Hindawi
Complexity
Volume 2021, Article ID 6911443, 11 pages
https://doi.org/10.1155/2021/6911443

Research Article

Some Further Results on the Reduction of Two-Dimensional Systems

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Received 5 August 2021; Accepted 29 October 2021; Published 17 November 2021

Academic Editor: Xiao Ling Wang

The reduction of two-dimensional systems plays an important role in the theory of systems, which is closely associated with the equivalence of the bivariate polynomial matrices. In this paper, the equivalence problems on several classes of bivariate polynomial matrices are investigated. Some new results on the equivalence of these matrices are obtained. These results are useful for reducing two-dimensional systems.

1. Introduction

Multidimensional (nD) systems, especially two-dimensional (2D) systems, are widely used in the field of circuits, image, signal processing, control systems, etc. [1–5, 12]. And the theory of 2D systems has received increasing attention, among which the reduction of 2D systems is an important research content. Usually, a given system is desirable to reduce to an equivalent system with fewer equations or unknowns (named the equivalence of 2D systems). In this way, the characteristics of the original system can be studied in a better and simpler way. A 2D system can be represented with two types of dynamical elements, so 2D systems are often described by bivariate polynomial matrices. Hence, the equivalence problems of 2D systems are usually transformed into the equivalence problems of polynomial matrices [6–11].

The equivalence problem of univariate polynomial matrices was solved by Rosenbrock in 1970 [6]. Using the Euclidean division property of the univariate polynomial ring, he proved that every univariate polynomial matrix in this ring is equivalent to the Smith form. The equivalence problem of the bivariate polynomial matrix is more complex. Lee and Zak gave an example of bivariate polynomial matrix $\begin{pmatrix} z_2 & -z_1 - 1 \\ -z_2^2 & z_2 \end{pmatrix}$, which is not equivalent to the Smith form [8]. Note that none of the bivariate polynomial rings is a Euclidean ring, and the Euclidean division property does not hold. So, many researchers study the equivalence of some special types of bivariate polynomial matrices. In 1986, Frost and Boudellioua proved that a full row rank bivariate polynomial matrix $T(z,w)$ is equivalent to the Smith form if and only if there exists a unimodular column vector U such that $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse [9]. In 2012, Boudellioua wrote an algorithm to enable the decomposition and equivalence of some bivariate polynomial matrices to be realized in Maple [10]. In 2019, Li et al. presented some criteria on an $l \times l$ bivariate polynomial matrix $F(z,w)$ which is equivalent to the Smith form $\mathrm{diag}(I_{l-1}, \det F(z,w))$ with $\det F(z,w)$ being an irreducible polynomial [11]. There are also some results on the equivalence of multivariate polynomial matrices in special cases [13, 14, 16, 18, 19], among which when an $l \times l$ polynomial matrix $F(z)$ which is equivalent to the Smith form diag $(I_{l-1}, \det F(z))$ is investigated, such as $\det F(z) = x_1 - f(x_2, \ldots, x_n)$ [13], $\det F(z) = (x_1 - f(x_2, \ldots, x_n))^q$ [14], and $\det F(z) = (x_1 - f_1(x_2, \ldots, x_n))^{q_1}(x_2 - f_2(x_2, \ldots, x_n))^{q_2}$ [19], diag $(I_{l-1}, \det F(z))$ is a very important kind of matrices in the equivalence of multidimensional systems [9, 15, 17].

In this paper, $K[z,w]$ denotes a bivariate polynomial ring with K being a field, and we consider arbitrary polynomial $f(z,w)$ in $K[z,w]$ as a polynomial in w with coefficient in $K[z]$, written as $f(z,w)=\sum_{i=0}^{n}\alpha_i w^i$, where $\alpha_i=\alpha_i(z)\in K[z]$. Note that the coefficient ring $K[z]$ is a Euclidean ring, and combined with the Euclidean division property of $K[z]$, we will investigate some classes of bivariate polynomial matrices with their entries in $K[z,w]$. The following three problems are also considered.

Problem 1. Let $F(z,w)\in K^{l\times l}[z,w]$, and $\det F(z,w)=h^r(z)$, where $h(z)\in K[z]$ is irreducible and r is a positive integer. When is $F(z,w)$ equivalent to

$$\begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix}? \tag{1}$$

Problem 2. Let $F(z,w)\in K^{l\times l}[z]$ with $\det F(z,w)=h^{q\cdot r}(z)$, $h(z)\in K[z]$ be irreducible, and q,r be positive integers. When is $F(z,w)$ equivalent to

$$\begin{pmatrix} I_{l-r} & & & \\ & h^q(z) & & \\ & & \ddots & \\ & & & h^q(z) \end{pmatrix}? \tag{2}$$

Problem 3. Let $F(z,w)\in K^{l\times m}[z]$ $(l\le m)$ with $d_l(F)=h^{q\cdot r}(z)$, where $d_l(F)$ denotes the greatest common divisor of the $l\times l$ minor of $F(z,w)$, $h(z)\in K[z]$ is irreducible, and q,r are positive integers. When is $F(z,w)$ equivalent to its Smith form?

2. Preliminaries

In the following, K is an arbitrary field, $K[z]$ is the univariate polynomial ring in variable z with coefficients in K, $K[z,w]$ is the bivariate polynomial ring in variables z,w whose coefficients are in K, $\overline{K}$ is the algebraic closed field of K, $0_{m,n}$ is the $m\times n$ zero matrix, and I_m is the $m\times m$ identity matrix. For $F(z,w)\in K^{l\times m}[z,w]$, $d_i(F)$ will be the greatest common divisor (g.c.d) of the $i\times i$ minors of $F(z,w)$, $i=1,\dots,l$. Set $F(z,w)=(f_{ij})\in K^{l\times m}[z,w]$, where $f_{ij}\in K[z,w]$, $h(z)\in K[z]$ is irreducible, and $\overline{F(z,w)}=(\overline{f_{ij}})\in K^{l\times m}[z,w]$, where $\overline{f_{ij}}$ denotes $f_{ij}\bmod h(z)$. For convenience, the argument (z,w) is omitted whenever its omission does not cause confusion throughout this paper.

Definition 1 (see [16]). Let $F(z,w)\in K^{l\times m}[z,w]$ $(l\le m)$ with rank r and Φ_i be a polynomial defined as follows:

$$\Phi_i=\begin{cases}\dfrac{d_i(F)}{d_{i-1}(F)}, & 1\le i\le r,\\[2ex] 0, & r<i\le l,\end{cases} \tag{3}$$

where $d_0(F)\equiv 1$, $d_i(F)$ is the g.c.d of the $i\times i$ minors of $F(z,w)$, and Φ_i satisfies

$$\Phi_1|\Phi_2|\cdots|\Phi_r. \tag{4}$$

Then, the Smith form of $F(z,w)$ is given by

$$S=\begin{pmatrix}\operatorname{diag}\{\Phi_i\} & 0_{r,m-r}\\ 0_{l-r,r} & 0_{l-r,m-r}\end{pmatrix}. \tag{5}$$

Definition 2 (see [16]). Let $F(z,w)\in K^{l\times m}[z,w]$ with full row rank; $F(z,w)$ is said to be zero left prime if the $l\times l$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$.

Definition 3. Let $U(z,w)\in K^{l\times l}[z,w]$; then, we say $U(z,w)$ to be a unimodular matrix if the determinant of $U(z,w)$ is a unit of K.

Definition 4 Let $F_1(z,w),F_2(z,w)\in K^{l\times m}[z,w]$; $F_1(z,w)$ is said to be equivalent to $F_2(z,w)$ if there are unimodular matrices $U(z,w)\in K^{l\times l}[z,w]$ and $V(z,w)\in K^{m\times m}[z,w]$ such that

$$F_1(z,w)=U(z,w)F_2(z,w)V(z,w). \tag{6}$$

Lemma 1 (see [16]). *Suppose $F(z,w),Q(z,w)\in K^{l\times l}[z,w]$; if $F(z,w)$ is equivalent to $Q(z,w)$, then $d_i(F(z,w))=d_i(Q(z,w))$, for $i=1,\dots,l$.*

Lemma 2 (see [16]). *Let $F(z,w)$, $F_1(z,w)$, $F_2(z,w)\in K^{l\times l}[z,w]$, and $F(z,w)=F_1(z,w)F_2(z,w)$. If the $(l-r)\times(l-r)$ $(r\le l)$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$, then the $(l-r)\times(l-r)$ minors of F_i have no common zero in $\overline{K}^2$ for $i=1,2$.*

3. Main Results

In this section, we investigate the three problems presented in Section 1 and give the main results of this paper.

In the following, for $f(z,w)\in K[z,w]$, we consider it as an element in $K[z][w]$, written as $f(z,w)=\sum_{i=0}^{n}\alpha_i w^i$, where $\alpha_i=\alpha_i(z)\in K[z]$. If $h(z)\in K[z]$ is irreducible, then $\overline{f(z,w)}$ denotes $f(z,w)\bmod h(z)$ and $\overline{f(z,w)}\equiv\sum_{i=0}^{n_1}\beta_i w^i \pmod{h(z)}$ or 0, where $\beta_i=\beta_i(z)\in K[z]$, $\deg\beta_i(z)<\deg h(z)$. Hence, $\beta_{n_1}(z)$ and $h(z)$ are relatively prime in $K[z]$; by Euclidean algorithm, there are $x(z),y(z)\in K[z]$ such that $x(z)\cdot\beta_{n_1}(z)=1-y(z)\cdot h(z)$, i.e., $x(z)\cdot\beta_{n_1}(z)\equiv 1\pmod{h(z)}$. Then, $x(z)\cdot\overline{f(z,w)}\equiv f^*(z,w)\pmod{h(z)}$, where $f^*(z,w)$ is monic. In other words, $x(z)\overline{f(z,w)}$ can be reduced to a monic polynomial $f^*(z,w)$ by $h(z)$.

Denote

$$P(z)=\begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix}. \tag{7}$$

First, we investigate Problem 1.

Theorem 1. *Let $F(z,w) \in K^{l\times l}[z,w]$ with $\det F(z,w) = h^r(z)$, where $h(z) \in K[z]$ is irreducible. If $h(z) | d_{l-r+1}(F(z,w))$, then $F(z,w)$ is equivalent to the Smith form $P(z)$.*

Proof. If r rows of $F(z,w)$ are zero vectors mod $h(z)$, that is, r rows of $\overline{F(z,w)}$ are zero vectors, we obtain that

$$F(z,w) = U(z,w)\begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix} Q(z,w), \quad (8)$$

and then $F(z,w) = U(z,w)P(z)Q(z,w)$, where $U(z,w)$ is unimodular; by computing, $\det Q(z,w) = 1$, so $Q(z,w)$ is unimodular. Hence, $F(z,w)$ is equivalent to the Smith form $P(z)$.

If $F(z,w)$ has no r rows which are the zero vectors mod $h(z)$, then $F(z,w)$ has r_1 rows of zero vectors mod $h(z)$; in other words, $\overline{F(z,w)}$ has r_1 rows of zero vectors, $0 \le r_1 \le r-1$. We premultiply and postmultiply $\overline{F(z,w)}$ by unimodular matrices U_{11} and V_{11} such that

$$U_{11}\overline{F(z,w)}V_{11} = \begin{pmatrix} \overline{f(z,w)} & \\ \overline{f_2(z,w)} & \\ \vdots & X \\ \overline{f_{l-r_1}(z,w)} & \\ 0_{r_1,1} & 0_{r_1,l-1} \end{pmatrix}, \quad (9)$$

where $\overline{f(z,w)} \ne 0$, $\deg_w \overline{f(z,w)} \le \deg_w \overline{f_j(z,w)}$, or $\overline{f_j(z,w)} = 0$, $j = 2,\dots,l-r_1$. Let $\overline{f(z,w)} = \alpha_0 + \alpha_1 w + \cdots + \alpha_n w^n$, where $\deg_z \alpha_i < \deg_z h(z)$, $i = 0,\dots,n$, and $\alpha_n \ne 0$. Note that α_n and $h(z)$ are relatively prime, so we can find $x(z), y(z) \in K[z]$ such that $x(z)\cdot\alpha_n + y(z)\cdot h(z) = 1$; then, $x(z)\cdot\alpha_n = 1 - y(z)\cdot h(z)$. We have $x(z)\cdot\overline{f(z,w)} \equiv f'(z,w) (\mathrm{mod}\, h(z))$, where $f'(z,w)$ is monic. There are $q_j(z,w), r_j(z,w) \in K[z,w]$ such that $\overline{f_j(z,w)} = q_j(z,w)\cdot f'(z,w) + r_j(z,w)$, where $\deg_w r_j(z,w) < \deg_w f'(z,w)$ or $r_j(z,w) = 0$, $j = 2,\dots,l-r_1$. Therefore,

$$\overline{f_j(z,w)} \equiv q_j(z,w)\cdot x(z)\cdot\overline{f(z,w)} + \overline{r_j(z,w)} (\mathrm{mod}\, h(z)), \quad (10)$$

where $\deg_w \overline{r_j(z,w)} < \deg_w \overline{f(z,w)}$ or $\overline{r_j(z,w)} = 0$, $j = 2,\dots,l-r_1$.

Let

$$U_{12} = \begin{pmatrix} 1 & 0_{1,l-1} \\ -x(z)q_2(z,w) & \\ \vdots & I_{l-1} \\ -x(z)q_{l-r_1}(z,w) & \\ 0_{r_1,l-r_1} & \end{pmatrix}. \quad (11)$$

Then,

$$U_{12}U_{11}\overline{F(z,w)}V_{11} = \begin{pmatrix} \overline{f(z,w)} & \\ \overline{r_2(z,w)} & \\ \vdots & X \\ \overline{r_{l-r_1}(z,w)} & \\ 0_{r_1,1} & 0_{r_1,l-1} \end{pmatrix} \mathrm{mod} h(z), \quad (12)$$

where $\deg_w \overline{r_j(z,w)} < \deg_w \overline{f(z,w)}$ or $\overline{r_j(z,w)} = 0$, $j = 2,\dots,l-r_1$. If some of $\overline{r_j(z,w)}$ are not 0, $j = 2,\dots,l-r_1$, do some row transformations to the matrix $U_{12}U_{11}\overline{F(z,w)}V_{11}$ such that the nonzero polynomial of the least degree in w among its first column is at position (1, 1). Repeating the previous steps, we obtain that

$$U_1\overline{F(z,w)}V_{11} = \begin{pmatrix} \overline{d}_{11} & X \\ 0_{l-1,1} & \overline{F_1(z,w)} \end{pmatrix} \mathrm{mod}\, h(z), \quad (13)$$

where $U_1 = U_{1t}\cdots U_{12}U_{11}$ is a unimodular matrix, the last r_1 rows of $\overline{F_1(z,w)}$ are zero vectors, and $\overline{d}_{11} \ne 0$.

If r rows of $\overline{F_1(z,w)}$ are zero vectors mod $h(z)$, we can find two $(l-1)\times(l-1)$ unimodular matrices U'_{20}, V'_{20} such that

$$U'_{20}\overline{F_1(z,w)}V'_{20} = \begin{pmatrix} B(z,w) \\ 0_{r,l-1} \end{pmatrix} \mathrm{mod}\, h(z), \quad (14)$$

where $B(z,w) \in K^{(l-r-1)\times(l-1)}[z,w]$.

Let $U_{20} = \begin{pmatrix} 1 & \\ & U'_{20} \end{pmatrix}$ and $V_{20} = \begin{pmatrix} 1 & \\ & V'_{20} \end{pmatrix}$; we have that

$$U_{20}U_1F(z,w)V_{11}V_{20} = \begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix} Q_1, \quad (15)$$

where $Q_1 \in K^{l\times l}[z,w]$ and $\det Q_1 = 1$. Then, $F(z,w) = UP(z)V$, where $U = U_1^{-1}U_{20}^{-1}$ and $V = Q_1V_{20}^{-1}V_{11}^{-1}$ are unimodular matrices, so $F(z,w)$ is equivalent to the Smith form $P(z)$.

If $\overline{F_1(z,w)}$ has no r rows of zero vectors mod $h(z)$, then $\overline{F_1(z,w)}$ has r_2 rows of zero vectors, $r_1 \le r_2 \le r-2$. Imitating the previous procedure to $\overline{F(z,w)}$, there are two $(l-2)\times(l-2)$ unimodular matrices U'_{21}, V'_{21} such that

$$U'_{21}\overline{F_1(z,w)}V'_{21} = \begin{pmatrix} \overline{d}_{22} & X \\ 0_{l-2,1} & \overline{F_2(z,w)} \end{pmatrix} \mathrm{mod}\, h(z), \quad (16)$$

where $\overline{d}_{22} \ne 0$ and the last r_2 rows of $\overline{F_1(z,w)}$ are zero vectors.

Let $U_{21} = \begin{pmatrix} 1 & \\ & U'_{21} \end{pmatrix}$ and $V_{21} = \begin{pmatrix} 1 & \\ & V'_{21} \end{pmatrix}$; then,

$$U_{21}U_1\overline{F(z,w)}V_{11}V_{21} = \begin{pmatrix} \overline{d}_{11} & * & X \\ & \overline{d}_{22} & \\ 0_{l-2,2} & & \overline{F_2(z,w)} \end{pmatrix} \mathrm{mod}\, h(z). \quad (17)$$

Repeating the procedure above successively, we obtain a series of $\overline{F_i(z,w)}$, $i = 3,\ldots,l-r-1$. If there is some $\overline{F_i(z,w)}$ which contains r rows of zero vectors mod $h(z)$, then the conclusion is straightforward. Otherwise, $\overline{F_i(z,w)}$ has no r rows of zero vectors mod $h(z)$.

Then, we consider the case that $\overline{F_i(z,w)}$ has no r rows of zero vectors mod $h(z)$, $i = 3,\ldots,l-r-1$. In this case, $\overline{F_{l-r-1}(z,w)} \in K^{(r+1)\times(r+1)}[z,w]$, and it has no r rows of zero vectors mod $h(z)$; there are $(r+1)\times(r+1)$ unimodular matrices $U'_{l-r,1}, V'_{l-r,1}$ such that

$$U'_{l-r,1}\overline{F_{l-r-1}(z,w)}V'_{l-r,1} = \begin{pmatrix} \overline{d}_{l-r,l-r} & X \\ 0_{r,l-r} & \overline{F_{l-r}(z,w)} \end{pmatrix} \bmod h(z), \quad (18)$$

where $\overline{d}_{l-r,l-r} \neq 0$.

Let $U_{l-r,1} = \begin{pmatrix} I_{l-r-1} & \\ & U'_{l-r,1} \end{pmatrix}$ and $V_{l-r,1} = \begin{pmatrix} I_{l-r-1} & \\ & V'_{l-r,1} \end{pmatrix}$; then,

$$U_{l-r,1}U_{l-r-1,1}\cdots U_{21}U_1\overline{F(z,w)}V_{11}V_{21}\cdots V_{l-r-1,1}V_{l-r,1} = \begin{pmatrix} \overline{d}_{11} & * & \cdots & & \\ & \overline{d}_{22} & * & \cdots & X \\ & & \ddots & & \\ & & & \overline{d}_{l-r,l-r} & \\ & 0_{r,l-r} & & & \overline{F_{l-r}(z,w)} \end{pmatrix} \bmod h(z). \quad (19)$$

Let

$$A(z,w) = \begin{pmatrix} \overline{d}_{11} & * & \cdots & & \\ & \overline{d}_{22} & * & \cdots & X \\ & & \ddots & & \\ & & & \overline{d}_{l-r,l-r} & \\ & 0_{r,l-r} & & & \overline{F_{l-r}(z,w)} \end{pmatrix}, \quad (20)$$

$$\overline{F_{l-r}(z,w)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix},$$

combined with $h(z)|d_{l-r+1}(F(z,w))$; we have that $d_{l-r+1}(\overline{F(z,w)}) \equiv 0 \bmod h(z)$ and $d_{l-r+1}(A(z,w)) \equiv 0 \bmod h(z)$. Considering the $(l-r+1)\times(l-r+1)$ minors of $A(z,w)$, we see that $\overline{d}_{11}\overline{d}_{22}\cdots\overline{d}_{l-r,l-r}\cdot a_{k,j} \equiv 0 \bmod h(z)$ for all $k,j = 1,\ldots,r$. Since $\overline{d}_{i,i} \neq 0$, $i = 1,\ldots,l-r$, $a_{k,j} \equiv 0 \bmod h(z)$, $k,j = 1,\ldots,r$. Hence, $\overline{F_{l-r}(z,w)} \equiv 0_{r,r} \bmod h(z)$. Hence,

$$U_{l-r,1}U_{l-r-1,1}\cdots U_{21}U_1F(z,w)V_{11}V_{21}\cdots V_{l-r-1,1}V_{l-r,1} = A(z,w) = \begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix} Q_2, \quad (21)$$

where $Q_2 \in K^{l\times l}[z,w]$. Note that $\det F(z,w) = h^r(z)$ and U_1, U_{i1}, V_{i1} are unimodular matrices, $i = 1,2,\ldots,l-r$; then, we obtain that $\det Q_2 = 1$ and Q_2 is a unimodular matrix. Let $U = U_{l-r,1}U_{l-r-1,1}\cdots U_1$ and $V = V_{11}V_{21}\cdots V_{l-r,1}Q_2^{-1}$; then, $F(z,w) = U^{-1}P(z)V^{-1}$ with U^{-1}, V^{-1} being unimodular matrices. Therefore, $F(z,w)$ is equivalent to the Smith form $P(z)$. □

Remark 1. Theorem 1 provides a positive answer to Problem 1.

Corollary 1. *Let $F(z,w) \in K^{l\times l}[z,w]$ and $h(z) \in K[z]$ be an irreducible polynomial. If $h(z)|d_{l-r+1}(F(z,w))$ and $h^r(z)|\det F(z,w)$, then $F(z,w)$ can be factorized as $T(z,w)P(z)Q(z,w)$, where $T(z,w), Q(z,w) \in K^{l\times l}[z,w]$ and $T(z,w)$ is unimodular.*

Theorem 2. *Let $F(z,w) \in K^{l\times l}[z,w]$ and $h(z) \in K[z]$ be an irreducible polynomial. Suppose $\det F(z,w) = h^{q\cdot r}(z)$, r and q are positive integers. If $h^q(z)|d_{l-r+1}(F(z,w))$, then $F(z,w)$ can be factorized as*

$$T_1(z,w)P(z)T_2(z,w)P(z)\cdots T_q(z,w)P(z)T(z,w), \quad (22)$$

where $P(z)$ is defined as above and $T(z,w), T_i(z,w)$ are unimodular matrices, $i = 1,2,\ldots,q$.

Proof. When $q = 1$, by Theorem 1, $F(z,w) = T_1(z,w)P(z)T(z,w)$, where $T(z,w), T_1(z,w)$ are unimodular matrices, so the conclusion is true.

When $q \geq 2$, since $h^r(z)$ is a factor of $\det F(z,w)$, according to Corollary 1, we see that $F(z,w)$ can be factorized as $T_1(z,w)P(z)F_1(z,w)$, where $T_1(z,w)$ is a unimodular matrix.

Then, we consider $F_1(z,w)$. For $\det(F_1(z,w)) = h^{(q-1)\cdot r}(z)$, according to Lemma 1, we obtain

$$d_i(F(z,w)) = d_i(P(z)F_1(z,w)), \quad i = 1,\ldots,l. \quad (23)$$

So,

$$d_{l-r+1}(F(z,w)) = d_{l-r+1}(P(z)F_1(z,w)) = d_{l-r+1}(P(z))d_{l-r+1}(F_1(z,w)) = h(z)d_{l-r+1}(F_1(z,w)). \quad (24)$$

Note that $h^q(z)|d_{l-r+1}(F(z,w))$; then, $h^{q-1}(z)|d_{l-r+1}(F_1(z,w))$. According to Corollary 1, $F_1(z,w)$ can be factorized as $T_2(z,w)P(z)F_2(z,w)$. Then,

$$F(z,w) = T_1(z,w)P(z)T_2(z,w)P(z)F_2(z,w). \quad (25)$$

Imitating the procedure above successively, we can obtain that

$$F(z,w) = T_1(z,w)P(z)T_2(z,w)P(z)\cdots T_q(z,w)P(z)T(z,w), \quad (26)$$

where $T(z,w), T_i(z,w)$ are unimodular matrices, $i = 1, 2, \ldots, q$. □

Lemma 3. *Let $P(z)$ be defined as above and $T(z,w) \in K^{l\times l}[z,w]$ be unimodular. Suppose $F(z,w) = P^{s_1}(z)T(z,w)P^{s_2}(z)$. If the $(l-r)\times(l-r)$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$, then $F(z,w)$ is equivalent to $P^{s_1+s_2}(z)$.*

Proof. Let

$$T(z,w) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad (27)$$

where $T_{11} \in K^{(l-r)\times(l-r)}[z,w]$, $T_{12} \in K^{(l-r)\times r}[z,w]$, $T_{21} \in K^{r\times(l-r)}[z,w]$, and $T_{22} \in K^{r\times r}[z,w]$.

Note that

$$P(z) = \begin{pmatrix} I_{l-r} & \\ & h(z)I_r \end{pmatrix}. \quad (28)$$

Then,

$$F(z,w) = P^{s_1}(z)T(z,w)P^{s_2}(z) = \begin{pmatrix} T_{11} & T_{12}h^{s_2}(z) \\ T_{21}h^{s_1}(z) & T_{22}h^{s_1+s_2}(z) \end{pmatrix}. \quad (29)$$

By computing, $d_{l-r+1}(F(z,w)) = h^{s_1+s_2}(z)\, d_{l-r+1}(T(z,w))$. Because $T(z,w)$ is unimodular and the $(l-r)\times(l-r)$ minors of $F(z,w)$ have no common zero,

$$\begin{aligned} &d_1(F(z,w)) = d_2(F(z,w)) = \cdots = d_{l-r}(F(z,w)) = 1, \\ &d_{l-r+k}(F(z,w)) = h^{k(s_1+s_2)}(z), \quad 1 \le k \le r. \end{aligned} \quad (30)$$

Since (T_{11}, T_{12}) is ZLP, the $(l-r)\times(l-r)$ minors of (T_{11}, T_{12}) have no common zero. Set the $(l-r)\times(l-r)$ minors of (T_{11}, T_{12}) to be $e_1, e_2, \ldots, e_t$, where $e_1 = \det(T_{11})$. Then, the $(l-r)\times(l-r)$ minors of $(T_{11}, T_{12}h^{s_2}(z))$ are $e_1, e_2h^{m_2}(z), \ldots, e_th^{m_t}(z)$, where $s_2 \le m_i \le r\cdot s_2$, $i = 2, \ldots, t$.

We prove that $(T_{11}, T_{12}h^{s_2}(z))$ is ZLP.

Suppose the $(l-r)\times(l-r)$ minors of $(T_{11}, T_{12}h^{s_2}(z))$ have a common zero a_0; then, a_0 is a common zero of e_1 and $h(z)$. Note that

$$F(z,w) = \begin{pmatrix} T_{11} & T_{12}h^{s_2}(z) \\ T_{21}h^{s_1}(z) & T_{22}h^{s_1+s_2}(z) \end{pmatrix}. \quad (31)$$

Then, the $(l-r)\times(l-r)$ minors of $F(z,w)$ have common zero a_0. This is a contradiction. So, the $(l-r)\times(l-r)$ minors of $(T_{11}, T_{12}h^{s_2}(z))$ have no common zero. Hence, $(T_{11}, T_{12}h^{s_2}(z))$ is ZLP. By the Quillen–Suslin theorem, we can find a unimodular matrix $M_1 \in K^{(l-r)\times(l-r)}[z,w]$ that satisfies

$$(T_{11}, T_{12}h(z))M_1 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \end{pmatrix}. \quad (32)$$

Then,

$$F(z,w)M_1 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \\ N_1 & N_2 \end{pmatrix}, \quad (33)$$

where $N_1 \in K^{r\times(l-r)}[z,w]$ and $N_2 \in K^{r\times r}[z,w]$. Let

$$M_2 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \\ -N_1 & I_r \end{pmatrix} \quad (34)$$

such that

$$M_2F(z,w)M_1 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \\ 0_{r,l-r} & N_2 \end{pmatrix}. \quad (35)$$

Let

$$M_3 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \\ 0_{r,l-r} & N_2 \end{pmatrix}. \quad (36)$$

We know M_3 is equivalent to $F(z,w)$.

According to Lemma 1, $d_i(M_3) = d_i(F(z,w))$, $i = 1, \ldots, l$. Hence,

$$\begin{aligned} &d_1(M_3) = d_2(M_3) = \cdots = d_{l-r}(M_3) = 1, \\ &d_{l-r+k}(M_3) = h^{k(s_1+s_2)}(z), \quad 1 \le k \le r. \end{aligned} \quad (37)$$

Notice that

$$M_3 = \begin{pmatrix} I_{l-r} & 0_{l-r,r} \\ 0_{r,l-r} & N_2 \end{pmatrix} = \begin{pmatrix} I_{l-r} & & & & \\ & b_{l-r+1,l-r+1} & b_{l-r+1,l-r+2} & \cdots & b_{l-r+1,l} \\ & b_{l-r+2,l-r+1} & b_{l-r+2,l-r+2} & \cdots & b_{l-r+2,l} \\ & \vdots & \vdots & \ddots & \vdots \\ & b_{l,l-r+1} & b_{l,l-r+1} & \cdots & b_{l,l} \end{pmatrix}. \quad (38)$$

We have that $h(z)$ divides every element of N_2, so

$$M_3 = \begin{pmatrix} I_{l-r} & & & \\ & h^{s_1+s_2}(z) & & \\ & & \ddots & \\ & & & h^{s_1+s_2}(z) \end{pmatrix} M_4. \quad (39)$$

Hence, $M_3 = P^{s_1+s_2}(z) \cdot M_4$, and $\det M_4 = 1$. Note that $M_2 \cdot F(z,w) \cdot M_1 = M_3$, so $M_2 \cdot F(z,w) \cdot M_1 = P^{s_1+s_2}(z) \cdot M_4$; then, $F(z,w) = M_2^{-1} P^{s_1+s_2}(z) \cdot M_4 M_1^{-1}$, where M_2^{-1} and $M_4 M_1^{-1}$ are unimodular matrices. Thus, $F(z,w)$ is equivalent to $P^{s_1+s_2}(z)$. □

Theorem 3. *Let $F(z,w) \in K^{l\times l}[z,w]$ and $h(z) \in K[z]$ be an irreducible polynomial. If $\det F(z,w) = h^{q\cdot r}(z)$ and $h^q(z) | d_{l-r+1}(F(z,w))$, where q and r are positive integers, then $F(z,w)$ is equivalent to the Smith form*

$$P^q(z) = \begin{pmatrix} I_{l-r} & & & \\ & h^q(z) & & \\ & & \ddots & \\ & & & h^q(z) \end{pmatrix}, \quad (40)$$

if and only if the $(l-r)\times(l-r)$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$.

Proof. Sufficiency: by Theorem 2, we have that

$$F(z,w) = T_1(z,w)P(z)T_2(z,w)P(z)\cdots T_q(z,w)P(z)T(z,w), \quad (41)$$

where

$$P(z) = \begin{pmatrix} I_{l-r} & & & \\ & h(z) & & \\ & & \ddots & \\ & & & h(z) \end{pmatrix}. \quad (42)$$

$T(z,w), T_i(z,w)$ are unimodular matrices, $i = 1, 2, \ldots, q$.

We first prove that $P(z)T_2(z,w)P(z)$ is equivalent to $P^2(z)$. From the $(l-r)\times(l-r)$ minors of $F(z,w)$ which have no common zero in $\overline{K}^2$, we know that the $(l-r)\times(l-r)$ minors of $P(z)T_2(z,w)P(z)$ have no common zero in $\overline{K}^2$ by combining with Lemma 2. Then, according to Lemma 3, $P(z)T_2(z,w)P(z)$ is equivalent to $P^2(z)$.

Repeating the procedure above, we obtain that $P(z)T_2(z,w)P(z)\cdots T_q(z,w)P(z)$ is equivalent to the matrix $P^q(z)$, so there exist $U_1(z,w), V_1(z,w)$ such that

$$P(z)T_2(z,w)P(z)\cdots T_q(z,w)P(z) = U_1(z,w)P^q(z)V_1(z,w), \quad (43)$$

where $U_1(z,w), V_1(z,w)$ are unimodular matrices. Hence, we have that

$$F(z,w) = U(z,w)P^q(z)V(z,w), \quad (44)$$

where $U(z,w) = T_1(z,w)U_1(z,w)$ and $V(z,w) = V_1(z,w)T(z,w)$ are unimodular matrices. Therefore, $F(z,w)$ is equivalent to the Smith form $P^q(z)$.

Necessity: since $F(z,w)$ is equivalent to $P^q(z)$, there exist unimodular matrices $U(z,w)$ and $V(z,w)$ such that $F(z,w) = U(z,w)P^q(z)V(z,w)$. For the $(l-r)\times(l-r)$ minors of $P^q(z)$ which have no common zero in $\overline{K}^2$, the $(l-r)\times(l-r)$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$ by Lemma 2. □

Remark 2. According to Theorem 3, Problem 2 is solved, and a criterion for discriminating this class of bivariate polynomial matrices to be equivalent to the Smith form is also presented.

Theorem 4. *Let $F(z,w) \in K^{l\times m}[z,w]$ $(l \le m)$ be of full row rank and $h(z) \in K[z]$ be irreducible. Suppose $d_l(F) = h^{q\cdot r}(z)$ and $h^q(z) | d_{l-r+1}(F)$, where q and r are positive integers. Then, $F(z,w)$ is equivalent to the Smith form*

$$Q(z) = \begin{pmatrix} P^q(z) & 0_{l\times(m-l)} \end{pmatrix}, \quad (45)$$

if and only if the $(l-r)\times(l-r)$ minors of $F(z,w)$ have no common zero in $\overline{K}^2$.

Proof. Sufficiency: by Theorem 3.3 in [17], there are $H(z,w) \in K^{l\times l}[z,w]$ and $F_1(z,w) \in K^{l\times m}[z,w]$ satisfying $F(z,w) = H(z,w)F_1(z,w)$, where $\det H(z,w) = d(F)$ and $F_1(z,w)$ is ZLP. Then, the $(l-r)\times(l-r)$ minors of $H(z,w)$ have no common zero by using Lemma 2. Combined with $h^q(z) | d_{l-r+1}(H)$ and Theorem 2, there are unimodular matrices $U_{11}(z,w), U_{12}(z,w) \in K^{l\times l}[z,w]$ such that $H(z,w) = U_{11}(z,w)P^q(z)U_{12}(z,w)$. Then,

$$F(z,w) = U_{11}(z,w)P^q(z)U_{12}(z,w)F_1(z,w). \quad (46)$$

We know that $U_{12}(z,w)F_1(z,w)$ is also ZLP ($U_{12}(z,w)$ is unimodular). According to the Quillen-Suslin theorem, there exists an $m\times m$ unimodular matrix $U(z,w)$ which satisfies that $U_{12}(z,w)F_1(z,w)U(z,w) = \begin{pmatrix} I_l & 0_{l,m-l} \end{pmatrix}$. Then,

$$F(z,w) = U_{11}(z,w)\begin{pmatrix} P^q(z) & 0_{l,m-l} \end{pmatrix} U(z,w) = U_1(z,w)Q(z)U(z,w). \quad (47)$$

According to the $(l-r)\times(l-r)$ minors of $F(z,w)$ which have no common zero and $h^q(z) | d_{l-r+1}(F)$, we see that $Q(z)$ is the Smith form of $F(z,w)$. Note that $U_1(z,w)$ and $U(z,w)$ are invertible matrices, so $F(z,w)$ is equivalent to the Smith form $Q(z)$.

Necessity: since $F(z,w)$ is equivalent to the Smith form $Q(z)$, it is easy to see that the $(l-r)\times(l-r)$ minors of $Q(z)$ have no common zero in $\overline{K}^2$. By Lemma 2, we have that the $(l-r)\times(l-r)$ minors of $F(z)$ have no common zero in $\overline{K}^2$. □

Remark 3. A positive answer to Problem 3 is presented in Theorem 4. And the equivalence of a rectangle matrix $F(z,w)$ and its Smith form is considered, which makes the result more general.

4. An Example

In this section, we use an example to illustrate our results and methods.

Example 1. Consider a 3×3 2D polynomial matrix of $\mathbb{R}^{3\times 3}[z,w]$:

$$F(z,w) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (48)$$

where

$$\begin{aligned}
a_{11} &= \left(z+1-w^2\right)(1-2w) - \left(2z^2+2zw+2\right)\left(z^2+z+1\right)^2, \\
a_{12} &= \left(z+1-w^2\right)(z-w) + [(zw+1-z)(w+z)+w-1]\left(z^2+z+1\right)^2, \\
a_{13} &= w\left(z+1-w^2\right) + \left(zw+z^2+1\right)\left(z^2+z+1\right)^2, \\
a_{21} &= -w(1-2w) - 2z\left(z^2+z+1\right)^2, \\
a_{22} &= -w(z-w) + (zw+1-z)\left(z^2+z+1\right)^2, \\
a_{23} &= -w^2 + z\left(z^2+z+1\right)^2, \\
a_{31} &= z(1-2w) - \left(2z^2+2\right)\left(z^2+z+1\right)^2, \\
a_{32} &= z(z-w) + [z(zw+1-z)+w-1]\left(z^2+z+1\right)^2, \\
a_{33} &= zw + \left(z^2+1\right)\left(z^2+z+1\right)^2.
\end{aligned} \quad (49)$$

By computing, $d_1(F(z,w)) = 1$, $d_2(F(z,w)) = (z^2+z+1)^2$, $d_3(F(z,w)) = (z^2+z+1)^4$, and $\det F(z,w) = (z^2+z+1)^4$. We know that the 1×1 minors of $F(z,w)$ have no common zero in $\mathbb{C}^2$. Combining $z^2+z+1 \in K[z]$ is irreducible and $(z^2+z+1)^2 | d_2(F(z,w))$; by Theorem 3, we have that $F(z,w)$ is equivalent to the Smith form

$$P^2(z) = \begin{pmatrix} 1 & & \\ & \left(z^2+z+1\right)^2 & \\ & & \left(z^2+z+1\right)^2 \end{pmatrix}. \quad (50)$$

Let $\overline{F(z,w)} = (\overline{f_{ij}})$ denote $F(z,w) = (f_{ij})$ mod (z^2+z+1); then,

$$\overline{F(z,w)} = \begin{pmatrix} \left(z+1-w^2\right)(1-2w) & \left(z+1-w^2\right)(z-w) & w\left(z+1-w^2\right) \\ 2w^2-w & w^2-zw & -w^2 \\ -2zw+z & -zw-z-1 & zw \end{pmatrix}, \quad (51)$$

and no row of $\overline{F(z,w)}$ is zero vector mod (z^2+z+1). Note that the nonzero polynomial of the least degree in w among the first column is $-2zw+z$, so we postmultiply $\overline{F(z,w)}$ by a unimodular matrix $U_{11}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ such that $-2zw+z$ can be changed to the position (1, 1). Then,

$$U_{11}\overline{F(z,w)}=\begin{pmatrix}-2zw+z & -zw-z-1 & zw\\ 2w^2-w & w^2-zw & -w^2\\ \left(z+1-w^2\right)(1-2w) & \left(z+1-w^2\right)(z-w) & w\left(z+1-w^2\right)\end{pmatrix}. \tag{52}$$

We consider the element $-2zw+z$; $-2z$ is the leading coefficient of $-2zw+z$. We know that $-2z$ and z^2+z+1 are relatively prime in $\mathbb{R}[z]$, so we can find $x_1(z)=(1/2)(z+1)$, $y_1(z)=1$ such that $x_1(z)\cdot(-2z)+y_1(z)\cdot(z^2+z+1)=1$ $((1/2)(z+1)\cdot(-2z)+(z^2+z+1)=1)$; then, $(1/2)(z+1)\cdot(-2z)=1-(z^2+z+1)\equiv 1 \bmod (z^2+z+1)$. So, $x_1(z)\cdot(-2zw+z)=(1/2)(z+1)\cdot(-2zw+z)\equiv(w-(1/2)) \bmod (z^2+z+1)$; that is, $-2zw+z$ can be reduced to a monic polynomial $(w-(1/2))$ by mod (z^2+z+1).

Then, we reduce other elements in the first column.

For the element $2w^2-w$, we can find $q_1(z,w)=2w$ and $r_1(z,w)=0$ such that $2w^2-w=(w-(1/2))\cdot q_1(z,w)+r_1(z,w)\equiv(1/2)(z+1)\cdot(-2zw+z)\cdot 2w+0 \bmod (z^2+z+1)$. In reality, $2w^2-w=(1/2)(z+1)\cdot(-2zw+z)\cdot 2w-(z^2+z+1)(w-2w^2)$.

And for the element $(z+1-w^2)(1-2w)$, we can find $q_2(z,w)=2w^2-2z-2$ and $r_2(z,w)=0$ such that

$$\begin{aligned}\left(z+1-w^2\right)(1-2w)&=\left(w-\frac{1}{2}\right)\cdot q_2(z,w)+r_2(z,w)=\left(w-\frac{1}{2}\right)\cdot\left(2w^2-2z-2\right)+\\ 0&\equiv\frac{1}{2}(z+1)\cdot(-2zw+z)\cdot\left(2w^2-2z-2\right)+0 \bmod\left(z^2+z+1\right).\end{aligned} \tag{53}$$

Actually, $(z+1-w^2)(1-2w)=(1/2)(z+1)\cdot(-2zw+z)\cdot(2w^2-2z-2)-(z^2+z+1)(w^2-z-1)(1-2w)$.

Let

$$U_{12}=\begin{pmatrix}1&0&0\\ -\frac{1}{2}(z+1)2w&1&0\\ -\frac{1}{2}(z+1)\left(2w^2-2z-2\right)&0&1\end{pmatrix}. \tag{54}$$

We have

$$U_{12}U_{11}\cdot\overline{F(z,w)}=\begin{pmatrix}-2zw+z & -zw-z-1 & zw\\ 0&0&0\\ 0&0&0\end{pmatrix}\bmod\left(z^2+z+1\right), \tag{55}$$

so

$$U_{12}U_{11}\cdot F(z,w)=\begin{pmatrix}1 & & \\ & z^2+z+1 & \\ & & z^2+z+1\end{pmatrix}F_1(z,w), \tag{56}$$

where

$$\begin{aligned}
F_1(z,w)&=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},\\
b_{11}&=z(1-2w)-\left(2z^2+2\right)\left(z^2+z+1\right)^2,\\
b_{12}&=z(z-w)+[z(zw+1-z)+w-1]\left(z^2+z+1\right)^2,\\
b_{13}&=zw+\left(z^2+1\right)\left(z^2+z+1\right)^2,\\
b_{21}&=2w^2-w+\left[-2z+(zw+w)\left(2z^2+2\right)\right]\left(z^2+z+1\right),\\
b_{22}&=\left(w^2-zw\right)-(zw+w)[z(zw+1-z)+w-1]\left(z^2+z+1\right)+(zw+1-z)\left(z^2+z+1\right),\\
b_{23}&=-w^2+\left[z-w(z+1)\left(z^2+1\right)\right]\left(z^2+z+1\right),\\
b_{31}&=(1-2w)\left(-w^2+1+z\right)+\left[-2z^2-2zw-2+(z+1)\left(-w^2+1+z\right)\left(-2z^2-2\right)\right]\left(z^2+z+1\right),\\
b_{32}&=\left(-w^2+1+z\right)(z-w)+[(zw+1-z)(w+z)+w-1]\left(z^2+z+1\right)+(z+1)\left(-w^2+1+z\right)\\
&\quad[z(zw+1-z)+w-1]\left(z^2+z+1\right),\\
b_{33}&=\left(-w^2+1+z\right)w+\left(z^2+zw+1\right)\left(z^2+z+1\right)+(z+1)\left(-w^2+z+1\right)\left(z^2+1\right)\left(z^2+z+1\right).
\end{aligned} \tag{57}$$

Let $U_1=U_{12}U_{11}$; then,

$$\begin{aligned}
F(z,w)&=U_1^{-1}\begin{pmatrix}1 & & \\ & z^2+z+1 & \\ & & z^2+z+1\end{pmatrix}F_1(z,w),\\
\overline{F_1(z,w)}&=\begin{pmatrix}-2zw+z & -zw-z-1 & zw\\ 2w^2-w & w^2-zw & -w^2\\ (1-2w)\left(-w^2+1+z\right) & \left(-w^2+z+1\right)(z-w) & w\left(-w^2+z+1\right)\end{pmatrix}.
\end{aligned} \tag{58}$$

Now, consider $\overline{F_1(z,w)}$, for none of its rows are zero vectors mod (z^2+z+1); repeating the steps above, we can obtain that

$$U_{13}\cdot\overline{F_1(z,w)}=\begin{pmatrix}-2zw+z & -zw-z-1 & zw\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\operatorname{mod}\left(z^2+z+1\right), \tag{59}$$

where

$$U_{13}=\begin{pmatrix}1 & 0 & 0\\ -\frac{1}{2}(z+1)2w & 1 & 0\\ -\frac{1}{2}(z+1)\left(2w^2-2z-2\right) & 0 & 1\end{pmatrix}. \quad (60)$$

Hence,

$$U_{13}F_1(z,w)=\begin{pmatrix}1 & & \\ & z^2+z+1 & \\ & & z^2+z+1\end{pmatrix}F_2(z,w), \quad (61)$$

where

$$\begin{aligned}
F_2(z,w)&=\begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix},\\
c_{11}&=z(1-2w)-\left(2z^2+2\right)\left(z^2+z+1\right)^2,\\
c_{12}&=z(z-w)+[z(zw+1-z)+w-1]\left(z^2+z+1\right)^2,\\
c_{13}&=zw+\left(z^2+1\right)\left(z^2+z+1\right)^2,\\
c_{21}&=2w^2-w-2z+(zw+w)\left(2z^2+2\right)\left(z^2+z+2\right),\\
c_{22}&=\left(w^2+1-z\right)-(zw+w)\left(z^2w+z-z^2\right)\left(z^2+z+2\right)-\left(w^2-w\right)(z+1)\left(z^2+z+2\right),\\
c_{23}&=z-w^2-w(z+1)\left(z^2+1\right)\left(z^2+z+2\right),\\
c_{31}&=(1-2w)\left(-w^2+1+z\right)-\left(2z^2+2zw+2\right)+(z+1)\left(-w^2+1+z\right)\left(-2z^2-2\right)\left(z^2+z+2\right),\\
c_{32}&=\left(-w^2+1+z\right)(z-w)+(zw+1-z)(w+z)+w-1+\left(z^2+z+2\right)(z+1)\left(-w^2+1+z\right)\\
&\quad(w-1)+\left(z^2+z\right)\left(-w^2+1+z\right)(zw+1-z)\left(z^2+z+2\right),\\
c_{33}&=\left(-w^2+1+z\right)w+\left(z^2+zw+1\right)+\left(z^2+z+2\right)(z+1)\left(-w^2+z+1\right)\left(z^2+1\right).
\end{aligned} \quad (62)$$

So,

$$F_1(z,w)=U_{13}^{-1}\begin{pmatrix}1 & & \\ & z^2+z+1 & \\ & & z^2+z+1\end{pmatrix}F_2(z,w). \quad (63)$$

Combining

$$F(z,w)=U_1^{-1}P(z)F_1(z,w), \quad (64)$$

we have that

$$F(z,w)=U_1^{-1}P(z)U_{13}^{-1}P(z)F_2(z,w). \quad (65)$$

Consider $P(z)U_{13}^{-1}P(z)$; combined with Lemma 3, we obtain

$$P(z)U_{13}^{-1}P(z)=\begin{pmatrix}1 & 0 & 0\\ (wz+w)\left(z^2+z+1\right) & \left(z^2+z+1\right)^2 & 0\\ -(z+1)\left(-w^2+1+z\right)\left(z^2+z+1\right) & 0 & \left(z^2+z+1\right)^2\end{pmatrix}. \quad (66)$$

Let

$$U_{14}=\begin{pmatrix}1 & 0 & 0\\ (wz+w)\left(z^2+z+1\right) & 1 & 0\\ -(z+1)\left(-w^2+1+z\right)\left(z^2+z+1\right) & 0 & 1\end{pmatrix}. \quad (67)$$

Then,

$$P(z)U_{13}^{-1}P(z)=U_{14}P^2(z). \quad (68)$$

So,

$$F(z,w)=U_1^{-1}U_{14}\begin{pmatrix}1 & & \\ & \left(z^2+z+1\right)^2 & \\ & & \left(z^2+z+1\right)^2\end{pmatrix}F_2(z,w), \quad (69)$$

and by computing, $U_1^{-1}U_{14}$ and $F_2(z,w)$ are unimodular matrices. Therefore, $F(z,w)$ is equivalent to $P^2(z)$.

5. Conclusions

In this paper, we have investigated the reduction of several kinds of bivariate polynomial matrices in $K[z,w]$, where K is an arbitrary field. Some new results on these matrices to be reduced to their Smith forms are presented. Furthermore, the conditions of these results are easily verified. An example is given to illustrate our method in the end of the article. All of these are useful for reducing 2D systems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11871207 and 11971161).

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