P_H(u,v)\colon\ u = x_1y_1 \sim \{z_j\colon\ 1\le j\le n\} \sim \{y_{n-j}\colon\ 0\le j\le n-2\} \sim \{x_j\colon\ 2\le j\le n-1\} \sim \{w_{n-j}\colon\ 0\le j\le n-1\} \sim x_n = v. \tag{56}$$

Case 12: $u = x_1$ and $v = w_i$, $1\le i\le n$.

Case 12.1: $i = 1$:

$$P_H(u,v)\colon\ u = x_1y_1z_1 \sim \{z_{n-j}\colon\ 0\le j\le n-2\} \sim \{y_j\colon\ 2\le j\le n\} \sim \{x_{n-j}\colon\ 0\le j\le n-2\} \sim \{w_j\colon\ 2\le j\le n\} \sim w_1 = v. \tag{57}$$

Case 12.2: $2\le i\le n-1$:

$$P_H(u,v)\colon\ u = x_1y_1z_1 \sim \{z_{n-j}\colon\ 0\le j\le n-2\} \sim \{y_j\colon\ 2\le j\le n\} \sim \{x_{n-j}\colon\ 0\le j\le n-i\} \sim \{w_j\colon\ i+1\le j\le n\} \sim w_1 \sim \{w_jx_j\colon\ 2\le j\le i-1\} \sim w_i = v. \tag{58}$$

Case 12.3: $i = n$:

$$P_H(u,v)\colon\ u = \{x_j\colon\ 1\le j\le n-1\} \sim \{y_{n-j}\colon\ 0\le j\le n-1\} \sim \{z_j\colon\ 1\le j\le n\} \sim y_nx_n \sim \{w_j\colon\ 1\le j\le n\} = v. \tag{59}$$

Case 13: $u = w_1$ and $v = z_i$, $1\le i\le n$.

Case 13.1: $i = 1$:

$$P_H(u,v)\colon\ u = \{w_j\colon\ 1\le j\le n\} \sim \{x_{n-j}\colon\ 0\le j\le n-1\} \sim \{y_j\colon\ 1\le j\le n\} \sim \{z_{n-j}\colon\ 0\le j\le n-1\} = v. \tag{60}$$

Case 13.2: $2\le i\le n$:

$$P_H(u,v)\colon\ u = \{w_jx_j\colon\ 1\le j\le i-1\} \sim \{w_j\colon\ i\le j\le n\} \sim \{x_{n-j}\colon\ 0\le j\le n-i\} \sim \{y_j\colon\ i\le j\le n\} \sim \{y_j\colon\ 1\le j\le i-1\} \sim \{z_{i-j}\colon\ 1\le j\le i-1\} \sim \{z_{n-j}\colon\ 0\le j\le n-i\} = v. \tag{61}$$

Case 14: $u = w_1$ and $v = y_i$, $1\le i\le n$.

Case 14.1: $i = 1$:

$$P_H(u,v):\ u = w_1 \sim \{w_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{z_j:\ 2 \le j \le n\} \sim z_1 y_1 = v. \tag{62}$$

Case 14.2: $2 \le i \le n-1$:

$$P_H(u,v):\ u = \{w_j x_j:\ 1 \le j \le i\} \sim \{w_j:\ i+1 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-i-1\} \sim \{y_j:\ 1 \le j \le i-1\} \sim \{z_{i-j}:\ 1 \le j \le i-1\} \sim \{z_{n-j}:\ 0 \le j \le n-i\} \sim y_i = v. \tag{63}$$

Case 14.3: $i = n$:

$$P_H(u,v):\ u = \{w_j:\ 1 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-1\} \sim \{y_j:\ 1 \le j \le n-1\} \sim \{z_{n-j}:\ 1 \le j \le n-1\} \sim z_n y_n = v. \tag{64}$$

Case 15: $u = w_1$ and $v = x_i$, $1 \le i \le n$.

Case 15.1: $i = 1$:

$$P_H(u,v):\ u = \{w_j:\ 1 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-2\} \sim \{y_j:\ 2 \le j \le n\} \sim \{z_{n-j}:\ 0 \le j \le n-1\} \sim y_1 x_1 = v. \tag{65}$$

Case 15.2: $2 \le i \le n-1$:

$$P_H(u,v): u = \{w_j x_j: 1 \le j \le i-1\} \sim \{w_j: i \le j \le n\} \sim \{x_{n-j}: 0 \le j \le n-i-1\} \sim \{y_j: i+1 \le j \le n\} \sim \{y_j: 1 \le j \le i-1\} \sim \{z_{i-j}: 1 \le j \le i-1\}\{z_{n-j}: 0 \le j \le n-i\} \sim y_i x_i = v. \tag{66}$$

Case 15.3: $i = n$:

$$P_H(u,v):\ u = w_1 \sim \{w_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n-1\} \sim \{y_{n-j}:\ 1 \le j \le n-1\} \sim \{z_j:\ 1 \le j \le n\} \sim y_n x_n = v. \tag{67}$$

Case 16: $u = w_1$ and $v = w_i$, $2 \le i \le n$.

Case 16.1: $2 \le i \le n-1$:

$$P_H(u,v):\ u = w_1 \sim \{w_{n-j}:\ 0 \le j \le n-i-1\} \sim \{x_j:\ i \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{z_j:\ 2 \le j \le n\} \sim z_1 y_1 x_1 \sim \{w_j x_j:\ 2 \le j \le i-1\} \sim w_i = v. \tag{68}$$

Case 16.2: $i = n$:

$$P_H(u,v):\ u = \{w_j:\ 1 \le j \le n-1\} \sim \{x_{n-j}:\ 1 \le j \le n-1\} \sim \{y_j:\ 1 \le j \le n-1\} \sim \{z_{n-j}:\ 1 \le j \le n-1\} \sim z_n y_n x_n w_n = v. \tag{69}$$

Existence of Hamiltonian paths between any two vertices of the n-dimensional convex polytope S_n completes the proof.

Using Theorems 1 and 3, the following proposition computes the detour index of S_n. □

Proposition 2. *Let $G = S_n$, where $n \ge 3$. Then, the detour index of G is*

$$\omega(G) = \frac{4n(4n-1)^2}{2}. \tag{70}$$

Proof. The number of vertices in graph G is $4n$. Replacing $4n$ with n in Theorem 1 completes the proof. □

4. Hamilton Connectivity and the Detour Index of K_n

This section shows that the n-dimensional convex polytope K_n is Hamilton-connected. Its Hamilton connectivity is then used to calculate exact analytical formula of the detour index of K_n.

Malik et al. [25] introduced the family of n-dimensional convex polytope K_n. We have

$$V(K_n) = \{v_i; w_j; x_j; y_j; z_j: 1 \le j \le n\},$$
$$E(K_n) = \{v_jv_{j+1}; w_jw_{j+1}; x_jx_{j+1}; y_jy_{j+1}; z_jz_{j+1}: 1 \le j \le n\} \cup \{v_jy_j; v_jw_{j+1}; w_jx_j; x_jy_j; x_jy_{j+1}; y_jz_j: 1 \le j \le n\}. \quad (71)$$

See Figure 3 to view the n-dimensional convex polytope K_n.

Theorem 4. *The graph n-dimensional convex polytope K_n, with $n \ge 5$, is Hamilton-connected.*

Proof. We prove this result by definition. For this, we have to show that there exists a Hamiltonian path between any pair of vertices of K_n.

Let $P_H(u', v')$ be a Hamiltonian path between vertices u' and v' in K_n. Let $V(R_n) = Z \cup Y \cup X \cup V$ such that $Z = \{z_1, z_2, \ldots, z_n\}$, $Y = \{y_1, y_2 \ldots y_n\}$, $X = \{x_1, x_2, \ldots, x_n\}$, and $V = \{v_1, v_2, \ldots, v_n\}$. We divide the construction of Hamiltonian paths in K_n into a number of cases as follows.

Case 1: $u' = z_1$ and $v' = z_i$, $2 \le i \le n$.

Case 1.1: $2 \le i \le n-1$:

$$P_H(u', v'):\ u' = z_1 \sim \{z_{n-j}:\ 0 \le j \le n-i-1\} \sim \{y_j:\ i+1 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-i\} \sim \{y_{i-j}x_{i-j-1}:\ 0 \le j \le i-3\}$$
$$\sim \{w_j:\ 2 \le j \le n\} \sim \{v_{n-j}:\ 0 \le j \le n-1\} \sim w_1x_1y_1y_2 \sim \{z_j:\ 2 \le j \le i\} = v'. \quad (72)$$

Case 1.2: $i = n$:

$$P_H(u', v'):\ u' = \{z_j:\ 1 \le j \le n-1\} \sim \{y_{n-j}:\ 1 \le j \le n-1\} \sim \{x_j:\ 1 \le j \le n-1\} \sim \{w_{n-j}:\ 1 \le j \le n-1\}$$
$$\sim \{v_j:\ 1 \le j \le n\} \sim w_nx_ny_nz_n = v'. \quad (73)$$

Case 2: $u' = z_1$ and $v' = y_i$, $1 \le i \le n$.

Case 2.1: $1 \le i \le n-1$:

$$P_H(u', v'):\ u' = \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-i-1\} \sim \{x_j:\ i+1 \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-i-1\}$$
$$\sim \{v_j:\ i+1 \le j \le n\} \sim \{v_jw_j:\ 1 \le j \le i\} \sim \{x_{i-j}:\ 0 \le j \le i-1\} \sim \{y_j:\ 1 \le j \le i\} = v'. \quad (74)$$

Case 2.2: $i = n$:

$$P_H(u', v'):\ u' = z_1 \sim \{z_{n-j}:\ 0 \le j \le n-2\} \sim \{y_j:\ 2 \le j \le n-1\} \sim \{x_{n-j}:\ 1 \le j \le n-2\} \sim \{w_j:\ 2 \le j \le n\}$$
$$\sim \{v_{n-j}:\ 0 \le j \le n-1\} \sim w_1x_1y_1x_ny_n = v'. \quad (75)$$

Case 3: $u' = z_1$ and $v' = x_i$, $1 \le i \le n$.

Case 3.1: $1 \le i \le n-1$:

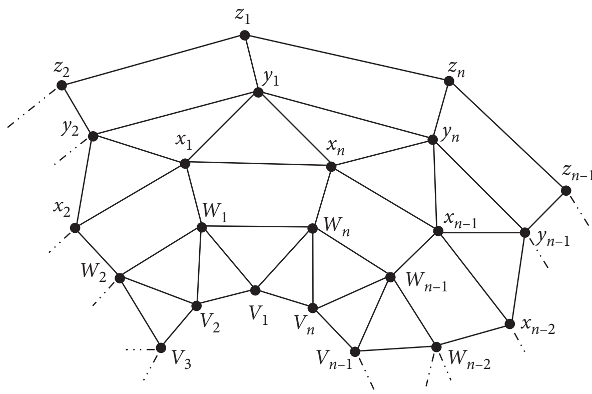

FIGURE 3: The graph of n-dimensional convex polytope K_n.

$$P_H(u',v'):\ u'=\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-1\}\sim\{x_{n-j}:\ 0\le j\le n-i-1\}\sim\{w_j:\ i+1\le j\le n\}\sim v_1 \sim\{v_{n-j}:\ 0\le j\le n-i-1\}\sim\{w_{i-j}v_{i-j}:\ 0\le j\le i-2\}\sim w_1\sim\{x_j:\ 1\le j\le i\}=v'. \tag{76}$$

Case 3.2: $i=n$:

$$P_H(u',v'):\ u'=\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-1\}\sim\{x_j:\ 1\le j\le n-1\}\sim\{w_{n-j}:\ 1\le j\le n-1\} \sim\{v_j:\ 1\le j\le n\}\sim w_nx_n=v'. \tag{77}$$

Case 4: $u'=z_1$ and $v'=w_i$, $1\le i\le n$.

Case 4.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u'=\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-1\}\sim\{x_j:\ 1\le j\le n\}\sim\{w_{n-j}:\ 0\le j\le n-i-1\}\sim\{v_j:\ i+1\le j\le n\} \sim\{v_jw_j:\ 1\le j\le i\}=v'. \tag{78}$$

Case 4.2: $i=n$:

$$P_H(u',v'):\ u'=\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-1\}\sim\{x_j:\ 1\le j\le n-2\}\sim\{w_{n-j}:\ 2\le j\le n-1\}\sim\{v_j:\ 1\le j\le n\} \sim w_{n-1}x_{n-1}x_nw_n=v'. \tag{79}$$

Case 5: $u'=z_1$ and $v'=v_i$, $1\le i\le n$.

Case 5.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u'=\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-1\}\sim\{x_j:\ 1\le j\le n\}\sim\{w_{n-j}:\ 0\le j\le n-i\}\sim\{v_j:\ i+1\le j\le n\} \sim\{v_jw_j:\ 1\le j\le i-1\}\sim v_i=v'. \tag{80}$$

Case 5.2: $i = n$:

$$P_H(u', v'):\ u' = \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-1\} \sim \{x_j:\ 1 \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-1\} \sim \{v_j:\ 1 \le j \le n\} = v'. \tag{81}$$

Case 6: $u' = y_1$ and $v' = z_i$, $1 \le i \le n$:

Case 6.1: $1 \le i \le n-1$:

$$\begin{aligned} P_H(u', v'):\ u' = {} & \{y_j x_j:\ 1 \le j \le i\} \sim \{x_j:\ i+1 \le j \le n-1\} \sim \{w_{n-j}:\ 1 \le j \le n-1\} \sim \{v_j:\ 1 \le j \le n\} \\ & \sim \{y_{n-j}:\ 0 \le j \le n-i-1\} \sim \{z_j:\ i+1 \le j \le n\} \sim \{z_j:\ 1 \le j \le i\} = v'. \end{aligned} \tag{82}$$

Case 6.2: $i = n$:

$$\begin{aligned} P_H(u', v'):\ u' = {} & y_1 \sim \{z_j:\ 1 \le j \le n-1\} \sim \{y_{n-j}:\ 1 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n-1\} \sim \{w_{n-j}:\ 1 \le j \le n-1\} \\ & \sim \{v_j:\ 1 \le j \le n\} \sim w_n x_n y_n z_n = v'. \end{aligned} \tag{83}$$

Case 7: $u' = y_1$ and $v' = y_i$, $2 \le i \le n$

Case 7.1: $2 \le i \le n-1$:

$$\begin{aligned} P_H(u', v'):\ u' = {} & y_1 \sim \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-i-1\} \sim \{x_j:\ i \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-2\} \\ & \sim \{v_j:\ 2 \le j \le n\} \sim v_1 w_1 \sim \{x_{j-1} y_j:\ 2 \le j \le i\} = v'. \end{aligned} \tag{84}$$

Case 7.2: $i = n$:

$$\begin{aligned} P_H(u', v'):u' = {} & y_1 z_1 \sim \{z_{n-j}:0 \le j \le n-2\} \sim \{y_j:2 \le j \le n-1\} \sim \{x_{n-j}:1 \le j \le n-1\} \sim \{w_j:1 \le j \le n-1\} \\ & \sim \{v_{n-j}:0 \le j \le n-1\} \sim w_n x_n y_n = v'. \end{aligned} \tag{85}$$

Case 8: $u' = y_1$ and $v' = x_i$, $1 \le i \le n$.

Case 8.1: $1 \le i \le n-1$:

$$\begin{aligned} P_H(u', v'):\ u' = {} & y_1 z_1 \sim \{z_{n-j}:\ 0 \le j \le n-2\} \sim \{y_j:\ 2 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-i-1\} \sim w_{i+1} v_{i+1} \\ & \sim \{v_j w_j:\ i+2 \le j \le n\} \sim \{v_j:\ 1 \le j \le i\} \sim \{w_{i-j}:\ 0 \le j \le i-1\}\{x_j:\ 1 \le j \le i\} = v'. \end{aligned} \tag{86}$$

Case 8.2: $i = n$:

$$P_H(u',v'):\ u' = y_1 \sim \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n-1\} \sim \{w_{n-j}:\ 1 \le j \le n-1\} \sim \{v_j:\ 1 \le j \le n\} \sim w_n x_n = v'. \tag{87}$$

Case 9: $u' = y_1$ and $v' = w_i$, $1 \le i \le n$.

Case 9.1: $1 \le i \le n-1$:

$$P_H(u',v'):\ u' = y_1 \sim \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-i-1\} \sim \{v_j:\ i+1 \le j \le n\} \sim \{v_j w_j:\ 1 \le j \le i\} = v'. \tag{88}$$

Case 9.2: $i = n$:

$$P_H(u',v'):\ u' = y_1 z_1 \sim \{z_{n-j}:\ 0 \le j \le n-2\} \sim \{y_j:\ 2 \le j \le n\} \sim \{x_{n-j}:\ 0 \le j \le n-1\} \sim \{w_j:\ 1 \le j \le n-1\} \sim \{v_{n-j}:\ 0 \le j \le n-1\} \sim w_n = v'. \tag{89}$$

Case 10: $u' = y_1$ and $v' = v_i$, $1 \le i \le n$.

Case 10.1: $1 \le i \le n-1$:

$$P_H(u',v'):\ u' = y_1 \sim \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-i\} \sim \{v_j:\ i+1 \le j \le n\} \sim \{v_j w_j:\ 1 \le j \le i-1\} \sim v_i = v'. \tag{90}$$

Case 10.2: $i = n$:

$$P_H(u',v'):\ u' = y_1 \sim \{z_j:\ 1 \le j \le n\} \sim \{y_{n-j}:\ 0 \le j \le n-2\} \sim \{x_j:\ 1 \le j \le n\} \sim \{w_{n-j}:\ 0 \le j \le n-1\} \sim \{v_j:\ 1 \le j \le n\} = v'. \tag{91}$$

Case 11: $u' = x_1$ and $v' = z_i$, $1 \le i \le n$.

Case 11.1: $1 \le i \le n-1$:

$$P_H(u',v'):\ u' = x_1 \sim \{y_j x_j:\ 2 \le j \le i\} \sim \{w_{i-j}:\ 0 \le j \le i-1\} \sim \{w_{n-j}:\ 0 \le j \le n-i-2\} \sim \{v_j:\ i+2 \le j \le n\} \sim \{v_j:\ 1 \le j \le i+1\} \sim w_{i+1} \sim \{x_j:\ i+1 \le j \le n\} \sim y_1 \sim \{y_{n-j}:\ 0 \le j \le n-i-1\} \sim \{z_j:\ i+1 \le j \le n\} \sim \{z_j:\ 1 \le j \le i\} = v'. \tag{92}$$

Case 11.2: $i = n$:

$$P_H(u',v'):\ u' = x_1w_1 \sim \{v_j:\ 1\le j\le n\} \sim \{w_{n-j}:\ 0\le j\le n-2\} \sim \{x_j:\ 2\le j\le n\} \sim \{y_{n-j}:\ 0\le j\le n-1\} \sim \{z_j:\ 1\le j\le n\} = v'. \tag{93}$$

Case 12: $u' = x_1$ and $v' = y_i$, $1\le i\le n$.

Case 12.1: $i = 1$:

$$P_H(u',v'):\ u' = x_1w_1v_1 \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 2\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-2\} \sim \{y_j:\ 2\le j\le n\} \sim \{z_{n-j}:\ 0\le j\le n-1\} \sim y_1 = v'. \tag{94}$$

Case 12.2: $2\le i\le n-1$:

$$P_H(u',v'):\ u' = x_1w_1v_1 \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 2\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-i\} \sim \{y_j:\ i+1\le j\le n\} \sim \{z_{n-j}:\ 0\le j\le n-1\} \sim y_1 \sim \{y_jx_j:\ 2\le j\le i-1\} \sim y_i = v'. \tag{95}$$

Case 12.3: $i = n$:

$$P_H(u',v'):\ u' = x_1w_1v_1 \sim \{z_{n-j}:\ 0\le j\le n-2\} \sim \{y_j:\ 2\le j\le n-1\} \sim \{x_{n-j}:\ 1\le j\le n-2\} \sim \{w_j:\ 2\le j\le n-1\} \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim w_1v_1w_nx_ny_n = v'. \tag{96}$$

Case 13: $u' = x_1$ and $v' = x_i$, $2\le i\le n$.

Case 13.1: $i = 2$:

$$P_H(u',v'):\ u' = x_1 \sim \{y_j:\ 1\le j\le n-1\} \sim \{z_{n-j}:\ 1\le j\le n-1\} \sim z_ny_n \sim \{x_{n-j}:\ 0\le j\le n-3\} \sim w_3v_3 \sim \{v_jw_j:\ 4\le j\le n\} \sim v_1w_1v_2w_2x_2 = v'. \tag{97}$$

Case 13.2: $3\le i\le n-1$:

$$P_H(u',v'):\ u' = x_1y_1 \sim \{y_jx_j:\ 2\le j\le i-1\} \sim \{y_j:\ i\le j\le n-1\} \sim \{z_{n-j}:\ 1\le j\le n-1\} \sim z_ny_n \sim \{x_{n-j}:\ 0\le j\le n-i-1\} \sim w_{i+1}v_{i+1} \sim \{v_jw_j:\ i+2\le j\le n\} \sim \{v_jw_j:\ 1\le j\le i\} \sim x_i = v'. \tag{98}$$

Case 13.3: $i = n$:

$$P_H(u',v'):\ u' = x_1w_1 \sim \{v_j:\ 1\le j\le n\} \sim \{w_{n-j}:\ 0\le j\le n-2\} \sim \{x_j:\ 2\le j\le n-1\} \sim \{y_{n-j}:\ 1\le j\le n-1\} \sim \{z_j:\ 1\le j\le n\} \sim y_nx_n = v'. \tag{99}$$

Case 14: $u' = x_1$ and $v' = w_i$, $1\le i\le n$.

Case 14.1: $1\le i\le n-1$:

$$P_H(u',v')\colon\ u' = x_1y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{v_j\colon\ i+1\le j\le n\right\} \sim \left\{v_jw_j\colon\ 1\le j\le i\right\} = v'. \tag{100}$$

Case 14.2: $i = n$:

$$P_H(u',v')\colon\ u' = x_1w_1 \sim \left\{v_j\colon\ 1\le j\le n\right\} \sim \left\{w_{n-j}\colon\ 1\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n-1\right\} \sim \left\{y_{n-j}\colon\ 1\le j\le n-1\right\} \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim y_nx_nw_n = v'. \tag{101}$$

Case 15: $u' = x_1$ and $v' = v_i$, $1\le i\le n$.

Case 15.1: $1\le i\le n-1$:

$$P_H(u',v')\colon\ u' = x_1y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-i\right\} \sim \left\{v_j\colon\ i+1\le j\le n\right\} \sim \left\{v_jw_j\colon\ 1\le j\le i-1\right\} \sim v_i = v'. \tag{102}$$

Case 15.2: $i = n$:

$$P_H(u',v')\colon\ u' = x_1y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-1\right\} \sim \left\{v_j\colon\ 1\le j\le n\right\} = v'. \tag{103}$$

Case 16: $u' = w_1$ and $v' = z_i$, $1\le i\le n$.

Case 16.1: $1\le i\le n-1$:

$$P_H(u',v')\colon\ u' = w_1v_1 \sim \left\{v_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{w_j\colon\ 2\le j\le n\right\} \sim x_n \sim \left\{x_j\colon\ 1\le j\le i\right\} \sim \left\{y_{i-j}\colon\ 0\le j\le i-1\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-i-1\right\} \sim \left\{z_j\colon\ i+1\le j\le n\right\} \sim \left\{z_j\colon\ 1\le j\le i\right\} = v'. \tag{104}$$

Case 16.2: $i = n$:

$$P_H(u',v')\colon\ u' = w_1 \sim \left\{v_j\colon\ 1\le j\le n\right\} \sim \left\{w_{n-j}\colon\ 0\le j\le n-2\right\} \sim \left\{x_j\colon\ 2\le j\le n\right\} \sim \left\{y_{n-j}\colon\ 0\le j\le n-2\right\} \sim x_1y_1 \sim \left\{z_j\colon\ 1\le j\le n\right\} = v'. \tag{105}$$

Case 17: $u' = w_1$ and $v' = y_i$, $1\le i\le n$.

Case 17.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u' = w_1 w_n v_1 \sim \left\{v_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n-1\right\} \sim \left\{x_{n-j} y_{n-j}:\ 1 \le j \le n-i-1\right\} \sim \left\{x_{i-j}:\ 0 \le j \le i-1\right\} \sim x_n y_n \sim \left\{z_{n-j}:\ 0 \le j \le n-1\right\} \sim \left\{y_j:\ 1 \le j \le i\right\} = v'. \tag{106}$$

Case 17.2: $i = n$:

$$P_H(u',v'):\ u' = w_1 v_1 \sim \left\{v_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-1\right\} \sim \left\{y_j:\ 1 \le j \le n-1\right\} \sim \left\{z_{n-j}:\ 1 \le j \le n-1\right\} \sim z_n y_n = v'. \tag{107}$$

Case 18: $u' = w_1$ and $v' = x_i$, $1 \le i \le n$.

Case 18.1: $1 \le i \le n-1$:

$$P_H(u',v'):\ u' = w_1 v_1 \sim \left\{v_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-i-1\right\} \sim \left\{y_j:\ i+1 \le j \le n\right\} \sim \left\{z_{n-j}:\ 0 \le j \le n-1\right\} \sim \left\{y_j x_j:\ 1 \le j \le i\right\} = v'. \tag{108}$$

Case 18.2: $i = n$:

$$P_H(u',v'):\ u' = w_1 \sim \left\{v_j:\ 1 \le j \le n\right\} \sim \left\{w_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{x_j:\ 2 \le j \le n-1\right\} \sim \left\{y_{n-j}:\ 1 \le j \le n-2\right\} \sim x_1 y_1 \sim \left\{z_j:\ 1 \le j \le n\right\} \sim y_n x_n = v'. \tag{109}$$

Case 19: $u' = w_1$ and $v' = w_i$, $2 \le i \le n$.

Case 19.1: $2 \le i \le n-1$:

$$P_H(u',v'):\ u' = w_1 x_1 y_1 \sim \left\{z_j:\ 1 \le j \le n\right\} \sim \left\{y_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{x_j:\ 2 \le j \le n\right\} \sim \left\{w_{n-j}:\ 0 \le j \le n-i-1\right\} \sim \left\{v_j:\ i+1 \le j \le n\right\} \sim v_1 \sim \left\{v_j w_j:\ 2 \le j \le i\right\} = v'. \tag{110}$$

Case 19.2: $i = n$:

$$P_H(u',v'):\ u' = w_1 x_1 y_1 z_1 \sim \left\{z_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{y_j:\ 2 \le j \le n\right\} \sim \left\{x_{n-j}:\ 0 \le j \le n-2\right\} \sim \left\{w_j:\ 2 \le j \le n-1\right\} \sim \left\{v_{n-j}:\ 1 \le j \le n-1\right\} \sim v_n w_n = v'. \tag{111}$$

Case 20: $u' = w_1$ and $v' = v_i$, $1 \le i \le n$.

Case 20.1: $i = 1$:

$$P_H(u',v'):\ u' = w_1x_1y_1 \sim \{z_j:\ 1\le j\le n\} \sim \{y_{n-j}:\ 0\le j\le n-2\} \sim \{x_j:\ 2\le j\le n\} \sim \{w_{n-j}:\ 0\le j\le n-2\} \sim \{v_j:\ 2\le j\le n\} \sim v_1 = v'. \tag{112}$$

Case 20.2: $2\le i\le n-1$:

$$P_H(u',v'):\ u' = w_1x_1y_1 \sim \{z_j:\ 1\le j\le n\} \sim \{y_{n-j}:\ 0\le j\le n-2\} \sim \{x_j:\ 2\le j\le n\} \sim \{w_{n-j}:\ 0\le j\le n-i\} \sim \{v_j:\ i+1\le j\le n\} \sim v_1 \sim \{v_jw_j:\ 2\le j\le i-1\} \sim v_i = v'. \tag{113}$$

Case 20.3: $i=n$:

$$P_H(u',v'):\ u' = w_1x_1y_1z_1 \sim \{z_{n-j}:\ 0\le j\le n-2\} \sim \{y_j:\ 2\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 2\le j\le n\} \sim \{v_j:\ 1\le j\le n\} = v'. \tag{114}$$

Case 21: $u' = v_1$ and $v' = z_i$, $1\le i\le n$.

Case 21.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u' = v_1 \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 1\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-i-1\} \sim \{y_j:\ i+2\le j\le n\} \sim \{y_jx_j:\ 1\le j\le i\} \sim y_{i+1} \sim \{z_j:\ i+1\le j\le n\} \sim \{z_j:\ 1\le j\le i\} = v'. \tag{115}$$

Case 21.2: $i=n$:

$$P_H(u',v'):\ u' = \{v_j:\ 1\le j\le n\} \sim \{w_{n-j}:\ 0\le j\le n-1\} \sim \{x_j:\ 1\le j\le n\} \sim \{y_{n-j}:\ 0\le j\le n-1\} \sim \{z_j:\ 1\le j\le n\} = v'. \tag{116}$$

Case 22: $u' = v_1$ and $v' = y_i$, $1\le i\le n$.

Case 22.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u' = v_1 \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 1\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-i\} \sim \{y_j:\ i+1\le j\le n\} \sim \{z_{n-j}:\ 0\le j\le n-1\} \sim \{y_jx_j:\ 1\le j\le i-1\} \sim y_i = v'. \tag{117}$$

Case 22.2: $i=n$:

$$P_H(u',v'):\ u' = v_1 \sim \{v_{n-j}:\ 0\le j\le n-2\} \sim \{w_j:\ 1\le j\le n\} \sim \{x_{n-j}:\ 0\le j\le n-1\} \sim \{y_j:\ 1\le j\le n-1\} \sim \{z_{n-j}:\ 1\le j\le n-1\} \sim z_ny_n = v'. \tag{118}$$

Case 23: $u' = v_1$ and $v' = x_i$, $1\le i\le n$.

Case 23.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u'=v_1\sim\{v_{n-j}:\ 0\le j\le n-2\}\sim\{w_j:\ 1\le j\le n\}\sim\{x_{n-j}:\ 0\le j\le n-i-1\}\sim\{y_j:\ i+1\le j\le n\}\sim\{z_{n-j}:\ 0\le j\le n-1\}\sim\{y_jx_j:\ 1\le j\le i\}=v'. \tag{119}$$

Case 23.2: $i=n$:

$$P_H(u',v'):\ u'=\{v_j:\ 1\le j\le n\}\sim\{w_{n-j}:\ 0\le j\le n-1\}\sim\{x_j:\ 1\le j\le n-1\}\sim\{y_{n-j}:\ 1\le j\le n-1\}\sim\{z_j:\ 1\le j\le n\}\sim y_nx_n=v'. \tag{120}$$

Case 24: $u'=v_1$ and $v'=w_i$, $1\le i\le n$.

Case 24.1: $1\le i\le n-1$:

$$P_H(u',v'):\ u'=\{v_jw_j:\ 1\le j\le i-1\}\sim\{v_j:\ i\le j\le n\}\sim\{w_{n-j}:\ 0\le j\le n-i-1\}\sim\{x_j:\ i+1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-2\}\sim\{z_j:\ 2\le j\le n\}\sim z_1y_1\sim\{x_j:\ 1\le j\le i\}\sim w_i=v'. \tag{121}$$

Case 24.2: $i=n$:

$$P_H(u',v'):\ u'=v_1\sim\{v_{n-j}:\ 0\le j\le n-2\}\sim\{w_j:\ 1\le j\le n-1\}\sim\{x_{n-j}:\ 1\le j\le n-1\}\sim\{y_j:\ 1\le j\le n-1\}\sim\{z_{n-j}:\ 1\le j\le n-1\}\sim z_ny_nx_nw_n=v'. \tag{122}$$

Case 25: $u'=v_1$ and $v'=v_i$, $2\le i\le n$.

Case 25.1: $2\le i\le n-1$:

$$P_H(u',v'):\ u'=v_1\sim\{v_{n-j}:\ 0\le j\le n-i-1\}\sim\{w_j:\ i\le j\le n\}\sim\{x_{n-j}:\ 0\le j\le n-2\}\sim\{y_j:\ 2\le j\le n\}\sim\{z_{n-j}:\ 0\le j\le n-1\}\sim y_1x_1\sim\{w_{j-1}v_j:\ 2\le j\le i\}=v'. \tag{123}$$

Case 25.2: $i=n$:

$$P_H(u',v'):\ u'=v_1w_1x_1y_1\sim\{z_j:\ 1\le j\le n\}\sim\{y_{n-j}:\ 0\le j\le n-2\}\sim\{x_j:\ 2\le j\le n\}\sim\{w_{n-j}:\ 0\le j\le n-2\}\sim\{v_j:\ 2\le j\le n\}=v'. \tag{124}$$

Existence of Hamiltonian paths between any two vertices of the n-dimensional convex polytope K_n completes the proof. □

Using Theorems 1 and 4, the following proposition computes the detour index of K_n. □

Proposition 3. *Let $G=K_n$, where $n\ge 3$. Then the detour index of G is*

$$\omega(G)=\frac{5n(5n-1)^2}{2}. \tag{125}$$

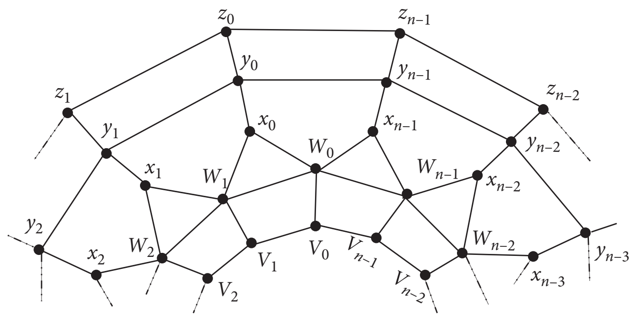

Figure 4: The graph of convex polytope B_n.

Proof. The number of vertices in graph G is $5n$. Replacing $5n$ with n in Theorem 1 completes the proof. □

5. A Family of Non-Hamilton-connected Convex Polytopes

For $n \geq 4$, by B_n, we denote the graph of convex polytope defined in [24] which consists of n number of 3-sided faces, $2n$ number of 4-sided faces, and n number of pentagonal faces, see Figure 4.

Mathematically, the vertex set of B_n consists of four layers of vertices, i.e., v_i, w_i, x_i, y_i, and z_i. That is to say that $V(B_n) = \{v_i, w_i, x_i, y_i, z_i\colon 0 \leq i \leq n-1\}$. Accordingly, the edge set of B_n is as follows:

$$E(B_n) = \{v_i v_{i+1}, v_i w_i, w_i w_{i+1}, w_i x_i, w_{i+1} x_i, x_i y_i, y_i y_{i+1}, y_i z_i, z_i z_{i+1}\colon 0 \leq i \leq n-1\}. \quad (126)$$

Note that arithmetics in the subscripts is performed modulo $n-1$. Layer comprising vertices $v_i\,(0 \leq i \leq n-1)$ is called the first layer. Similarly, layers composed by vertices $w_i\,(0 \leq i \leq n-1)$, $x_i\,(0 \leq i \leq n-1)$, $y_i\,(0 \leq i \leq n-1)$, and $z_i\,(0 \leq i \leq n-1)$ are called second layer, third layer, fourth layer, and fifth layer, respectively. Note that first, second, fourth, and fifth layers form cycles of length n.

The following result shows that the infinite family B_n of convex polytopes is non-Hamilton-connected.

Theorem 5. *The n-dimensional convex polytope B_n, with $n \geq 4$, is non-Hamilton-connected.*

Proof. It is enough to show that there exist two vertices in the n-dimensional convex polytope B_n such that no Hamiltonian path exists between them.

It is easy to see that there exists no Hamiltonian path between any two vertices at distance two on the outer layer, i.e., between z_i and z_{i+2} if n is even and between consecutive vertices on the outer layer, i.e., z_i and z_{i+1} if n is odd, where $0 \leq i \leq n-1$ such that subscripts are taken modulo $n-1$. This shows that the n-dimensional convex polytope B_n is non-Hamilton-connected. □

6. Conclusion

Determining whether or not a graph is Hamilton-connected is NP-complete graph. Thus, it is natural to study the Hamilton-connectivity of infinite families of graphs. In this paper, we have studied Hamilton-connectivity of certain infinite families of convex polytopes. We construct three infinite families of convex polytopes which have been shown to be Hamilton-connected. Moreover, one infinite family is shown to be non-Hamilton-connected which shows that not all the convex polytopes are Hamilton-connected. As a by-product, we compute the detour index of Hamilton-connected families of convex polytopes.

As future directions, we propose the following problems:

Problem 1. Is there any other way to show Hamilton-connectivity of a given graph?

Problem 2. Baca [18] introduced a family D_n of convex polytopes. Determine whether D_n is Hamilton-connected.

Problem 3. Imran et al. [57] introduced the family T_n of convex polytopes. Determine whether or not T_n is Hamilton-connected.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the article.

Authors' Contributions

S. H. and A. K. devised the methodology and acquired funding. S. H. and S. K. carried out the formal analysis and data curation. S. H. and J.-B. L. wrote the original draft, reviewed the writing, and edited the manuscript. S. H., A. K., and J.-B. L. proofread the manuscript before its final submission. S. H. and A. K. contributed equally to this work.

Acknowledgments

Sakander Hayat was grateful to Dr. Muhammad Imran for providing the registration details to Mayura software. Sakander Hayat was supported by the Higher Education Commission, Pakistan, under Grant no. 20-11682/NRPU/RGM/R&D/HEC/2020. J-B. Liu was supported by Anhui Provincial Natural Science Foundation under Grant 2008085J01 and Natural Science Fund of Education Department of Anhui province under Grant no. KJ2020A0478.

References

[1] O. Ore, "Hamilton-connected graphs," *Journal de Mathématiques Pures et Appliquées*, vol. 42, pp. 21–27, 1963.

[2] R. Frucht, "A canonical representation of trivalent Hamiltonian graphs," *Journal of Graph Theory*, vol. 1, no. 1, pp. 45–60, 1976.

[3] V. S. Gordon, Y. L. Orlovich, and F. Werner, "Hamiltonian properties of triangular grid graphs," *Discrete Mathematics*, vol. 308, no. 24, pp. 6166–6188, 2008.

[4] S. Qiang, Z. Qain, and A. Yahui, "The Hamiltonicity of generalized honeycomb torus networks," *Information Processing Letters*, vol. 115, no. 2, pp. 104–111, 2005.

[5] I. A. Stewart, "Sufficient conditions for Hamiltonicity in multiswapped networks," *Journal of Parallel and Distributed Computing*, vol. 101, pp. 17–26, 2017.

[6] B. Wei, "Hamiltonian paths and Hamiltonian connectivity in graphs," *Discrete Mathematics*, vol. 121, no. 1-3, pp. 223–228, 1993.

[7] X. Yang, D. J. Evans, H. Lai, and G. M. Megson, "Generalized honeycomb torus is Hamiltonian," *Information Processing Letters*, vol. 92, no. 1, pp. 31–37, 2004.

[8] G. Chartrand, A. M. Hobbs, H. A. Jung, S. F. Kapoor, and C. S. J. A. Nash-Williams, "The square of a block is Hamiltonian connected," *Journal of Combinatorial Theory, Series B*, vol. 16, no. 3, pp. 290–292, 1974.

[9] C. Thomassen, "Hamiltonian-connected tournaments," *Journal of Combinatorial Theory, Series B*, vol. 28, no. 2, pp. 142–163, 1980.

[10] J.-M. Chang, J.-S. Yang, Y.-L. Wang, and Y. Cheng, "Panconnectivity, fault-tolerant Hamiltonicity and Hamiltonian-connectivity in alternating group graphs," *Networks*, vol. 44, no. 4, pp. 302–310, 2004.

[11] Z. Kewen, H.-J. Lai, and J. Zhou, "Hamiltonian-connected graphs," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2707–2714, 2008.

[12] Q. Zhou and L. Wang, "Some sufficient spectral conditions on Hamilton-connected and traceable graphs," *Linear and Multilinear Algebra*, vol. 65, no. 2, pp. 224–234, 2017.

[13] Q. Zhou, L. Wang, and Y. Lu, "Wiener index and Harary index on Hamilton-connected graphs with large minimum degree," *Discrete Applied Mathematics*, vol. 247, pp. 180–185, 2018.

[14] J. Wei, Z. You, and H.-J. Lai, "Spectral analogues of Erdős' theorem on Hamilton-connected graphs," *Applied Mathematics and Computation*, vol. 340, pp. 242–250, 2019.

[15] R. W. Hung, F. Keshavarz-Kohjerdi, C. B. Lin, and J. S. Chen, "The Hamiltonian connectivity of alphabet supergrid graphs," *International Journal of Applied Mathematics*, vol. 49, no. 1, pp. 1–10, 2019.

[16] Q. Zhou, L. Wang, and Y. Lu, "Signless Laplacian spectral conditions for Hamilton-connected graphs with large minimum degree," *Linear Algebra and Its Applications*, vol. 592, pp. 48–64, 2020.

[17] A. Shabbir, M. F. Nadeem, and T. Zamfirescu, "The property of Hamiltonian connectedness in Toeplitz graphs," *Complexity*, vol. 2020, Article ID 5608720, 6 pages, 2020.

[18] M. Bača, "Face anti-magic labelings of convex polytopes," *Utilitas Mathematica*, vol. 55, pp. 221–226, 1999.

[19] M. Baca, "Labelings of two classes of convex polytopes," *Utilitas Mathematica*, vol. 34, pp. 24–31, 1988.

[20] M. Bača, "On magic labellings of convex polytopes," *Annals of Discrete Mathematics*, vol. 51, pp. 13–16, 1992.

[21] M. Miller, M. Baca, and J. A. MacDougall, "Vertex-magic total labeling of generalized Petersen graphs and convex polytopes," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 59, pp. 89–99, 2006.

[22] M. Imran, A. Q. Baig, and A. Ahmad, "Families of plane graphs with constant metric dimension," *Utilitas Mathematica*, vol. 88, pp. 43–57, 2012.

[23] M. Imran, S. A. U. H. Bokhary, and A. Q. Baig, "On the metric dimension of rotationally-symmetric convex polytopes," *Journal of Algebra Combinatorics Discrete Structures and Applications*, vol. 3, pp. 45–59, 2015.

[24] M. Imran and H. M. A. Siddiqui, "Computing the metric dimension of convex polytopes generated by wheel related graphs," *Acta Mathematica Hungarica*, vol. 149, no. 1, pp. 10–30, 2016.

[25] M. A. Malik and M. Sarwar, "On the metric dimension of two families of convex polytopes," *Afrika Matematika*, vol. 27, no. 1-2, pp. 229–238, 2016.

[26] M. Imran, S. A. u. H. Bokhary, A. Ahmad, and A. Semaničová-feňovčíková, "On classes of regular graphs with constant metric dimension," *Acta Mathematica Scientia*, vol. 33, no. 1, pp. 187–206, 2013.

[27] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, and M. Stojanović, "Minimal doubly resolving sets and the strong metric dimension of some convex polytopes," *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9790–9801, 2012.

[28] H. Raza, S. Hayat, and X.-F. Pan, "On the fault-tolerant metric dimension of convex polytopes," *Applied Mathematics and Computation*, vol. 339, pp. 172–185, 2018.

[29] H. Raza, J.-B. Liu, and S. Qu, "On mixed metric dimension of rotationally symmetric graphs," *IEEE Access*, vol. 8, pp. 11560–11569, 2020.

[30] M. Abas and T. Vetrík, "Metric dimension of Cayley digraphs of split metacyclic groups," *Theoretical Computer Science*, vol. 809, pp. 61–72, 2020.

[31] T. Vetrik, "On the metric dimension of directed and undirected circulant graphs," *Discussiones Mathematicae Graph Theory*, vol. 40, no. 1, pp. 67–76, 2020.

[32] A. Simić, M. Bogdanović, and J. Milošević, "The binary locating-dominating number of some convex polytopes," *Ars Mathematica Contemporanea*, vol. 13, pp. 367–377, 2017.

[33] H. Raza, S. Hayat, and X.-F. Pan, "Binary locating-dominating sets in rotationally-symmetric convex polytopes," *Symmetry*, vol. 10, no. 12, pp. 727–745, 2018.

[34] A. Savic, Z. Maksimovic, and M. Bogdanovic, "The open-locating-dominating number of some convex polytopes," *FILOMAT*, vol. 32, no. 2, pp. 635–642, 2018.

[35] J.-B. Liu, J. Cao, A. Alofi, A. AL-Mazrooei, and A. Elaiw, "Applications of Laplacian spectra for n-prism networks n-Prism Networks," *Neurocomputing*, vol. 198, pp. 69–73, 2016.

[36] J.-B. Liu and J. Cao, "The resistance distances of electrical networks based on Laplacian generalized inverse," *Neurocomputing*, vol. 167, pp. 306–313, 2015.

[37] J.-B. Liu, X.-F. Pan, L. Yu, and D. Li, "Complete characterization of bicyclic graphs with minimal Kirchhoff index," *Discrete Applied Mathematics*, vol. 200, pp. 95–107, 2016.

[38] J.-B. Liu, C. Wang, S. Wang, and B. Wei, "Zagreb indices and multiplicative Zagreb indices of Eulerian graphs," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 1, pp. 67–78, 2019.

[39] J.-B. Liu, J. Zhao, and Z. Cai, "On the generalized adjacency, Laplacian and signless Laplacian spectra of the weighted edge corona networks," *Physica A*, vol. 540, p. 12307, 2020.

[40] I. Lukovits, "The detour index," *Croatica Chemica Acta*, vol. 69, pp. 873–882, 1996.

[41] I. Lukovits, "Indicators for atoms included in cycles," *Journal of Chemical Information and Computer Sciences*, vol. 36, no. 1, pp. 65–68, 1996.

[42] N. Trinajstić, S. Nikolić, B. Lučić, D. Amić, and Z. Mihalić, "The detour matrix in chemistry," *Journal of Chemical Information and Modeling*, vol. 37, pp. 631–638, 1997.

[43] G. Rücker and C. Rücker, "Symmetry-aided computation of the detour matrix and the detour index," *Journal of Chemical Information and Computer Sciences*, vol. 38, no. 4, pp. 710–714, 1998.

[44] I. Lukovits and M. Razinger, "On calculation of the detour index," *Journal of Chemical Information and Computer Sciences*, vol. 37, no. 2, pp. 283–286, 1997.

[45] N. Trinajstić, S. Nikolić, and Z. Mihalić, "On computing the molecular detour matrix," *International Journal of Quantum Chemistry*, vol. 65, pp. 415–419, 1998.

[46] F. Harary, *Graph Theory*, p. 203, Addison-Wesley, Reading, MA, USA, 1969.

[47] A. Mahmiani, O. Khormali, and A. Iranmanesh, "The edge versions of detour index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 62, no. 2, pp. 419–431, 2009.

[48] B. Zhou and X. Cai, "On detour index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 63, pp. 199–210, 2010.

[49] X. Qi and B. Zhou, "Detour index of a class of unicyclic graphs," *Filomat*, vol. 24, no. 1, pp. 29–40, 2010.

[50] C. Du, "Minimum detour index of bicyclic graphs, MATCH Commun," *Mathematical and in Computer Chemistry*, vol. 68, no. 1, pp. 357–370, 2012.

[51] W. Fang, Z. Q. Cai, and X. X. Li, "Minimum detour index of tricyclic graphs," *Journal of Chemistry*, vol. 2019, Article ID 6031568, 8 pages, 2019.

[52] A. Karbasioun, A. R. Ashrafi, and M. V. Diudea, "Distance and detour matrices of an infinite class of dendrimer nanostars," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 63, no. 1, pp. 239–246, 2010.

[53] R. Wu and H. Deng, "On the detour index of a chain of C20 fullerenes," *Journal of Optoelectronic and Biomedical Materials*, vol. 8, no. 2, pp. 45–48, 2016.

[54] V. Kaladevi and A. Abinayaa, "On detour distance Laplacian energy," *Journal of Informatics and Mathematical Sciences*, vol. 9, no. 3, pp. 721–732, 2017.

[55] H. O. Abdullah and Z. I. Omar, "Edge restricted detour index of some graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 23, no. 4, pp. 861–877, 2020.

[56] A. A. Dobrynin, R. Entringer, and I. Gutman, "Wiener index of trees: theory and applications," *Acta Applicandae Mathematicae*, vol. 66, no. 3, pp. 211–249, 2001.

[57] M. Imran, S. A. Ul Haq Bokhary, and A. Q. Baig, "On families of convex polytopes with constant metric dimension," *Computers & Mathematics with Applications*, vol. 60, no. 9, pp. 2629–2638, 2010.