

Research Article

Enhance the Transfer Capacity of Multiplex Networks

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Most complex real systems are found to have multiple layers of connectivity and required to be modelled as multiplex networks. One of the extremely critical problems is to reduce the congestion and enhance the transfer capacity, especially in real communication networks with a big data environment. A novel and effective strategy to improve traffic and control congestion is proposed by adding edges according to their weights which are defined by the topology structural properties. Furthermore, which layer is more effective when our strategy is applied is discussed based on its topology structure. Adding edges between nodes whose product of multiplex network betweenness is the highest is confirmed to be more effective, particularly in the layer with stronger community structure. Simulation experiments on both computer-generated and real-world networks demonstrate that our strategy can enhance the transfer capacity of multiplex networks significantly, which is in good agreement with our analysis.

1. Introduction

The complex real systems can be depicted as complex networks [1, 2] with small-world effects [1] or scale-free properties [2]. These complex systems, such as online social networks, biological systems, and the Internet, are abstracted as a graph, while nodes represent individuals and edges represent the interactions among them. In recent decades, due to the ever-increasing amount of information transfer on the Internet and online social networks, it is of paramount interest to explore the transfer capacity of complex networks [3–7]. Models of packet transfer process in computer networks are presented in many previous studies. In most of the models presented, all nodes in the network are both hosts and routers. Hosts generate packets to be sent to destination addresses and receive packets from other hosts or routers, while routers forward the packets to their destinations according to the specified routing strategy. At each time step, every node in the network generates a packet with a constant packet generated rate λ , with randomly chosen source and destination addresses. Once a packet is generated, it is placed at the tail of the queue with packets generated by itself or transferred from neighbours in the previous time steps. Meanwhile, the first C_i packets at the

head of the queue of each node i are forwarded one step to their destinations and placed at the tail of the queues of the selected nodes. When a packet arrives at its final destination, it will be expelled from the network. The transfer capacity of each node is defined in different ways. In some models, it is considered to be the same in all nodes while to be proportional to its topology metrics such as degree or betweenness in other models. These former studies are focused on the critical value λ_c of the packet-generated rate [3, 5, 8]. We can obtain a phase transition from free state to congestion at the critical point λ_c along with the increase of the packet-generated rate. For $\lambda < \lambda_c$, the amount of generated and transferred packets is balanced and the whole network is in a steady free state. For $\lambda > \lambda_c$, it turns into a jammed state because the node cannot transfer the packet beyond its limited transfer capacity. This critical packet-generated rate λ_c can best reflect the maximum packet transfer capacity of a network. Therefore, it is used to estimate which routing strategy is of more validity. It may have a certain reference value for the real-world networks.

Along with the development of studies on the topology of complex networks, a mesoscopic description, community structure, is found in many networks [9–14]. It is presented that there is a tendency for nodes to divide into communities

within which node-node connections are dense but between which connections are sparse. The modularity Q [12] is defined as a measure of the community structure which can specify a certain mesoscopic description of the network in terms of communities being more or less accurate. The larger value of Q means a more accurate division of community. The influence of community structure on packet transport and how to enhance the transfer capacity based on community structure are also investigated [4, 15, 16].

Traditional studies of complex networks usually assume that all nodes are linked to each other by a certain type of edge to produce a single-layer network. Recently, it has been recognized that lots of complex real systems are not composed by single network primitively, but by multiplex network [17–24]. They consist of a series of N nodes linked by L different kinds of interactions. All interactions of the same kind determine a unique layer/network of the multiplex network. Nodes have diverse neighbours in each layer. For example, the multiplex network of relationships existing among employees of the Computer Science Department of Aarhus University [23] contains 61 nodes (employees) and five different layers (five kinds of online or offline relationships): Facebook, Leisure, Work, Co-authorship, and Lunch. Generally, the routing strategy is based on the shortest-path algorithm in the single-layer network. However, in a multiplex network, there are two distinct kinds of shortest paths: paths that only use a single layer and paths that use more than one layer. The dynamic process in multiplex networks is becoming a hot spot of current research [17–20, 25–27].

Adjusting the network topology structure, such as adding or deleting some edges, is effective to improve the network transfer capability of the single-layer network [4, 28, 29]. The multiplex network consists of two or more single-layer networks with different community structures. How to enhance the transfer capacity based on the interior topology characteristics attracts our attention. By adding edges in the light of the different definitions of edge weight, we present strategies to enhance the transfer capacity of the multiplex network. And the impacts of the nodes number N and the modularity Q of different layers on our strategies will be discussed.

The rest of this paper is organized as follows. Section 2 presents the proposed routing strategies method. Extensive simulation experiments are conducted to validate our routing strategies and make comparisons, and the results are reported in Section 3. Eventually, conclusions are made in Section 4.

2. Materials and Methods

The node betweenness b_i is widely used to access the possible traffic through node i under a specified routing strategy, in general, the shortest-path routing algorithm [30]. It is usually defined as follows:

$$b_i = \sum_{s,t} \frac{\sigma(s,i,t)}{\sigma(s,t)}, \quad (1)$$

where $\sigma(s,i,t)$ is the amount of shortest paths connected nodes s and t while passing through node i and $\sigma(s,t)$ is the total amount of shortest paths connected nodes s and t . The likelihood that a generated packet will travel through node i is $b_i / \sum_{j=1}^N b_j$. Accordingly, the mean number of packets that node i receives at a certain time step is $N * \lambda * b_i / (N * (N - 1)) = \lambda * b_i / (N - 1)$. When the number of incoming packets is no less than the packet transfer capacity of node i , that is, $\lambda * b_i / (N - 1) \geq C_i$, the node is unable to forward all packets to their destination and it will produce a congestion in the network gradually. The critical packet generated rate λ_c is [3, 4]

$$\lambda_c = \min \frac{C_i * (N - 1)}{b_i}. \quad (2)$$

When it turns to a multiplex network, there are two different types of shortest paths: intralayer paths and interlayer paths. When we investigate the dynamics of multiplex network, the number of shortest paths must be considered together with the intralayer paths and interlayer paths through a certain node. The critical packet injection rate of the multiplex network is as follows [19, 25, 27]:

$$\rho_c = \min \frac{\tau_i * (N - 1)}{L * B_i}, \quad (3)$$

where τ_i is the max node processing rate of node i , which is similar to the packet delivery capability C_i in the previous studies. In general, we suppose that all nodes have the same maximum processing rate τ_i and we set $\tau_i = 1$ for simplicity. L is the layer number and B_i is the multiplex betweenness of node i . The critical injection rate ρ_c is used to estimate the maximum transfer capacity of a multiplex network.

In a system divided into m communities, a $m \times m$ symmetric matrix E is used to calculate the modularity whose element e_{ij} is the portion of all edges in the system that connect nodes in two different communities, community i and community j . The modularity measure, Q , is as follows [12, 13]:

$$Q = \sum_i (e_{ii} - a_i^2), \quad (4)$$

where $a_i = \sum_j e_{ij}$.

Different divisions of the network result in different Q , and the maximum is symbolized by Q_{\max} . The greater modularity Q_{\max} implies the stronger community structure inside the system. Multiplex networks always contain two or more single-layer networks with different community structures. Our strategies are as follows:

- (1) In a certain single-layer network of multiplex network, we calculate the weight of all node pairs, W_{ij} , between nodes i and j , in different ways according to different definitions. In the MBP strategy, W_{ij} is equal to the product of their multiplex betweenness $B_i * B_j$. In the SBP strategy, W_{ij} is equal to the product of their node betweenness $b_i * b_j$ using the shortest-path routing algorithm. In the DDP strategy, W_{ij} is equal to the product of their degree $k_i * k_j$.

(2) We sort the weights in decreasing order and add the edge between the node pair whose weight is at the top. If adding an edge will cause multiple edges between the node pair, we will not add the edge, but cope with the node pair rank next.

(3) We renew the weight W_{ij} and duplicate step 2 until a certain f_e of edges are added.

In a simple network with N nodes, $N*(N-1)/2$ edges are the maximum. If the mean degree is $\langle k \rangle$, there exist $N * \langle k \rangle / 2$ edges approximately. The number of edges we can add is $N * (N-1)/2 - N * \langle k \rangle / 2$. We only add 10 percent of the edges we can add at most. A fraction $f_e = 1$ means $0.1 * (N * (N-1)/2 - N * \langle k \rangle / 2)$ edges are added. When each edge is added, we calculate the critical packet injection rate ρ_c of the new multiplex system and record the enhancement as ρ_c / ρ_{c0} where ρ_{c0} is the critical packet injection rate of the original multiplex system.

For comparison, we define the RAN strategy as adding edges randomly (also without multiple edges). And, the validity of our strategies on different layers will be discussed by adapting our strategies in different layers.

3. Results and Discussion

We generate a multiplex network that only consists of two Erdős-Rényi single-layer networks. In each single-layer network, we utilize a series of pseudorandom networks with N nodes and separate these nodes into m communities. Each node has the average number of edges, Z_{in} , connected to the nodes in the same community and Z_{out} connected to the nodes of any other communities, while the average degree $\langle k \rangle = Z_{in} + Z_{out}$ is constant. We can adjust Z_{in} for different community structures. In all simulations, we generate 100 multiplex networks randomly to calculate the average.

Firstly, we generate a multiplex network with 128 nodes which are separated into 4 equal communities while the average degree $\langle k \rangle$ is fixed to be 16. In layer 1, Z_{in} is fixed to be 8, which means the single-layer network in layer 1 is a totally random network. In layer 2, we change Z_{in} from 8 to 11 and 14 to get increasingly pronounced community structure. The results are exhibited in Figure 1.

As shown in Figure 1, when f_e is larger than zero, ρ_c / ρ_{c0} is greater than 1. It means that when some edges are added, the critical packet injection rate of the new multiplex network is greater than that of the original multiplex network. Therefore, our strategies can enhance the transfer capacity of the multiplex network. In Figures 1(a)–1(c), the MBP strategy, adding edges with the highest product of multiplex betweenness, is the most effective one, while the enhancement as ρ_c / ρ_{c0} of the MBP strategy is the highest. The modularity Q_{max} of the single network in layer 1 is about 0.2245. In layer 2, when Z_{in} is 8, the modularity Q_{max} is close to 0.2245. When Z_{in} increases to 11, the modularity Q_{max} is about 0.4404 and 0.6288 for $Z_{in} = 14$. By comparing

Figures 1(a) with 1(b) and 1(c), we can discover that they are more effective in the network with stronger community structure, especially the MBP strategy.

Then, we apply our strategies in different layers to check the impact of community structure on our strategies. In layer 1, we set Z_{in} as 14, which means the single-layer network in layer 1 has a strong community structure. In layer 2, we change Z_{in} from 8 to 14 to obtain the results presented in Figure 2.

From Figure 2, we can observe that the MBP strategy is also the most effective among these strategies. And, when the layer where our strategies are applied has stronger community structure, our strategies are more effective. Comparing Figure 2(a) with Figure 1(c), we can discover that in a multiplex network consisting of two layers with different community structures, applying our strategies in the one with pronounced community structure yields better results.

Afterwards, we double the average degree $\langle k \rangle$ to 32 to check the influence of average degree $\langle k \rangle$. Results are illustrated in Figure 3.

Due to the increase of the average degree $\langle k \rangle$, the absolute number of edges we can add is rising. However, we keep the relative proportion of added edges fixed to 10 percent. The change of average degree almost has no influence on our strategies. Figure 3 also proves that the MBP strategy works better than the other strategies, particularly in the layer with strong community structure.

The impact of communities number m is presented in Figure 4.

From Figures 4(a) and 4(b), we can detect the enhancement of the MBP strategy is still the highest. The increase of communities number results in more accurate division of the network and the stronger community structure. In Figure 4(a), the modularity of layer 2 is about 0.3234, while in Figure 4(b), it is 0.6982. Hence, the enhancement in Figure 4(b) is nearly 268.48%, which is greater than 198.37% in Figure 4(a).

Finally, we check the influence of network size. We generate two-layer multiplex network with $N = 256$ nodes in $m = 8$ communities. The average degree $\langle k \rangle$ and the adjusting parameter of community structure Z_{in} are also increased accordingly. Results are shown in Figure 5.

The enhancement ρ_c / ρ_{c0} of the MBP strategy is the highest as usual regardless of the expansion of network scale. The increase of adjusting parameter of community structure Z_{in} leads to stronger community structure. Figure 5 also validates that our strategies work more efficiently in the network with distinct community structures.

The incessant expansion of the network poses a new challenge to our strategies. Since most of the real networks are rather large with massive number of nodes and edges, it is of vital importance to consider the computational cost of the algorithm required to compute the multiplex network betweenness. The computational complexity of our strategies is mainly dominated by the calculation of the multiplex betweenness. The time complexity of

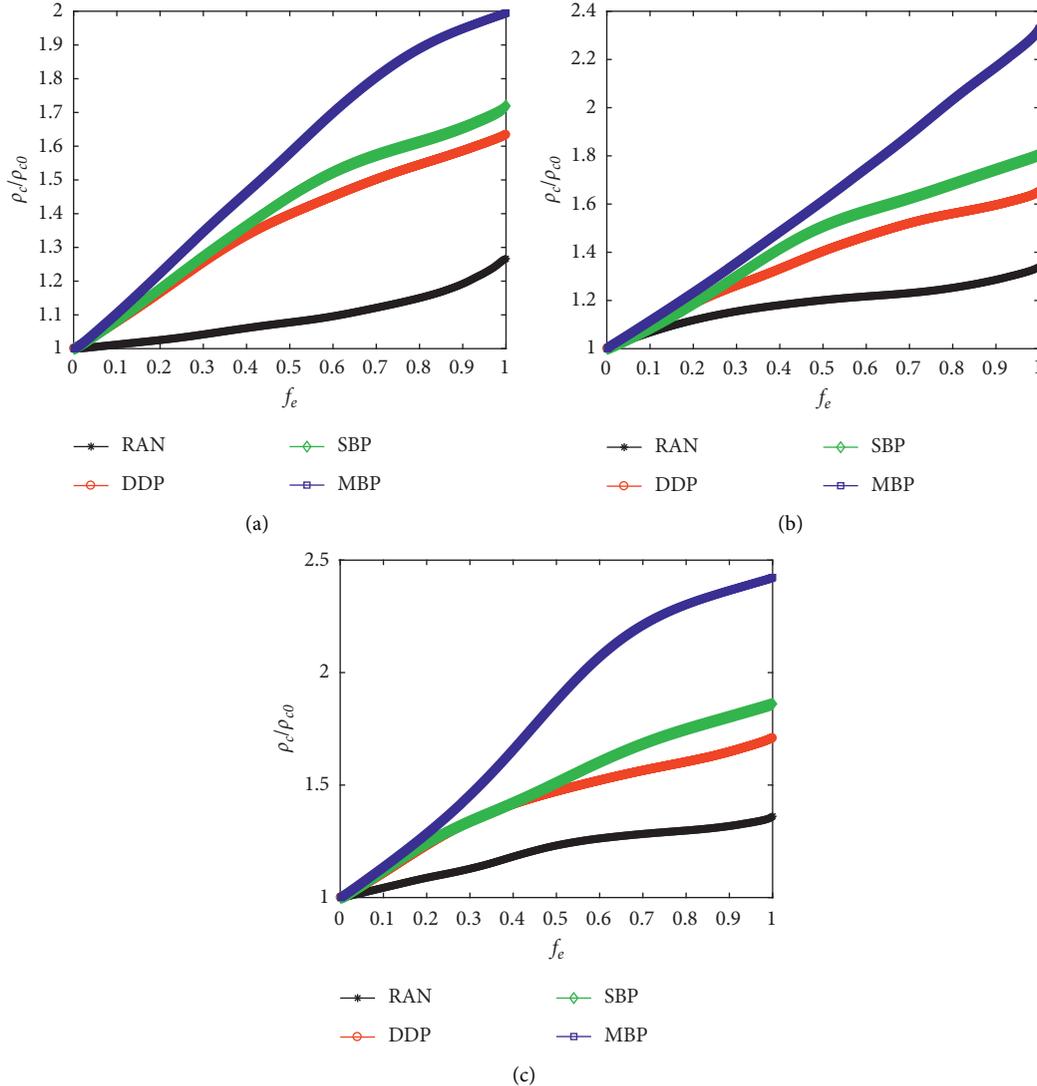


FIGURE 1: The enhancement of our strategies in multiplex network: $(N) = 128$, $(m) = 4$, $\langle(k)\rangle = 16$, and $(Z)_{in} = 8$ in layer 1; in layer 2, (a) $(Z)_{in} = 8$, (b) $(Z)_{in} = 11$, and (c) $(Z)_{in} = 14$.

computing betweenness centrality of unweighted multiplex networks is $O(L*N*M)$ by using Breadth First Search [17] (M is the number of edges) which is acceptable under current circumstances.

The above simulations are run in computer-generated multiplex networks. Then, we verify our three strategies on real-world systems. We apply our strategies to the multiplex network of relationships among employees of the Computer Science Department of Aarhus University [23]. We choose two layers with Giant Components: the “Lunch” network and the “Work” network. The “Lunch” network has 193 edges and the modularity is 0.6548, while the “Work” network has 194 edges and the modularity is 0.4587. We apply our strategies in each layer to test them and get the simulations shown in Figure 6.

As shown in Figure 6(a), the MBP strategy can almost triple the transfer capacity when it is applied in the “Work” network whose modularity is 0.4587. In Figure 6(b), the

enhancement is up to 333.4% when the MBP strategy is used in the “Lunch” network with stronger modularity of 0.6548. It indicates that our strategies are of practical value in enhancing transfer capacity in real multiplex networks.

Adding edges between nodes with the highest product of multiplex betweenness will make the two nodes to connect directly to each other. In general, it will reduce the load of those nodes with high multiplex betweenness. Stronger community structure will result in more nodes with high multiplex betweenness. That is why our strategies are more effective in the network with pronounced community structure. To uncover how our strategies work, we conduct further research on the initial and final multiplex betweenness. We employ the MBP strategy on the computer-generated multiplex network with $N = 128$, $m = 4$, $\langle k \rangle = 16$, and $Z_{in} = 8$ in layer 1 and $Z_{in} = 14$ in layer 2 (the same simulation environment as Figure 1(c)) and the real CS-Aarhus multiplex network (the same simulation

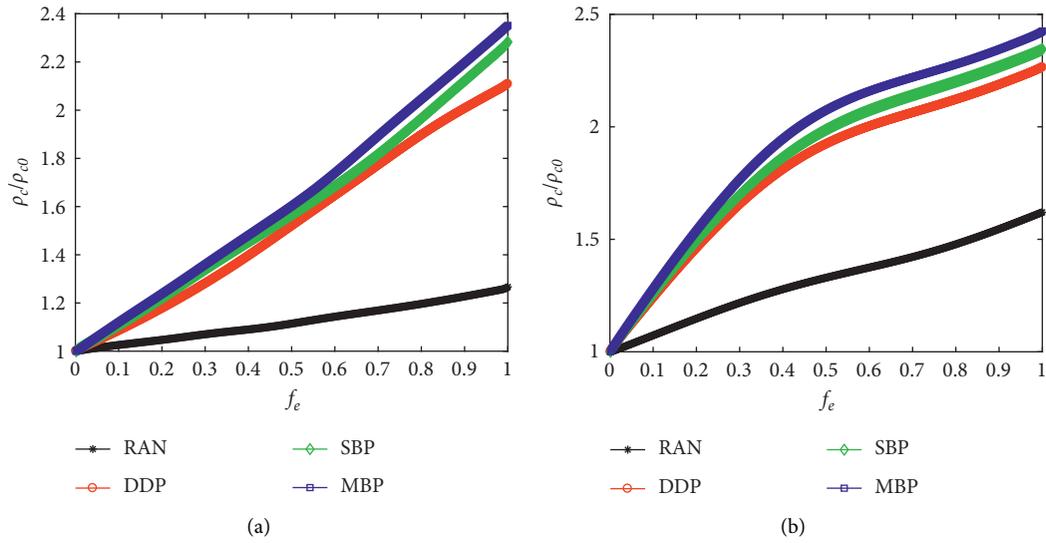


FIGURE 2: The enhancement of our strategies in multiplex network: $(N) = 128$, $(m) = 4$, $\langle k \rangle = 16$, and $(Z)_{in} = 14$ in layer 1; in layer 2, (a) $(Z)_{in} = 8$ and (b) $(Z)_{in} = 14$.

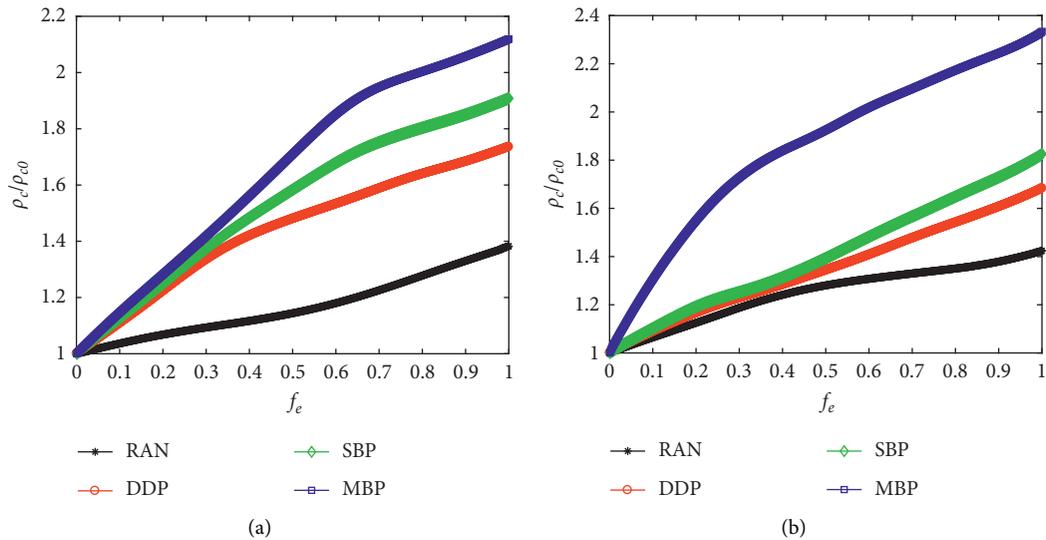


FIGURE 3: The enhancement of our strategies in multiplex network: $(N) = 128$, $(m) = 4$, $\langle k \rangle = 32$, and $(Z)_{in} = 16$ in layer 1; in layer 2, (a) $(Z)_{in} = 16$ and (b) $(Z)_{in} = 28$.

environment as Figure 6(b)). The multiplex betweennesses plot against the node index is presented in Figure 7.

Initially, in the original multiplex network, the multiplex betweennesses are distributed over a pretty wide range (see the square in Figure 7). After applying the MBP strategy, in

the ultimate multiplex network, the multiplex betweennesses are restricted to an extremely narrow region (see the plus sign in Figure 7). There is a significant decrease in the highest multiplex betweenness of all nodes. The multiplex betweennesses are more uniformly distributed after our

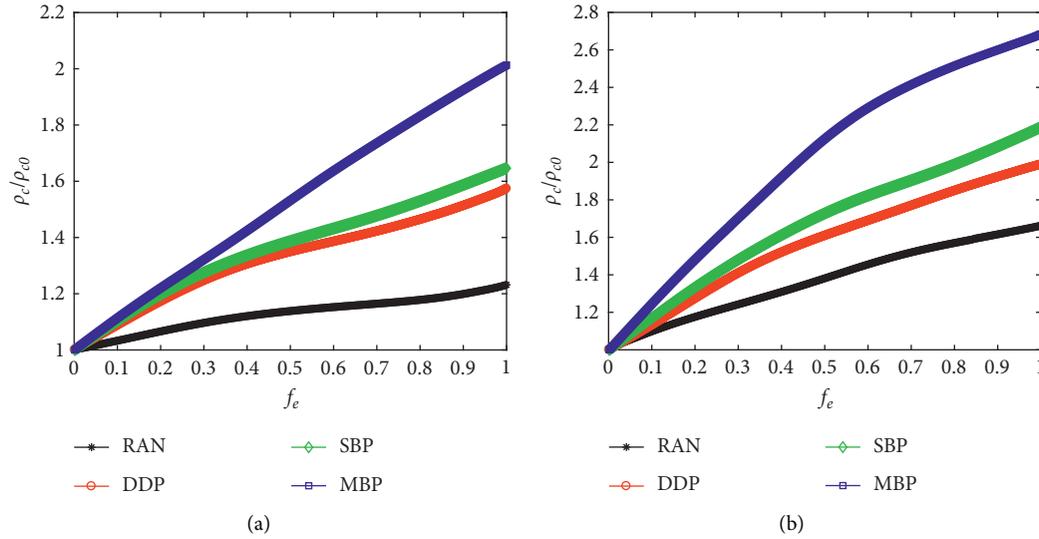


FIGURE 4: The enhancement of our strategies in multiplex network: $(N) = 128$, $(m) = 8$, $\langle(k)\rangle = 16$, and $(Z)_{in} = 8$ in layer 1; in layer 2, (a) $(Z)_{in} = 8$ and (b) $(Z)_{in} = 14$.

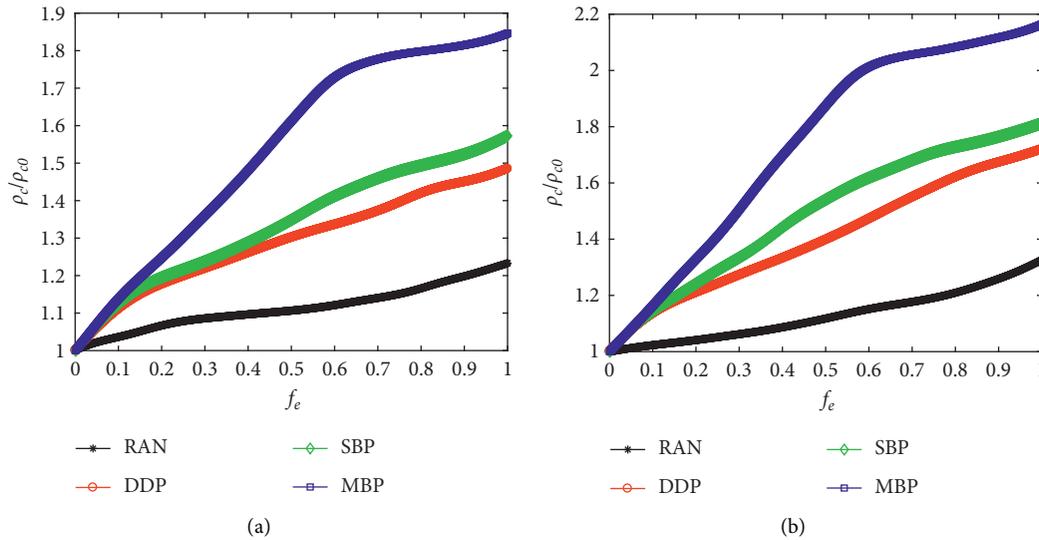


FIGURE 5: The enhancement of our strategies in multiplex network: $(N) = 256$, $(m) = 8$, $\langle(k)\rangle = 32$, and $(Z)_{in} = 16$ in layer 1; in layer 2, (a) $(Z)_{in} = 16$ and (b) $(Z)_{in} = 28$.

strategies are applied. In Figure 7(b), there are a few nodes with comparatively high multiplex betweenness. The sharp declines of these nodes lead to a higher increase of network transfer capacity.

In our strategies, we add edges between nodes which will result in shortcuts between nodes. Therefore, the average path length will be reduced and the small-world phenomenon is still maintained.

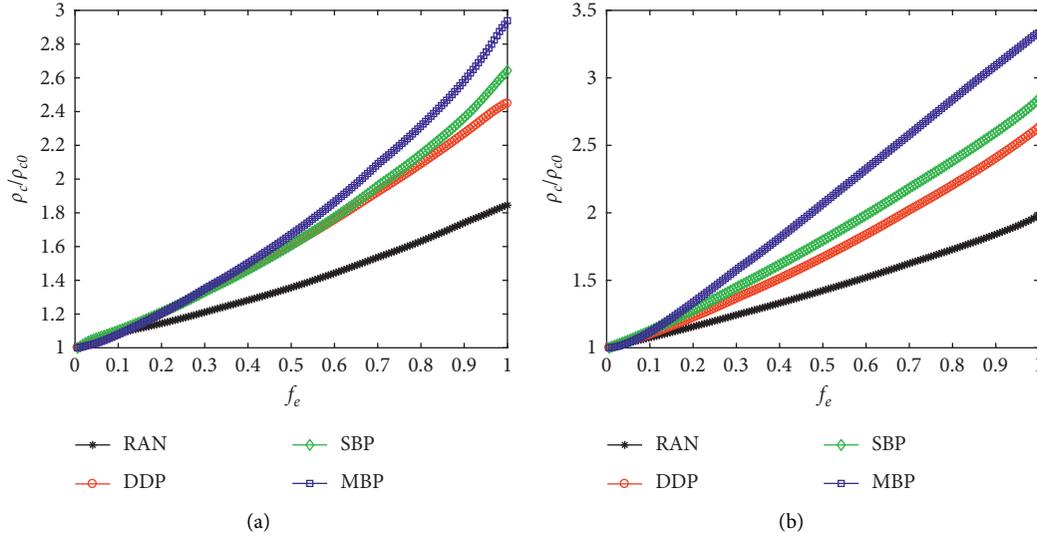


FIGURE 6: The enhancement of our strategies in a real multiplex network. (a) In “Work” network. (b) In “Lunch” network.

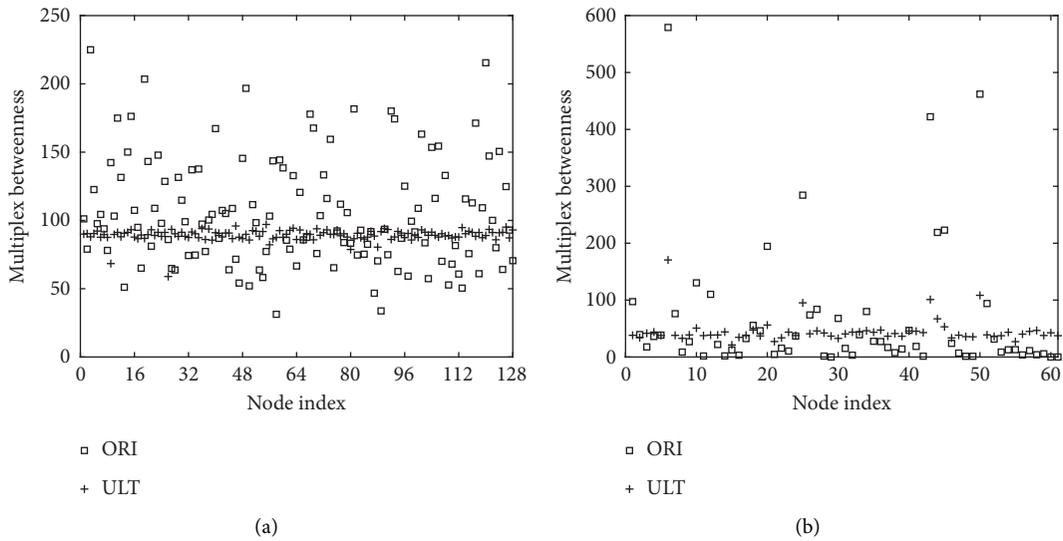


FIGURE 7: The multiplex betweenness of each node before and after applying MBP strategy. (a) The computer-generated multiplex network with $(N) = 128$, $(m) = 4$, $\langle(k)\rangle = 16$, and $(Z)_{in} = 8$ in layer 1 and $(Z)_{in} = 14$ in layer 2. (b) The real CS-Aarhus multiplex network.

4. Conclusions

In order to enhance the transfer capacity of multiplex systems, we propose some strategies by adding edges according to different weights. By checking the critical packet injection rate of the multiplex network, we discover that our strategies are capable of enhancing the transfer capacity significantly. The MBP which adds edges with the highest product of multiplex betweenness is more effective than the others. The impacts of different topology characteristics, such as the community structure, the average

degree, the communities number, and the nodes number, are explored to find that our strategies are more effective in multiplex networks with pronounced community structure. And in a two-layer network with different community structures, applying our strategies in the layer with the stronger community structure can yield better results. Our strategies are proved to be very effective by simulation results of computer-generated networks and real-world systems. The time complexity of our strategies is also acceptable which means our strategies might be helpful in developing more efficient transfer networks and routing strategies.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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