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Research Article

An Implementation of Lipschitz Simple Functions in Computer Algebra System Singular

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A complete classification of simple function germs with respect to Lipschitz equivalence over the field of complex numbers $\mathbb{C}$ was given by Nguyen et al. The aim of this article is to implement a classifier in terms of easy computable invariants to compute the type of the Lipschitz simple function germs without computing the normal form in the computer algebra system singular.

1. Introduction and Preliminaries

Let K be a field and O_n be the collection of all germs of smooth function at $0 \in K^n$. The collection of all germs $f \in O_n$ such that $f(0) = 0$ is denoted by $\mathfrak{m}$. If $K = \mathbb{C}$, then it is equivalent to consider O_n as $\mathbb{C}[[x_1, \ldots, x_n]]$, the local ring of formal power series in n variables and $\mathfrak{m}$ its maximal ideal. Let $\mathscr{R} = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[[x_1, \ldots, x_n]])$, the set of all $\mathbb{C}$-automorphisms of $\mathbb{C}[[x_1, \ldots, x_n]]$, and $\mathscr{H} = \mathrm{Hom}_{\mathbb{C}}\ (\mathbb{C}\ [[x_1, \ldots, x_n]])$, the set of all $\mathbb{C}$-bi-Lipschitz homeomorphisms of $\mathbb{C}[[x_1, \ldots, x_n]]$. Two smooth germs f and $g \in \mathfrak{m}$ are right equivalent (resp., bi-Lipschitz right equivalent) denoted by $f \sim_r g$ (resp., $f \sim_L g$) if there exists an automorphism $\phi \in \mathscr{R}$ (resp., a bi-Lipschitz homeomorphisms $\phi \in \mathscr{H}$) such that $\phi(f) = g$. In case of two variables (resp., three variables), we will later use $\mathbb{C}[[x, y]]$ (resp., $\mathbb{C}[[x, y, z]]$) instead of $\mathbb{C}[[x_1, x_2]]$ (resp., $\mathbb{C}[[x_1, x_2, x_3]]$) [13].

In seventies, Arnold [1–3] introduced the notion of modality for singularities over the fields $\mathbb{R}$ and $\mathbb{C}$. He gave a classification of hypersurfaces of modality 0, 1, and 2 under right equivalence. These are also classifications with respect to contact equivalence. Also we have the contributions by Guisti [4], Wall [5], and many others [6–11]. Mostowski [12] showed that germs of complex analytic set do not admit moduli with respect to bi-Lipschitz equivalence. Note that the result of Mostowski does not hold for function germs. Henry and Parusiński [13] proved that function germs do admit moduli under bi-Lipschitz right equivalence. Nguyen et al. [14] gave the classification of Lipschitz simple function germs. The aim of this article is to implement a classifier for this classification in the computer algebra system SINGULAR [15].

Let $f \in \mathfrak{m}$; then k-jet of f denoted by $j^k(f)$ is the Taylor expansion of f up to degree k terms. Let $f - g \in m^{k+1}$ for all $g \in \mathbb{C}[[x_1, \ldots, x_n]]$ and $f - g \sim_r f$, then f is called k-determined. A finitely determined germ f is Lipschitz 0-modal if there is a neighborhood $j^k(f)$, the $k-$th jet of f for sufficiently large $k \in \mathbb{N}$ that meets only finitely many bi-Lipschitz equivalence classes. We use the following invariants for our classifier.

Definition 1. The Taylor expansion of f at 0 is an expression of the form

$$Tf(0) = f_k + f_{k+1} + \cdots, \quad (1)$$

where each f_i is a homogeneous polynomial of degree i and $f_k \neq 0$. In this expansion, f_k denotes the homogenous polynomial of lowest degree, and we denote it by $H(f)$. Then $m = k$ is the multiplicity of f at 0.

Definition 2. Let $J(f) = \langle f_{x_1}, \ldots, f_{x_n} \rangle$ be a Jacobian ideal and $M_f = \mathbb{C}[[x]]/J(f)$ be the Milnor algebra over $\mathbb{C}$−algebra. Then $\mu(f) = \dim(M_f)$ is called the codimension of f.

Definition 3. Let $f \in \mathfrak{m}$ and $\operatorname{Hess}(f) = ((\partial^2 f/\partial x_i \partial x_j)(0))_{1 \le i,j \le n}$ be its corresponding Hessian matrix. Then n−rank(Hess(f)) is called the corank of f and is denoted by corank$f$.

Remark 1. Lemma 4.2, Lemma 4.3, and Theorem 4.7 of [14] give the bi-Lipschitz invariants $m(f)$, $H(f)$, and corank(f).

2. A Classifier for Simple Function Germs under Lipschitz Equivalence

In this section, we present propositions and algorithms deduced from propositions to characterize the simple functions germs with respect to Lipschitz equivalence in terms of certain invariants such as multiplicity, corank, and codimension of f. To differentiate some cases, we use locus of $H(f)$. For a proof of the following results, see Theorem 8.4, Theorem 8.5, and Theorem 8.7 of [14].

Theorem 1 (see [14]). *A germ f is Lipschitz simple if and only if it is bi-Lipschitz equivalent to one of the normal forms given in Tables 1 and 2.*

Theorem 2 (see [14]). *Every germ with corank greater or equal to 4 is the Lipschitz modal.*

2.1. Lipschitz Simple Function Germs of Corank 1 and 2. In the following, Propositions 1 and 2 give the possible overlapping of Lipschitz simple germs in case of corank 2.

Proposition 1. *Let $f \in \mathbb{C}[[x, y]]$ be a map germ of corank 2 and multiplicity 3.*

(1) If codimension of f is 6, then f is Lipschitz simple of type D_6 or E_6.

(2) If codimension of f is 7, then f is Lipschitz simple of type D_7 or E_7.

(3) If codimension of f is 8, then f is Lipschitz simple of type D_8 or E_8.

Table 1: Lipschitz Simple Function Germs of corank 1 and 2.

Type	Normal form	Codimension
A_k	$x^{k+1} + y^2$	$k \ge 1$
D_k	$x^2 y + y^{k-1}$	$k \ge 4$
E_6	$x^3 + y^4$	6
E_7	$x^3 + xy^3$	7
E_8	$x^3 + y^5$	8
X_9	$x^4 + y^4 + x^2 y^2$	9
$T_{2,4,5}$	$x^4 + y^5 + x^2 y^2$	10
$T_{2,5,5}$	$x^5 + y^5 + x^2 y^2$	11
Z_{11}	$x^3 y + y^5 + xy^4$	11
W_{12}	$x^4 + y^5 + x^2 y^3$	12

Table 2: Lipschitz simple function germs of corank 3.

Type	Normal form for $f \in \mathbb{C}[[x, y, z]]$	Codimension
$P_8 = T_{3,3,3}$	$x^3 + y^3 + z^3 + xyz$	8
$T_{3,3,4}$	$x^3 + y^3 + z^4 + xyz$	9
$T_{3,3,5}$	$x^3 + y^3 + z^5 + xyz$	10
$T_{3,4,5}$	$x^3 + y^4 + z^5 + xyz$	11
$T_{3,5,5}$	$x^3 + y^5 + z^5 + xyz$	12
$T_{4,4,4}$	$x^4 + y^4 + z^4 + xyz$	11
$T_{4,4,5}$	$x^4 + y^4 + z^5 + xyz$	12
$T_{4,5,5}$	$x^4 + y^5 + z^5 + xyz$	13
$T_{5,5,5}$	$x^5 + y^5 + z^5 + xyz$	14
Q_{10}	$x^3 + y^4 + yz^2 + xy^3$	10
Q_{11}	$x^3 + y^5 + yz^2 + xy^3$	11
S_{11}	$x^4 + y^2 z + xz^2 + x^3 z$	11
S_{12}	$x^2 y + y^2 z + xz^3 + z^5$	12

Proposition 2. *Let $f \in \mathbb{C}[[x, y]]$ be a map germ of corank 2 and multiplicity 4. If codim$(f) = 11$, then f is Lipschitz simple of type $T_{2,5,5}$ or Z_{11}.*

Proof. The statements of Propositions 1 and 2 follow Theorem 8.4 of [14], and these overlappings can be differentiated by computing the zero set of $H(f)$. □

2.2. Lipschitz Simple Function Germs of Corank 3. The following Proposition 3 describes the possible overlappings of Lipschitz simple germs in case of corank 3.

Proposition 3. *Let $f \in \mathbb{C}[[x, y, z]]$ be a map germ of corank 3 and multiplicity 3.*

(1) If codimension of f is 10, then f is Lipschitz simple of type $T_{3,3,5}$ or Q_{10}.

(2) If codimension of f is 11, then f is Lipschitz simple of type $T_{3,4,5}$, $T_{4,4,4}$, Q_{11}, or S_{11}.

(3) If codimension of f is 12, then f is Lipschitz simple of type $T_{3,5,5}$, $T_{4,4,5}$, or S_{12}.

Proof. The statement follows from Theorem 8.5 of [14], and the overlappings can be differentiated by computing the zero set of the $H(f)$. □

Input: $f \in \mathbb{C}[[x, y]]$.
Output: Type of f w.r.t. Lipschitz equivalence.

Compute c, the corank of f;
Compute k, the codimension of f;
Compute $m =$, the multiplicity of f;
if $c = 1$**, then**
  return f is of type A_k;
end if
if $c = 2$ and $m = 3$, **then**
    Compute $V(H(f))$, the zero set of the lowest degree homogeneous part of f;
    if $V(H(f))$ is the intersection of a line and a double line, **then**
      return f is of type D_k;
end if
    if $V(H(f))$ is a triple line and $k = 6, 7$, or 8, **then**
      return f is of type E_k;
end if
end if
    if $c = 2$ and $m = 4$, **then**
  if $k = 9$**, then**
      return f is of type X_9;
end if
    if $k = 10$**, then**
      return f is of type $T_{2,4,5}$;
end if
    if $k = 11$**, then**
    Compute $V(H(f))$, the zero set of the lowest degree homogeneous part of f;
    if $V(H(f))$ is the intersection of a line and a triple line, **then**
      return f is of type Z_{11};
end if
if $V(H(f))$ is the intersection of two double lines, **then**
      return f is of type $T_{2,5,5}$;
end if
end if
  if $k = 12$**, then**
    return f is of type W_{12};
end if
end if

Algorithm 1: Type of f, when corank of $f = 1$ or 2.

2.3. Singular Examples. We have implemented the Algorithm in the computer algebra system SINGULAR [15]. Code can be downloaded from https://www.mathcity.org/files/ahsan/classiFyLip-Procrdure.txt.

We give some examples:

```
ring R=0, (x, y), ds;
  poly  f=x4−x3y−3x2y2+5xy3−2y4+2x5+17x4y+
59x3y2+88x2y3+55xy4+22y5
        +2x4y2+5x3y3+30x2y4+34xy5+10y6+9x5
y2+60x4y3+147x3y4+138x2y5
        +52xy6+2y7+4x4y4+24x3y5+51x2y6+29xy
7+16x5y4+78x4y5+120x3y6
        +56x2y7+xy8+6x4y6+24x3y7+24x2y8+14x
5y6+44x4y7+32x3y8+4x4y8
        +8x3y9+6x5y8+9x4y9+x4y10+x5y10;
> LclassiFy2(f);
f is of type Z11
  poly   g=x3−3xy2+2y3+2x2y2+2xy3−4y4+x3y2
−2x2y3+2xy4+2y5+2x2y4−2xy5
        +x7+14x6y+84x5y2+280x4y3+560x3y4+67
2x2y5+449xy6+128y7+7x7y2
        +84x6y3+420x5y4+1120x4y5+1680x3y6+
1344x2y7+448xy8+21x7y4

        +210x6y5+840x5y6+1680x4y7+1680x3y8+67
2x2y9+35x7y6+280x6y7
        +840x5y8+1120x4y9+560x3y10+35x7y8+
210x6y9+420x5y10+280x4y11
        +21x7y10+84x6y11+84x5y12+7x7y12+14x
6y13+x7y14;
> LclassiFy2(g);
f is of type D8
```

Input: $f \in \mathbb{C}[[x, y, z]]$.
Output: Type of f w.r.t. Lipschitz equivalence.

Compute c, the corank of f;
Compute k, the codimension of f;
Compute m, the multiplicity of f;
if $c = 8$**, then**
return f is of type $T_{3,3,3}$;
end if
if $c = 9$**, then**
return f is of type $T_{3,3,4}$;
end if
if $c = 10$**, then**
Compute $V(H(f))$, the zero set of the lowest degree homogeneous part of f;
if $V(H(f))$ is irreducible and has in the singular locus a fat point of multiplicity 6, **then**
return f is of type $T_{3,3,5}$;
end if
if $V(H(f))$ is irreducible and has in the singular locus a fat point of multiplicity 4, **then**
return f is of type Q_{10};
end if
end if
if $c = 11$**, then**
Compute $V(H(f))$, the zero set of the lowest degree homogeneous part of f;
if $V(H(f))$ is the intersection of a plane and a node and has as a singular locus, the union of the lines $V(x, y) \cup V(x, z)$ and the point $V(x, y, z)$ as embedded point, **then**
return f is of type $T_{3,4,5}$;
end if
if $V(H(f))$ is the intersection of 3 planes and has as singular locus, the union of three lines $V(x, y) \cup V(x, z) \cup V(y, z)$, **then**
return f is of type $T_{4,4,4}$;
end if
if $V(H(f))$ is irreducible and has in the singular locus a fat point of multiplicity 4, **then**
return f is of type Q_{11};
end if
if $V(H(f))$ is the intersection of a plane and a node and has as a singular locus, the line $V(x, z)$, **then**
return f is of type S_{11};
end if
end if

Algorithm 2: Type of f, when corank of $f = 3$.

if $c = 12$**, then**
Compute $V(H(f))$, the zero set of the lowest degree homogeneous part of f;
if $V(H(f))$ is the intersection of a plane and a node and has as a singular locus, the union of the lines $V(x, y) \cup V(x, z)$ and the point $V(x, y, z)$ as embedded point, **then**
return f is of type $T_{3,5,5}$;
end if
if $V(H(f))$ is the intersection of 3 planes and has as singular locus, the union of three lines $V(x, y) \cup V(x, z) \cup V(y, z)$, **then**
return f is of type $T_{4,4,5}$;
end if
if $V(H(f))$ is the intersection of a plane and a node and has as a singular locus, the line $V(x, z)$, **then**
return f is of type S_{12};
end if
end if
if $c = 13$**, then**
return f is of type $T_{4,5,5}$;
end if
if $c = 14$**, then**
return f is of type $T_{5,5,5}$;
end if

Algorithm 3: Continuation of Algorithm 2.

```
ring R=0,(x,y,z),ds;
  poly    h=2x3+5x2y+4xy2+y3+5x2z+9xyz+4y
2z+4xz2+4yz2+z3+x4+4x3z+6x2z2
        +4xz3+z4+x5+5x4y+10x3y2+10x2y3+5xy4
+y5;
> LclassiFy3(h);
f is of type T_3, 4, 5
  poly   p=2x3+3x2y+xy2+5x2z+6xyz+y2z+4xz2
+3yz2+z3+x4+4x3y+6x2y2+4xy3
        +y4+x3z+3x2yz+3xy2z+y3z+x5+5x4y+10x
3y2+10x2y3+5xy4+y5;
> LclassiFy3(p);
f is of type S12
```

3. Conclusion

The aim of this paper is to implement the classification of simple function germs with respect to the Lipschitz equivalence given by Nguyen et al. in computer algebra system Singular. The proposed algorithms compute the type of the Lipschitz simple function germs without computing its normal form.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] V. I. Arnold, "Normal form of functions near degenerate critical points," *Functional Analysis and Its Applications*, vol. 6, pp. 254–272, 1972.

[2] V. I. Arnold, "Classification of unimodal critical points of functions," *Functional Analysis and Its Applications*, vol. 7, pp. 230-231, 1973.

[3] V. I. Arnold, "Local normal forms of functions," *Inventiones Mathematicae*, vol. 35, no. 1, pp. 87–109, 1976.

[4] M. Giusti, "Classification des singularités isolées simples d'intersections complètes," *Singularities*, vol. 40, pp. 457–494, 1983.

[5] C. T. C. Wall, "Classification of unimodal isolated singularities complete intersections," in *Proceedings of Symposia in Pure Mathematics*, vol. 40, pp. 625–640, 1983.

[6] D. Afzal, M. A. Binyamin, and F. K. Janjua, "On the classification of simple singularities in positive characteristic," *Analele Universitatii "Ovidius" Constanta—Seria Matematica*, vol. 22, no. 2, pp. 5–20, 2014.

[7] M. A. Binyamin, J. A. Khan, K. Mahmood, A. Shehzad, and U. Shuaib, "Classification of right unimodal and bimodal hypersurface singularities in positive characteristic by invariants," *Bulletin Mathematique de la Societe des Sciences*, vol. 61, no. 109, pp. 333–343, 2018.

[8] G.-M. Greuel and H. D. Nguyen, "Right simple singularities in positive characteristic," *Journal für die Reine und Angewandte Mathematik*, vol. 712, pp. 81–106, 2016.

[9] G.-M. Greuel and H. Kröning, "Simple singularities in positive characteristic," *Mathematische Zeitschrift*, vol. 203, no. 1, pp. 339–354, 1990.

[10] H. D. Nguyen, "Classification of singularities in positive characteristic," Dissertation, University of Kaiserlautern, Kaiserslautern, Germany, 2012.

[11] H. D. Nguyen, "Right unimodal and bimodal singularities in positive characteristic," *International Mathematics Research Notices*, vol. 2019, no. 6, pp. 1612–1641, 2019.

[12] T. Mostowski, "Lipschitz equisingularity," *Dissertationes Math. (Rozprawy Mat.)*, vol. 243, p. 46, 1985.

[13] J.-P. Henry and A. Parusiński, "Existence of moduli for bi-Lipschitz equivalence of analytic functions," *Compositio Mathematica*, vol. 136, no. 2, pp. 217–235, 2003.

[14] N. Nguyen, M. Ruas, and S. Trivedi, "Classification of Lipschitz simple function germs," *Proceedings of the London Mathematical Society*, vol. 121, no. 1, pp. 51–82, 2020.

[15] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, "Singular 4-1-2—a computer algebra system for polynomial computations," 2019, http://www.singular.uni-kl.de.