Research Article

Blow-Up of Solutions for a Coupled Nonlinear Viscoelastic Equation with Degenerate Damping Terms: Without Kirchhoff Term

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In this work, we consider a quasilinear system of viscoelastic equations with degenerate damping and source terms without the Kirchhoff term. Under suitable hypothesis, we study the blow-up of solutions.

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds + (|u|^k + |v|^l)|u_t|^{j-1}u_t \\ = f_1(u, v), (x, t) \in \Omega \times (0, T), \\ |v_t|^\eta v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds + (|v|^\theta + |u|^\gamma)|v_t|^{s-1}v_t \\ = f_2(u, v), (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \\ v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in \Omega, \end{cases} \quad (1)$$

where $k, l, \theta, \gamma \geq 0$; $j, s \geq 1$ for $N = 1, 2$, and $0 \leq j, s \leq (N + 2/N - 2)$ for $N \geq 3$; and $\eta \geq 0$ for $N = 1, 2$, and $0 < \eta \leq (2/N - 2)$ for $N \geq 3$; and $h_i(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ($i = 1, 2$) are positive relaxation functions which will be specified later. $(|\cdot|)^a + |\cdot|^b)(\cdot)_t|^{r-1}(\cdot)_t$ is the degenerate damping term, and

$$\begin{cases} f_1(u, v) = a_1|u + v|^{2(p+1)}(u + v) + b_1|u|^p \cdot u \cdot |v|^{p+2}, \\ f_2(u, v) = a_1|u + v|^{2(p+1)}(u + v) + b_1|v|^p \cdot v \cdot |u|^{p+2}. \end{cases} \quad (2)$$

The motivation of our problem firstly is by the initial boundary value problem for the quasilinear equation of the form

$$|u_t|^\eta u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds + g(u, u_t) = f(u). \quad (3)$$

This type of problem is frequently found in some mathematical models in applied sciences, especially in the theory of viscoelasticity. Problem (3) has been studied by various authors, and several results concerning asymptotic behavior and blow-up have been studied (case $\eta \geq 0$). For example, in the case $(g(u, u_t) = 0)$, problem (3) has been investigated in [1] and the author proved the blow-up result. In the case $(g(u, u_t) = 0)$ of boundary value problem and in the presence of the dispersion term $(-\Delta u_{tt})$, Liu [2] studied a general decay of solutions. And, in [3], the authors applied the potential well method to indicate the global existence and uniform decay of solutions $(g(u, u_t) = 0)$ instead of Δu_t . Furthermore, the authors obtained a blow-up result. In the case $(g(u, u_t) = |u_t|^m u_t)$, in [4], Wu studied a general decay of solution. Later, the same author in [5] considered the same problem but $(g(u, u_t) = u_t)$ and discussed the decay rate of solution. Recently, in [6], the authors proved the existence of global solution and a general stability result.

There are several works in case $(\eta = 0)$, where the authors have studied the blow-up of solutions of problem (3) (for example, see [3, 7–12]).

For a coupled system, He [13] considered the following problem:

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + g_1(u, u_t) \\ = f_1(u, v), \\ |v_t|^\eta v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + g_2(v, v_t) \\ = f_2(u, v), \end{cases} \quad (4)$$

where $\eta > 0$; $j, s \geq 2$; and $g_1(u, u_t) = |u_t|^{j-2} u_t$ and $g_2(v, v_t) = |v_t|^{s-2} v_t$. The author proved general and optimal decay of solutions. Then, in [14], the author investigated the same problem without damping term and established a general decay of solutions. Furthermore, the author obtained a blow-up of solutions. In addition, in problem (1) with $\eta = 0$, in [15], Wu proved a general decay of solutions. Later, in [16], Piskin and Ekinçi established a general decay and blow-up of solutions with nonpositive initial energy for problem (1) case (Kirchhoff type). In recent years, some other authors investigate the hyperbolic type system with degenerate damping terms (see [17–20]). Very recently, in the presence of the dispersion term $(-\Delta u_{tt})$, our problem (1) has been studied in [21]. Under some restrictions on the initial datum and standard conditions on relaxation functions, the authors have established the global existence and proved the general decay of solutions.

Based on all of the abovementioned discussion, we believe that the combination of these terms of damping (memory term, degenerate damping, and source terms) constitutes a new problem worthy of study and research, different from the above that we will try to shed light on, especially the blow-up of solutions.

Our paper is divided into several sections: In Section 2, we lay down the hypotheses, concepts, and lemmas we need.

In Section 3, we prove our main result. Finally, we give some concluding remarks in the last section.

2. Preliminaries

We prove the blow-up result under the following suitable assumptions:

(A1) $h_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are differentiable and decreasing functions such that

$$h_i(t) \geq 0, 1 - \int_0^\infty h_i(s)ds = l_i > 0, \quad i = 1, 2. \quad (5)$$

(A2) There exist a constants $\xi_1, \xi_2 > 0$ such that

$$h_i'(t) \leq -\xi_i h_i(t), \quad t \geq 0, i = 1, 2. \quad (6)$$

Lemma 1. *There exists a function $F(u, v)$ such that*

$$\begin{aligned} F(u, v) &= \frac{1}{2(p+2)} [u f_1(u, v) + v f_2(u, v)], \\ \frac{1}{2(p+2)} [a_1 |u+v|^{2(p+2)} + 2b_1 |uv|^{p+2}] &\geq 0, \end{aligned} \quad (7)$$

where

$$\begin{aligned} \frac{\partial F}{\partial u} &= f_1(u, v), \\ \frac{\partial F}{\partial v} &= f_2(u, v). \end{aligned} \quad (8)$$

We take $a_1 = b_1 = 1$ for convenience.

Lemma 2 (see [18]). *There exist two positive constants c_0 and c_1 such that*

$$\begin{aligned} \frac{c_0}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) &\leq F(u, v) \\ &\leq \frac{c_1}{2(p+2)} (|u|^{2(p+2)} + |v|^{2(p+2)}). \end{aligned} \quad (9)$$

Now, we state the local existence theorem that can be established by combining arguments of [13, 16].

Theorem 1. *Assume (5) and (6) hold. Let*

$$\begin{cases} -1 < p < \frac{4-n}{n-2}, & n \geq 3, \\ p \geq -1, & n = 1, 2. \end{cases} \quad (10)$$

Then, for any initial datum,

$$(u_0, u_1, v_0, v_1) \in \mathcal{H}. \quad (11)$$

Problem (1) has a unique solution, for some $T > 0$:

$$\begin{aligned}
u, v &\in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)), \\
u_t &\in C([0, T]; H_0^1(\Omega) \cap L^{j+1}(\Omega)), \\
v_t &\in C([0, T]; H_0^1(\Omega) \cap L^{s+1}(\Omega)),
\end{aligned} \tag{12}$$

where

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega). \tag{13}$$

Now, we define the energy functional.

Lemma 3. Assume (5), (6), and (10) hold; let (u, v) be a solution of (1); then, $E(t)$ is nonincreasing, that is,

$$\begin{aligned}
E(t) &= \frac{1}{\eta+2} \left[\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right] \\
&\quad + \frac{1}{2} \left[\left(1 - \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 \right. \\
&\quad \left. + \left(1 - \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 \right], \\
&\quad + \frac{1}{2} [(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)] - \int_{\Omega} F(u, v) dx,
\end{aligned} \tag{14}$$

which satisfies

$$\begin{aligned}
E'(t) &\leq \frac{1}{2} [(h_1' \circ \nabla u)(t) + (h_2' \circ \nabla v)(t)] \\
&\quad - \frac{1}{2} [h_1(t) \|\nabla u\|_2^2 + h_2(t) \|\nabla v\|_2^2], \\
&\quad - \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx - \int_{\Omega} (|v|^\theta + |u|^\varrho) |v_t|^{s+1} dx, \\
&\leq 0.
\end{aligned} \tag{15}$$

Proof. By multiplying the first and second equations in (1) by u_t, v_t and integrating over Ω , we get

$$\begin{aligned}
\frac{d}{dt} &\left\{ \frac{1}{\eta+2} \|u_t\|_{\eta+2}^{\eta+2} + \frac{1}{\eta+2} \|v_t\|_{\eta+2}^{\eta+2} + \frac{1}{2} \left(-1 \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} \left(-1 \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 + \frac{1}{2} (h_1 \circ \nabla u)(t) \right. \\
&\quad \left. + \frac{1}{2} (h_2 \circ \nabla v)(t) - \int_{\Omega} F(u, v) dx \right\}, \\
&= - \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx - \int_{\Omega} (|v|^\theta + |u|^\varrho) |v_t|^{s+1} dx + \frac{1}{2} (h_1' \circ \nabla u) - \frac{1}{2} h_1(t) \|\nabla u\|_2^2 + \frac{1}{2} (h_2' \circ \nabla v) - \frac{1}{2} h_2(t) \|\nabla v\|_2^2.
\end{aligned} \tag{16}$$

We obtain (14) and (15). $\square$

First, we define the functional as

3. Blow-Up

In this section, we prove the blow-up result of solution of problem (1).

$$\begin{aligned}
\mathbb{H}(t) &= -E(t) = -\frac{1}{\eta+2} \left[\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right] - \frac{1}{2} \left(1 - \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} \left(1 - \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 \\
&\quad - \frac{1}{2} [(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)] \\
&\quad + \frac{1}{2(p+2)} \left[\|u + v\|_{2(p+2)}^{2(p+2)} + \|uv\|_{2(p+2)}^{2(p+2)} \right].
\end{aligned} \tag{17}$$

Theorem 4. Assume that (5), (6), and (10) hold, and suppose that $E(0) < 0$ and

$$2(p+2) > \max\{k+j+1; l+j+1; \theta+s+1; \varrho+s+1\}. \quad (18)$$

Then, the solution of problem (1) blows up in finite time.

Proof. From (14), we have

$$E(t) \leq E(0) \leq 0. \quad (19)$$

Therefore,

$$\begin{aligned} \mathbb{H}'(t) = -E'(t) &\geq \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx \\ &+ \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx. \end{aligned} \quad (20)$$

Hence,

$$\begin{aligned} \mathbb{H}'(t) &\geq \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx \geq 0, \\ \mathbb{H}'(t) &\geq \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx \geq 0. \end{aligned} \quad (21)$$

By (9) and (17), we have

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(p+2)} \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)} \right], \\ &\leq \frac{c_1}{2(p+2)} \left[\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right]. \end{aligned} \quad (22)$$

We set

$$\mathcal{K}(t) = \mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\eta+1} \int_{\Omega} [u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t] dx, \quad (23)$$

where $\varepsilon > 0$ is to be assigned later and

$$0 < \alpha < \min \left\{ \left(1 - \frac{1}{2(p+2)} - \frac{1}{\eta+2} \right), \frac{2p+3-(k+j)}{2j(p+2)}, \frac{2p+3-(l+j)}{2j(p+2)}, \frac{2p+3-(\theta+s)}{2s(p+2)}, \frac{2p+3-(\varrho+s)}{2s(p+2)} \right\} < 1. \quad (24)$$

By multiplying the first and second equations in (1) by u, v and with a derivative of (23), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1} \left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right) + \underbrace{\varepsilon \int_{\Omega} \nabla u \int_0^t g(t-s)\nabla u(s)ds dx}_{J_1} + \underbrace{\varepsilon \int_{\Omega} \nabla v \int_0^t h(t-s)\nabla v(s)ds dx}_{J_2} \\ &\quad - \underbrace{\varepsilon \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j-1} u_t \cdot u dx}_{J_3} - \underbrace{\varepsilon \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s-1} v_t \cdot v dx}_{J_4} - \varepsilon (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \underbrace{\varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)} \right]}_{J_5}, \end{aligned} \quad (25)$$

where we have

$$\begin{aligned} J_1 &= \varepsilon \int_0^t h_1(t-s)ds \int_{\Omega} \nabla u \cdot (\nabla u(s) - \nabla u(t)) dx ds \\ &\quad + \varepsilon \int_0^t h_1(s)ds \|\nabla u\|_2^2, \\ &\geq \frac{\varepsilon}{2} \int_0^t h_1(s)ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2} (h_1 \circ \nabla u), \end{aligned} \quad (26)$$

$$\begin{aligned} J_2 &= \varepsilon \int_0^t h_2(t-s)ds \int_{\Omega} \nabla v \cdot (\nabla v(s) - \nabla v(t)) dx ds \\ &\quad + \varepsilon \int_0^t h_2(s)ds \|\nabla v\|_2^2, \\ &\geq \frac{\varepsilon}{2} \int_0^t h_2(s)ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2} (h_2 \circ \nabla v). \end{aligned} \quad (27)$$

From (25), we find that

$$\begin{aligned} \mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1} \left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right) \\ &\quad - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t h_1(s)ds \right) \|\nabla u\|_2^2 \right. \\ &\quad \left. + \left(1 - \frac{1}{2} \int_0^t h_2(s)ds \right) \|\nabla v\|_2^2 \right] \\ &\quad - \frac{\varepsilon}{2} (h_1 \circ \nabla u) - \frac{\varepsilon}{2} (h_2 \circ \nabla v) - J_3 - J_4 + J_5. \end{aligned} \quad (28)$$

At this point, we use Young's inequality; for $\delta > 0$,

$$XY \leq \frac{\delta^{\alpha} X^{\alpha}}{\alpha} + \frac{\delta^{-\beta} Y^{\beta}}{\beta}, \alpha, \beta > 0, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad (29)$$

We get that for $\delta_1, \delta_2 > 0$,

$$\begin{aligned}
|u|u_t|^{j-1}u_t| &\leq \frac{\delta_1^{j+1}}{j+1}|u_t|^{j+1} + \frac{j}{j+1}\delta_1^{-(j+1/j)}|u_t|^{j+1}, \\
|v|v_t|^{s-1}v_t| &\leq \frac{\delta_2^{s+1}}{s+1}|v_t|^{s+1} + \frac{s}{s+1}\delta_2^{-(s+1/s)}|v_t|^{s+1}.
\end{aligned} \tag{30}$$

Hence, we have

$$\begin{aligned}
J_3 &\leq \varepsilon \frac{\delta_1^{j+1}}{j+1} \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx \\
&\quad + \varepsilon \frac{j\delta_1^{-(j+1/j)}}{j+1} \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx, \\
J_4 &\leq \varepsilon \frac{\delta_2^{s+1}}{s+1} \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx \\
&\quad + \varepsilon \frac{s\delta_2^{-(s+1/s)}}{s+1} \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx.
\end{aligned} \tag{31}$$

Therefore, using (21) and by setting δ_1, δ_2 so that

$$\begin{aligned}
\frac{j\delta_1^{-(j+1/j)}}{j+1} &= \frac{\kappa \mathbb{H}^{-\alpha}(t)}{2}, \\
\frac{s\delta_2^{-(s+1/s)}}{s+1} &= \frac{\kappa \mathbb{H}^{-\alpha}(t)}{2},
\end{aligned} \tag{32}$$

and substituting in (28), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon \kappa] \mathbb{H}^{-\alpha} \mathbb{H}'(t) + \frac{\varepsilon}{\eta+1} \left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right) - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 + \left(1 - \frac{1}{2} \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 \right] \\
&\quad - \frac{\varepsilon}{2} (h_1 \circ \nabla u) - \frac{\varepsilon}{2} (h_2 \circ \nabla v) - \varepsilon C_1(\kappa) \mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx - \varepsilon C_2(\kappa) \mathbb{H}^{\alpha j}(t) \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx + J_5,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
C_1(\kappa) &:= \left(\frac{2j}{\kappa(j+1)} \right)^{j+1} \frac{1}{j+1}, \\
C_2(\kappa) &:= \left(\frac{2s}{\kappa(s+1)} \right)^{s+1} \frac{1}{s+1}.
\end{aligned} \tag{34}$$

We have

$$\begin{aligned}
\int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx &= \|u\|_{k+j+1}^{k+j+1} + \int_{\Omega} |v|^l |u_t|^{j+1} dx, \\
\int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx &= \|v\|_{\theta+s+1}^{\theta+s+1} + \int_{\Omega} |u|^{\varrho} |v_t|^{s+1} dx.
\end{aligned} \tag{35}$$

By Young's inequality, we find that for $\delta_3, \delta_4 > 0$,

$$\begin{aligned}
\int_{\Omega} |v|^l |u_t|^{j+1} dx &\leq \frac{l}{l+j+1} \delta_3^{(l+j+1/l)} \|v\|_{l+j+1}^{l+j+1} \\
&\quad + \frac{j+1}{l+j+1} \delta_3^{-(l+j+1/l)} \|u\|_{l+j+1}^{l+j+1}, \\
\int_{\Omega} |u|^{\varrho} |v_t|^{s+1} dx &\leq \frac{\varrho}{\varrho+s+1} \delta_4^{(\varrho+s+1/\varrho)} \|u\|_{\varrho+s+1}^{\varrho+s+1} \\
&\quad + \frac{s+1}{\varrho+s+1} \delta_4^{-(\varrho+s+1/\varrho)} \|v\|_{\varrho+s+1}^{\varrho+s+1}.
\end{aligned} \tag{36}$$

Hence,

$$\begin{aligned}
\mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx &\leq \mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} + \frac{l \mathbb{H}^{\alpha j}(t)}{l+j+1} \delta_3^{(l+j+1/l)} \|v\|_{l+j+1}^{l+j+1} + \frac{(j+1) \mathbb{H}^{\alpha j}(t)}{l+j+1} \delta_3^{-(l+j+1/l)} \|u\|_{l+j+1}^{l+j+1}, \\
\mathbb{H}^{\alpha j}(t) \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx &\leq \mathbb{H}^{\alpha s}(t) \|v\|_{\theta+s+1}^{\theta+s+1} + \frac{\varrho \mathbb{H}^{\alpha s}(t)}{\varrho+s+1} \delta_4^{(\varrho+s+1/\varrho)} \|u\|_{\varrho+s+1}^{\varrho+s+1} + \frac{(s+1) \mathbb{H}^{\alpha s}(t)}{\varrho+s+1} \delta_4^{-(\varrho+s+1/\varrho)} \|v\|_{\varrho+s+1}^{\varrho+s+1}.
\end{aligned} \tag{37}$$

Since (10) holds, we obtain the following by using (22) and (24):

$$\begin{aligned}
\mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} &\leq c_1 \left(\|u\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} + \|v\|_{2(p+2)}^{2\alpha j(p+2)} \|u\|_{k+j+1}^{k+j+1} \right), \\
\mathbb{H}^{\alpha j}(t) \|v\|_{k+j+1}^{k+j+1} &\leq c_2 \left(\|v\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} + \|u\|_{2(p+2)}^{2\alpha j(p+2)} \|v\|_{k+j+1}^{k+j+1} \right), \\
\mathbb{H}^{\alpha s}(t) \|v\|_{\theta+s+1}^{\theta+s+1} &\leq c_3 \left(\|v\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} + \|u\|_{2(p+2)}^{2\alpha s(p+2)} \|v\|_{\theta+s+1}^{\theta+s+1} \right), \\
\mathbb{H}^{\alpha s}(t) \|u\|_{\theta+s+1}^{\theta+s+1} &\leq c_4 \left(\|u\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} + \|v\|_{2(p+2)}^{2\alpha s(p+2)} \|u\|_{\theta+s+1}^{\theta+s+1} \right),
\end{aligned} \tag{38}$$

for some positive constants $c_i, i = 1, \dots, 4$. By using (24) and the algebraic inequality

$$B^\zeta \leq (B+1) \leq \left(1 + \frac{1}{b}\right) (B+b), \forall B > 0, 0 < \zeta < 1, b > 0, \tag{39}$$

we have, $\forall t > 0$,

$$\begin{aligned}
\|u\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} &\leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0) \right) \leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\|v\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} &\leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0) \right) \leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\|v\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} &\leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0) \right) \leq d \left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\|u\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} &\leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0) \right) \leq d \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right),
\end{aligned} \tag{40}$$

where $d = 1 + (1/\mathbb{H}(0))$. Also, since

$$(X+Y)^\gamma \leq C(X^\gamma + Y^\gamma), X, Y > 0, \gamma > 0, \tag{41}$$

we conclude that

$$\begin{aligned}
\|v\|_{2(p+2)}^{2\alpha j(p+2)} \|u\|_{k+j+1}^{k+j+1} &\leq c_5 \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} \right), \\
\|u\|_{2(p+2)}^{2\alpha j(p+2)} \|v\|_{k+j+1}^{k+j+1} &\leq c_6 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right), \\
\|u\|_{2(p+2)}^{2\alpha s(p+2)} \|v\|_{\theta+s+1}^{\theta+s+1} &\leq c_7 \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right), \\
\|v\|_{2(p+2)}^{2\alpha s(p+2)} \|u\|_{\theta+s+1}^{\theta+s+1} &\leq c_8 \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} \right).
\end{aligned} \tag{42}$$

Substituting (40) and (42) in (38), we get

$$\begin{aligned}
\mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} &\leq c_9 \left(\|v\|_{2(p+2)}^{2(p+2)} \right) + \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\mathbb{H}^{\alpha j}(t) \|v\|_{k+j+1}^{k+j+1} &\leq c_{10} \left(\|v\|_{2(p+2)}^{2(p+2)} \right) + \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\mathbb{H}^{\alpha j}(t) \|v\|_{k+j+1}^{k+j+1} &\leq c_{11} \left(\|v\|_{2(p+2)}^{2(p+2)} \right) + \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} &\leq c_{12} \left(\|v\|_{2(p+2)}^{2(p+2)} \right) + \left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right).
\end{aligned} \tag{43}$$

Hence, by fixing $\delta_3, \delta_4 > 0$, we get

$$\begin{aligned}
\mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx &\leq M_1 \left(1 + \frac{l\delta_3^{(l+j+1/l)}}{l+j+1} + \frac{(j+1)\delta_3^{-(l+j+1/l)}}{l+j+1} \right) \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right), \\
\mathbb{H}^{\alpha s}(t) \int_{\Omega} (|v|^\theta + |u|^\varrho) |v_t|^{s+1} dx &\leq M_2 \left(1 + \frac{\varrho\delta_4^{(\varrho+s+1/\varrho)}}{\varrho+s+1} + \frac{(s+1)\delta_4^{-(\varrho+s+1/\varrho)}}{\varrho+s+1} \right) \left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t) \right),
\end{aligned} \tag{44}$$

for some constants $M_1, M_2 > 0$.

Now, for $0 < a < 1$, from (17),

$$\begin{aligned}
J_5 &= \varepsilon \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] = \varepsilon a \left[\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right] + \frac{2\varepsilon(p+2)(1-a)}{\eta+2} \left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right) \\
&\quad + \varepsilon(p+2)(1-a) \left(1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 + \varepsilon(p+2)(1-a) \left(1 - \int_0^t h(s)ds \right) \|\nabla v\|_2^2 \\
&\quad - \varepsilon(p+2)(1-a) ((h_1 \circ \nabla u) + (h_2 \circ \nabla v)) + \varepsilon 2(p+2)(1-a) \mathbb{H}(t).
\end{aligned} \tag{45}$$

Substituting in (33) and by using (9), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq \{(1-a) - \varepsilon\kappa\} \mathbb{H}^{-\alpha} \mathbb{H}'(t) + \varepsilon \left\{ \frac{2\varepsilon(p+2)(1-a)}{\eta+2} + \frac{1}{\eta+1} \right\} \left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} \right), \\
&+ \varepsilon \left\{ (p+2)(1-a) \left(1 - \int_0^t h_1(s) ds \right) - \left(1 - \frac{1}{2} \int_0^t h_1(s) ds \right) \right\} \|\nabla u\|_2^2 \\
&+ \varepsilon \left\{ (p+2)(1-a) \left(1 - \int_0^t h_2(s) ds \right) - \left(1 - \frac{1}{2} \int_0^t h_2(s) ds \right) \right\} \|\nabla v\|_2^2 \\
&+ \varepsilon \left\{ (p+2)(1-a) - \frac{1}{2} \right\} (h_1 o \nabla u + h_2 o \nabla v) + \varepsilon \{c_0 a - (M_3 C_1(\kappa) + M_4 C_2(\kappa))\} \left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right) \\
&+ \varepsilon \{2(p+2)(1-a) - (M_3 C_1(\kappa) + M_4 C_2(\kappa))\} \mathbb{H}(t),
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
M_3 &:= M_1 \left(1 + \frac{l\delta_3^{(l+j+1/l)}}{l+j+1} + \frac{(j+1)\delta_3^{-(l+j+1/l)}}{l+j+1} \right) > 0, \\
M_4 &:= M_2 \left(1 + \frac{\varrho\delta_4^{(\varrho+s+1/\varrho)}}{\varrho+s+1} + \frac{(s+1)\delta_4^{-(\varrho+s+1/\varrho)}}{\varrho+s+1} \right) > 0.
\end{aligned} \tag{47}$$

In this stage, we take $a > 0$ small enough so that

$$\lambda_1 = (p+2)(1-a) - 1 > 0, \tag{48}$$

and we assume that

$$\begin{aligned}
\max \left\{ \int_0^\infty h_1(s) ds, \int_0^\infty h_2(s) ds \right\} &< \frac{(p+2)(1-a) - 1}{((p+2)(1-a) - (1/2))} \\
&= \frac{2\lambda_1}{2\lambda_1 + 1}
\end{aligned} \tag{49}$$

gives

$$\begin{aligned}
\lambda_2 &= \left\{ ((p+2)(1-a) - 1) - \int_0^t h_1(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0, \\
\lambda_3 &= \left\{ \left((p+2)(1-a) - 1 \right) - \int_0^t h_2(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0.
\end{aligned} \tag{50}$$

Then, we choose κ so large such that

$$\begin{aligned}
\lambda_4 &= ac_0 - (M_3 C_1(\kappa) + M_4 C_2(\kappa)) > 0, \\
\lambda_5 &= 2(p+2)(1-a) - (M_3 C_1(\kappa) + M_4 C_2(\kappa)) > 0.
\end{aligned} \tag{51}$$

Finally, we fix κ and a and we appoint ε small enough so that

$$\lambda_6 = (1-\alpha) - \varepsilon\kappa > 0. \tag{52}$$

Thus, for some $\beta > 0$, estimate (46) becomes

$$\mathcal{K}'(t) \geq \beta \left\{ \mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (h_1 o \nabla u) + (h_2 o \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right\}. \tag{53}$$

By (9), for some $\beta_1 > 0$, we obtain

$$\mathcal{K}'(t) \geq \beta_1 \left\{ \mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (h_1 o \nabla u) + (h_2 o \nabla v) + \|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{(p+2)}^{(p+2)} \right\} \tag{54}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, t > 0. \tag{55}$$

Next, using Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\left| \int_{\Omega} (u|u_t|^{\eta} u_t + v|v_t|^{\eta} v_t) dx \right|^{(1/(1-\alpha))} \\
&\leq C \left[\|u\|_{2(p+2)}^{(\theta/(1-\alpha))} + \|u_t\|_{\eta+2}^{(\mu/(1-\alpha))} + \|v\|_{2(p+2)}^{(\theta/(1-\alpha))} + \|v_t\|_{\eta+2}^{(\mu/(1-\alpha))} \right],
\end{aligned} \tag{56}$$

where $(1/\mu) + (1/\theta) = 1$.

We take $\mu = (\eta + 2)(1 - \alpha)$, to get

$$\frac{\theta}{1 - \alpha} = \frac{\eta + 2}{(1 - \alpha)(\eta + 2) - 1} \leq 2(p + 2). \quad (57)$$

Subsequently, by using (24), (22), and (39), we obtain

$$\begin{aligned} \|u\|_{2(p+2)}^{(\eta+2/(1-\alpha)(\eta+2)-1)} &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\right), \\ \|v\|_{2(p+2)}^{(\eta+2/(1-\alpha)(\eta+2)-1)} &\leq d\left(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)\right), \forall t \geq 0. \end{aligned} \quad (58)$$

Therefore,

$$\begin{aligned} \left| \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right|^{(1/1-\alpha)} \\ \leq c_{13} \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + H(t) \right]. \end{aligned} \quad (59)$$

Hence, by substituting (59) into (23), we get

$$\begin{aligned} \mathcal{K}^{(1/1-\alpha)}(t) &= \left(\mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\eta+1} \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right)^{(1/1-\alpha)}, \\ &\leq c \left(\mathbb{H}(t) + \left| \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right|^{(1/1-\alpha)} \right), \\ &\leq c \left(\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right), \\ &\leq c \left(\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (h_1 o \nabla u) + (h_2 o \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right). \end{aligned} \quad (60)$$

From (53) and (60), we get

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{(1/1-\alpha)}(t), \quad (61)$$

where $\lambda > 0$, and this depends only on β and c .

By integration of (61), we obtain

$$\mathcal{K}^{(\alpha/1-\alpha)}(t) \geq \frac{1}{\mathcal{K}^{(-\alpha/1-\alpha)}(0) - \lambda(\alpha/1-\alpha)t}. \quad (62)$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1 - \alpha}{\lambda \alpha \mathcal{K}^{(\alpha/1-\alpha)}(0)}. \quad (63)$$

Then, the proof is completed. $\square$

4. Conclusion

The objective of this work is the study of the blow-up of solutions for a quasilinear viscoelastic system with degenerate damping. This type of problem is frequently found in some mathematical models in applied sciences, especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (memory term, degenerate damping, and source terms), which dictates the emergence of these terms in the system.

In the next work, we will try using the same method with the same problem in addition to other damping terms (dispersion term, Balakrishnan–Taylor damping, and delay term).

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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