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Research Article

Solitons, Breathers, and Lump Solutions to the $(2+1)$-Dimensional Generalized Calogero–Bogoyavlenskii–Schiff Equation

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In this paper, a generalized $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff equation is considered. Based on the Hirota bilinear method, three kinds of exact solutions, soliton solution, breather solutions, and lump solutions, are obtained. Breathers can be obtained by choosing suitable parameters on the 2-soliton solution, and lump solutions are constructed via the long wave limit method. Figures are given out to reveal the dynamic characteristics on the presented solutions. Results obtained in this work may be conducive to understanding the propagation of localized waves.

1. Introduction

Nonlinear subject is a new interdisciplinary subject which studies the common properties of nonlinear phenomena. The theory of solitons, as one of the three branches of nonlinear science, has wild applications in many fields of natural science such as fluids, plasmas, nonlinear optics, field optics, solid-state physics, and marine science. Hence, it is very important and meaningful to study the exact solution of the nonlinear system. By far, researchers have established several effective methods to search exact solutions of soliton equations, including the Bäcklund transformation [1–7], the Darboux transformation [8–13], the Riemann Hilbert approach [14], Hirota's bilinear method [15–20], tanh-function method [21–24], and so on [25]. Among these methods, the Hirota bilinear transformation is widely used by scholars because of its simplicity and directness.

The Hirota bilinear transformation method can be used to find the soliton, breather, lump, and rouge wave solutions of the equation. Solitons, breathers, lumps, and rogue waves are four types of nonlinear localized waves, which have some physical applications in nonlinear optics, plasmas, shallow water waves, and Bose–Einstein condensate. Solitons [26–29] are the stable nonlinear waves. Lump and lump-type are a kind of rational function. Lump [30–39] is a rational function solution and localized in all space directions. Rogue waves [40–43] are localized in both space and time and appear from nowhere, and disappear without a trace. Breathers [44–49] are the partially localized breathing waves with a periodic structure in a certain direction. Rogue waves and breathers are localized structures under the background of the instability. In recent years, the study of nonlinear localized waves and interaction solutions among them is one of the important research subjects. For example, based on the Hirota bilinear method, Yue et al. [50] obtained the N-solitons, breathers, lumps, and rogue waves of the $(3+1)$-dimensional nonlinear evolution equation and analysed the impacts of the parameters on these solutions. Based on the Hirota bilinear method, Liu et al. [51] constructed the N-soliton solution for the $(2+1)$-dimensional generalized Hirota–Satsuma–Ito equation, from which some localized waves such as line solitons, lumps, periodic solitons, and their interactions are obtained by choosing special parameters. Hossen et al. [52] derived multisolitons, breather solutions, lump soliton, lump kink waves, and multilumps for the $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Veselov equation based on the bilinear formalism and with the aid of symbolic computation.

In this paper, we study soliton solution and local wave solution of generalized $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff equation:

$$u_t + u_{xxy} + 3uu_y + 3u_x v_y + \delta_1 u_y + \delta_2 v_{yy} = 0, \quad v_x = u, \tag{1}$$

which was constructed by Bogoyavlenskii [53] and Schiff [54] in different ways. This is a generalization of a $(2+1)$-dimensional CBS equation considered in [55]:

$$v_{xt} + v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y = 0, \tag{2}$$

whose coefficients have a different pattern from the original one (4.2) (see, e.g., [56] and references therein). In [57], Toda and Yu derived the $(2+1)$-dimensional CBS equation from the Korteweg–de Vries equation. Based on the Hirota bilinear formulation, Chen and Ma [30] explored lump solutions, through Maple symbolic computations by using quadratic polynomial, to a generalized Calogero–Bogoyavlenskii–Schiff equation. Wazwaz [55] derived multiple-soliton solutions and multiple singular soliton solutions for the $(2+1)$ and $(3+1)$-dimensional CBS equations, based on the Cole–Hopf transformation and the Hirota bilinear method. Bruzon et al. [56] used classical and nonclassical methods to obtain symmetry reductions and exact solutions of the $(2+1)$-dimensional integrable Calogero–Bogoyavlenskii–Schiff equation. Very recently, Roshid [58, 59] gave the general formula of n-soliton and found the various dynamics.

The structure of this paper is as follows. In Section 2, we introduce the bilinear form of a generalized Calogero–Bogoyavlenskii–Schiff equation. Then, based on the Hirota bilinear method, we will get the soliton solutions of equation (1). In Section 3, by choosing suitable parameters on the two-soliton solution, breather solutions can be obtained. Moreover, we will get the y-periodic soliton structures of solutions and the (x, y)-periodic soliton structures by choosing different parameters on the breather solution, In Section 4, in order to obtain the lump solution, we can choose suitable parameters on the two-soliton solution. We shall give our conclusions in Section 5.

2. The Soliton Solutions

By using transformation,

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_x. \tag{3}$$

Equation (1) is converted into the following bilinear formulism [30]:

$$\left(D_t D_x + D_x^3 D_y + \delta_1 \cdot D_x D_y + \delta_2 \cdot D_y^2\right) f \cdot f = 0. \tag{4}$$

That is,

$$2\left[f_{tx} f - f_t f_x + f_{xxxy} f - f_{xxx} f_y - 3f_{xxy} f_x + 3f_{xx} f_{xy} + \delta_1\left(f_{xy} f - f_x f_y\right) + \delta_2\left(f_{yy} f - f_y^2\right)\right] = 0, \tag{5}$$

where $f = f(x, y, t)$, and the derivatives $D_t D_x, D_x^3 D_y, D_x D_y, D_y^2$ are all bilinear derivative operators [15] defined by

$$D_x^m D_y^n D_t^p (f \cdot g) = (\partial_x - \partial_{x'})^m \left(\partial_y - \partial_{y'}\right)^n (\partial_t - \partial_{t'})^p f \cdot (x, y, t) g(x', y', t')|_{x=x', y=y', t=t'}. \tag{6}$$

It is clear that if f solves equation (5), then $u = u(x, y, t)$ is a solution of equation (1) through transformation (3).

2.1. The 1-Soliton Solution. In order to find one soliton solution of generalized Calogero–Bogoyavlenskii–Schiff equation, suppose

$$f = 1 + e^{\eta_1}, \tag{7}$$

where

$$\eta_1 = a_1 x + b_1 y + c_1 t + \eta_{01}, \tag{8}$$

where the parameters a_1, b_1, c_1, and η_{01} are arbitrary constants.

Substituting equations (7) and (8) into equation (5), we have

$$c_1 = -\frac{b_1\left(a_1^3 + a_1\delta_1 + b_2\delta_2\right)}{a_1}. \tag{9}$$

Then, substituting equations (7) to (9) into equation (3), we have

$$u = \frac{2a_1^2 \exp(\eta_1)}{(1 + \exp(\eta_1))^2}, \quad v = \frac{2a_1 \exp(\eta_1)}{1 + \exp(\eta_1)}, \tag{10}$$

while

$$\eta_1 = a_1 x + b_1 y + c_1 t + \eta_{01}. \tag{11}$$

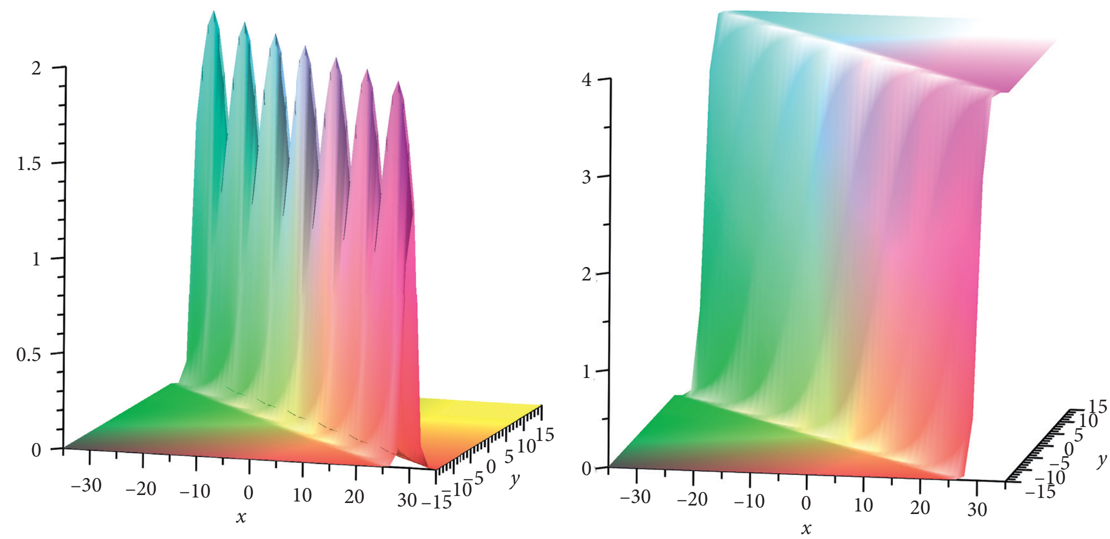
If we take $a_1 = 2, b_1 = 4, \delta_1 = 1, \delta_2 = -1, \eta_{01} = 0$, one-soliton solution can be obtained about equation (1), which is shown in Figure 1 at $t = 0$. In the process of wave propagation, we can observe that the velocity, amplitude, and shape of u, v are always consistent.

2.2. The 2-Soliton Solution. Set

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \tag{12}$$

where

$$\eta_i = a_i x + b_i y + c_i t + \eta_{0i}, \quad (i = 1, 2), \tag{13}$$



(a) (b)

FIGURE 1: One-soliton solution (u) v of equation (1) with the parameter selections $a_1 = 2$, $b_1 = 4$, $\delta_1 = 1$, $\delta_2 = -1$, and $\eta_{01} = 0$, at $t = 0$. (a) u. (b) v.

where the parameters a_i, b_i, c_i, and η_{0i} are arbitrary constants.

Substituting equations (12) and (13) into equation (4). Through maple software calculation, we can get

$$c_i = -\frac{b_i\left(a_i^3 + a_i\delta_1 + b_i\delta_2\right)}{a_i}, \quad (i = 1, 2), \tag{14}$$

$$A_{12} = -\frac{\left(a_1^3 - a_1^2 a_2\right)\left(a_1 a_2 - 2a_2^2\right)b_2 + \left(a_2^3 - a_1 a_2^2\right)\left(a_1 a_2 - 2a_1^2\right)b_1 - \delta_2\left(a_1 b_2 - a_2 b_1\right)^2}{\left(a_1^3 + a_1^2 a_2\right)\left(a_1 a_2 + 2a_2^2\right)b_2 + \left(a_2^3 + a_1 a_2^2\right)\left(a_1 a_2 + 2a_1^2\right)b_1 - \delta_2\left(a_1 b_2 - a_2 b_1\right)^2}. \tag{15}$$

Substituting equations (12) and (15) into equation (3). Through maple software calculation, we can obtain the two-soliton solution.

If we take $a_1 = 2$, $b_1 = 4$, $a_2 = 3$, $b_2 = 2$, $\delta_1 = 1$, $\delta_2 = -1$, $\eta_{01} = 0$, and $\eta_{02} = 0$ in Figure 2. We can observe that u is the two bell-shaped waves and v is two-kink soliton. This is an elastic collision, because their velocity, amplitude, and shape did not change during the wave propagation.

3. The Breather Solutions

Breather solutions of equation (1) can be obtained in the (x, y) plane, by choosing suitable parameters on the two-soliton solution, where the parameters in equation (3) meet the following conditions:

$$\begin{aligned} a_1 &= a_2 = m, \\ b_1 &= p + ik, \\ b_2 &= p - ik, \\ \eta_{01} &= \eta_{02} = 0. \end{aligned} \tag{16}$$

Equation (12) can be rewritten as

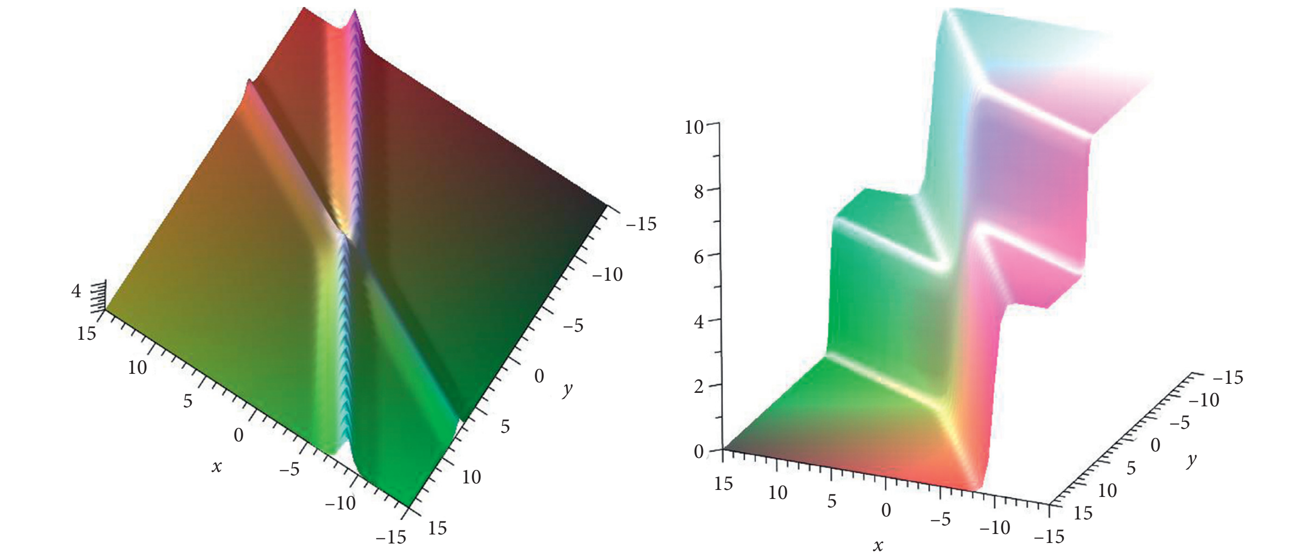
$$f = 1 + 2e^{\xi}\cos(ky + \omega t) + A_{12}e^{2\xi}, \tag{17}$$

with

$$\begin{aligned} \xi &= mx + py - \left(m^2 p + p\delta_1 + \frac{\left(p^2 - k^2\right)\delta_2}{m}\right)t, \\ \omega &= -\left(m^2 k + \delta_1 k + \frac{2kp\delta_2}{m}\right), \end{aligned} \tag{18}$$

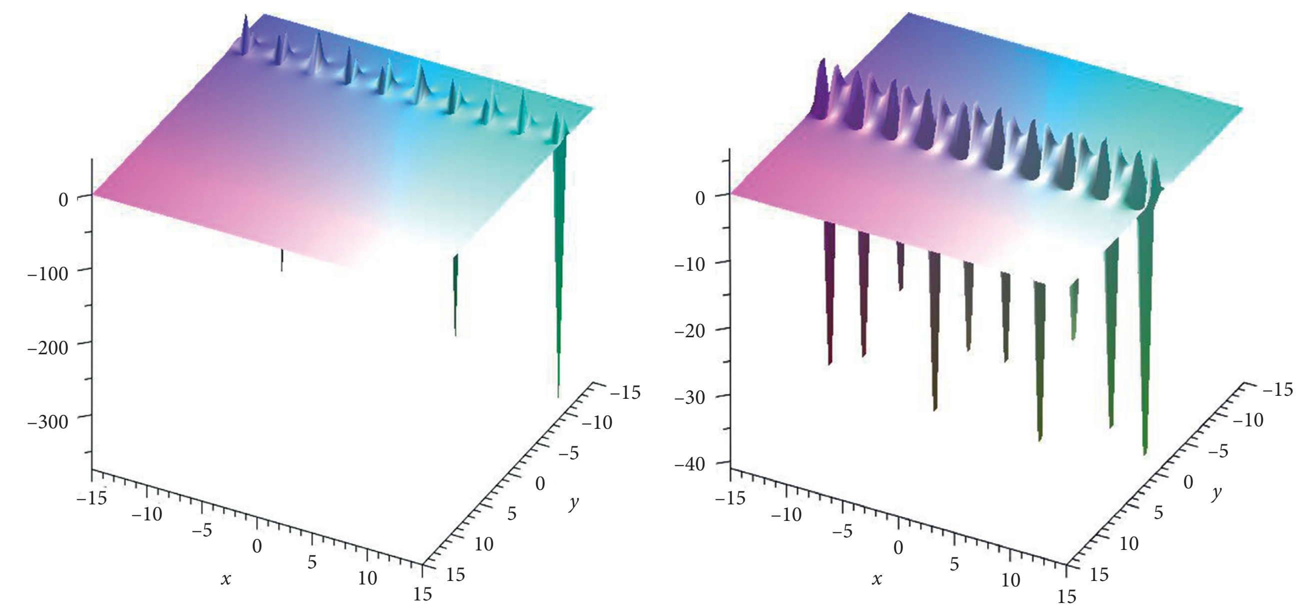
$$A_{12} = \frac{k^2\delta_2}{-3m^2 p + k^2\delta_2}.$$

If taking $m = 2$, $p = 0$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$, we have the y-periodic soliton structures of solutions u, v as shown in Figures 3 and 4 and their directions are perpendicular to the x-axis. Taking $t = -10$, $t = 0$, and $t = 10$, we can obtain the dynamic behavior of solving u with time as shown in Figure 3. The dynamic behavior of solving v with time is shown in Figure 4. The line of breathers can be obtained in the (x, y) plane. When $t = 0$, the alternation of light and dark of soliton can be observed from Figures 3(b) and 4.



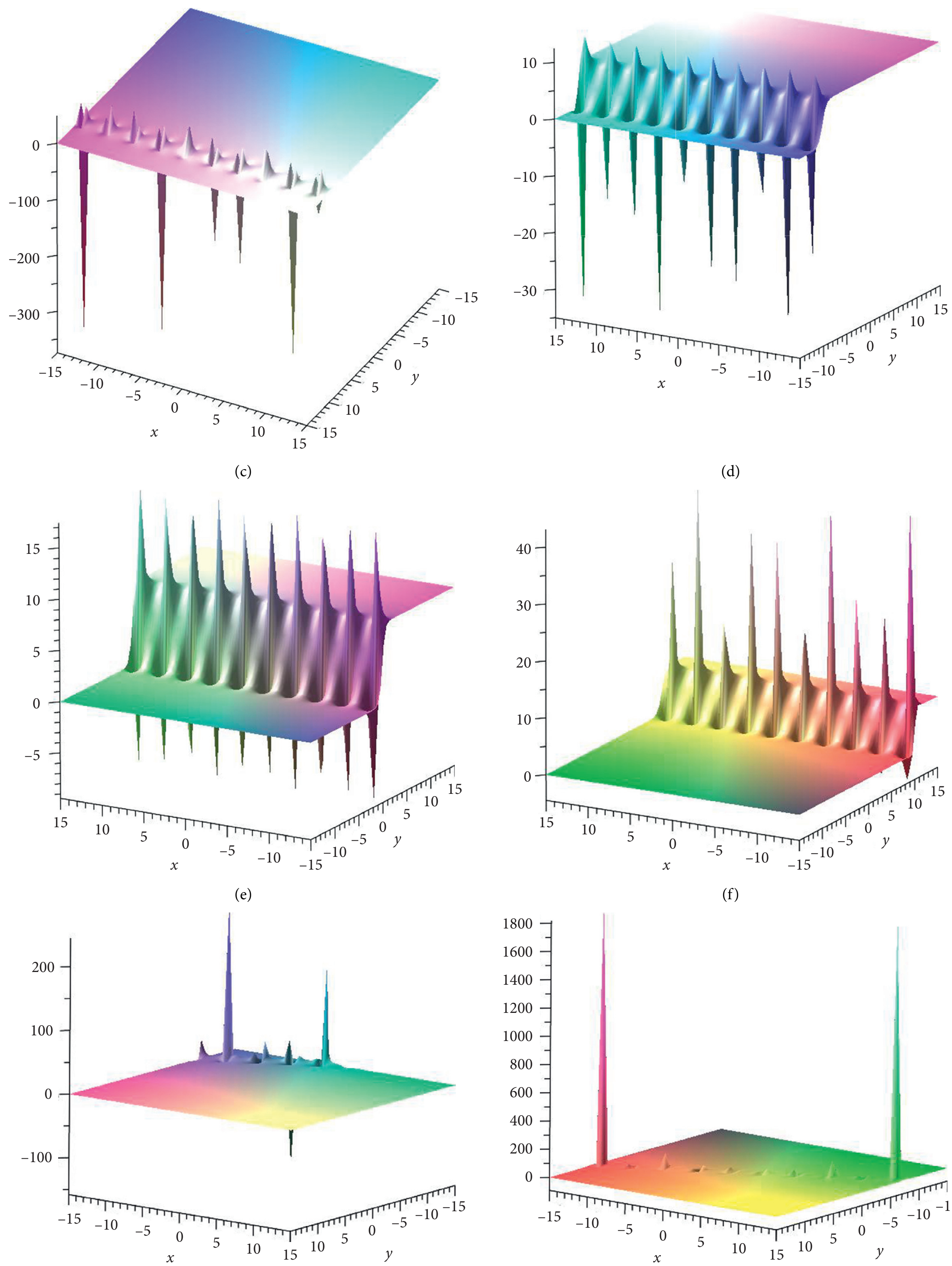
(a) (b)

Figure 2: One soliton solution u, v of equation (1) with the parameter selections $a_1 = 2$, $b_1 = 4$, $a_2 = 3$, $b_2 = 2$, $\delta_1 = 1$, $\delta_2 = -1$, $\eta_{01} = 0$, and $\eta_{02} = 0$, at $t = 0$. (a) u. (b) v.



(a) (b)

Figure 3: Continued.



(c) (d) (e) (f) (g) (h)

Figure 3: Continued.

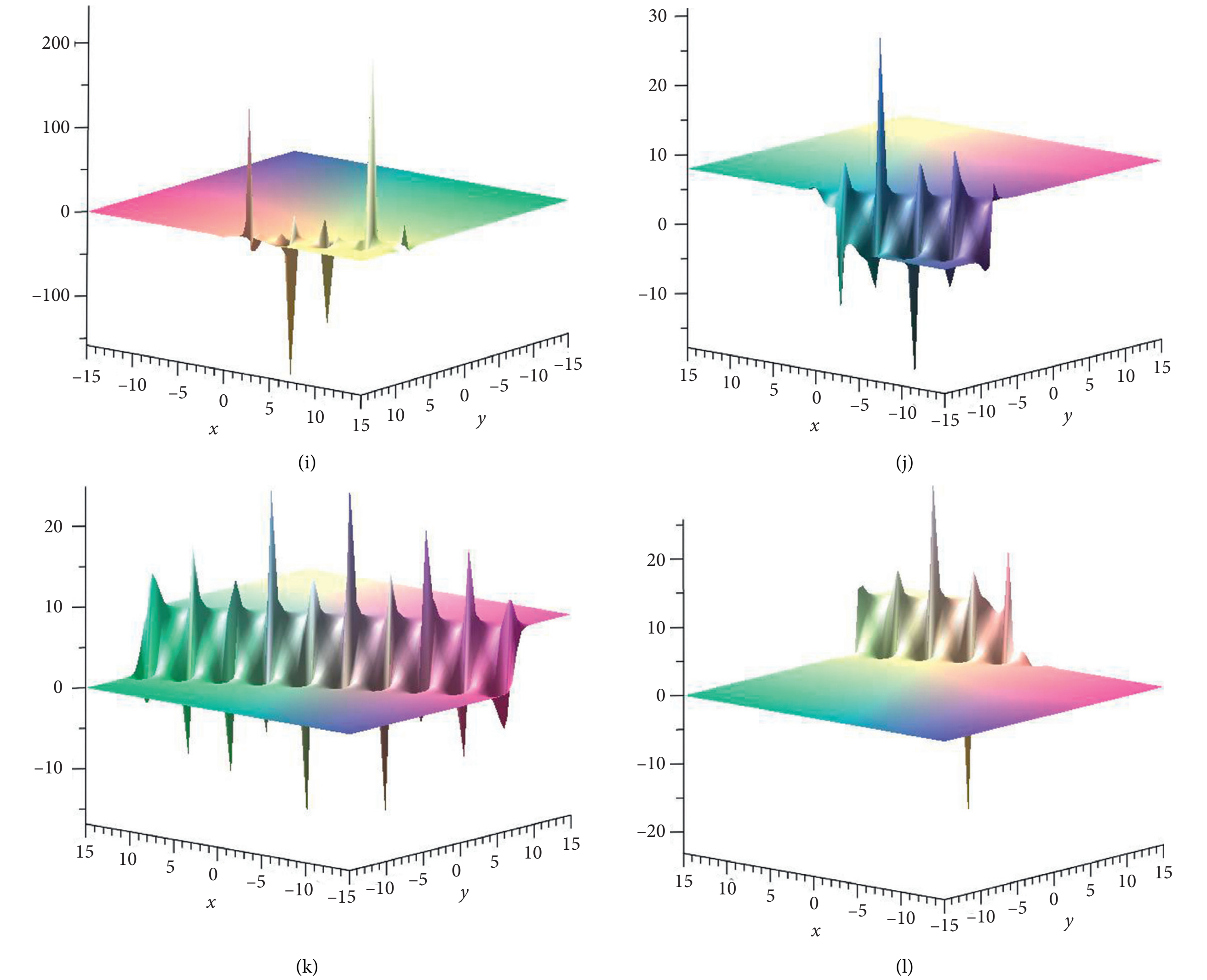


Figure 3: The breather solution u of equation (1) with the parameter selections $m = 2$, $p = 0$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$.

If taking $m = 2$, $p = 1$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$, we have the (x, y)-periodic soliton structures as shown in Figures 5 and 6. Their shapes remain the same during the propagation. Taking $t = -5, t = 0$, and $t = 5$, we can obtain the dynamic behavior of solving u with time as shown in Figure 5. The dynamic behavior of solving v with time is shown in Figure 6.

4. The Lump Solutions

In order to obtain the lump solution, we can choose suitable parameters on the two-soliton solution. Setting parameters

$$\begin{aligned} a_1 &= l_1 \cdot \varepsilon, \\ a_2 &= l_2 \cdot \varepsilon, \\ b_1 &= n_1 \cdot a_1, \\ b_2 &= n_2 \cdot a_2, \\ \eta_{01} &= \eta_{02}^{*} = l \cdot \pi, \end{aligned} \tag{19}$$

in equation (12) and taking the limit as $\varepsilon \longrightarrow 0$, the function f converted into the following form:

$$f = (\theta_1\theta_2 + \theta_0) l_1 l_2 \varepsilon^2 + o(\varepsilon^3), \tag{20}$$

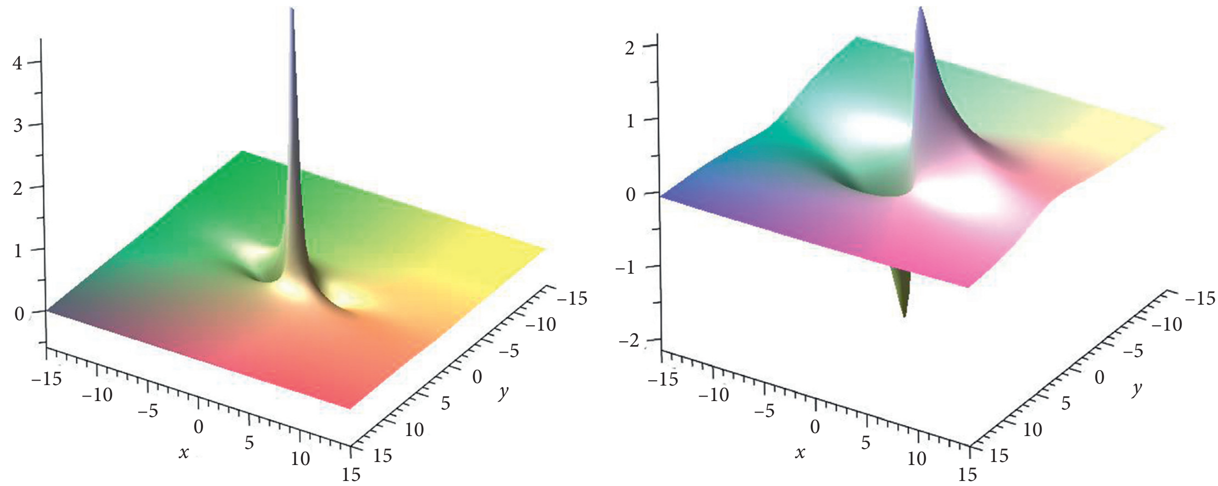
with

$$\begin{aligned} \theta_1 &= -x - n_1 y + \left(\delta_1 n_1 + \delta_1 n_1^2\right)t, \\ \theta_2 &= -x - n_2 y + \left(\delta_1 n_2 + \delta_1 n_2^2\right)t, \\ \theta_0 &= \frac{6(n_1 + n_2)}{\delta_2 (n_1 - n_2)^2}. \end{aligned} \tag{21}$$

Substituting equations (20) and (21) into equation (3), we can obtain

$$u = \frac{4}{\theta_1\theta_2} - \frac{2(\theta_1 + \theta_2)^2}{(\theta_1\theta_2 + \theta_0)^2}. \tag{22}$$

If taking $n_1 = a + ib$ and $n_2 = a - ib$, a, b are all real constants. We have the lump soliton of solutions u and v shown in Figure 7. The lump solutions u have one global maximum point and two global minimum points in



(a) (b)

Figure 4: The breather solution v of equation (1) with the parameter selections $m = 2$, $p = 0$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$.

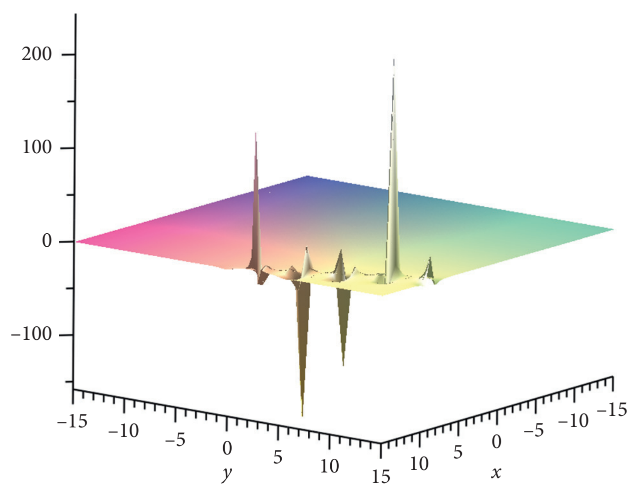


Figure 5: The breather solution u of equation (1) with the parameter selections $m = 2$, $p = 1$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$.

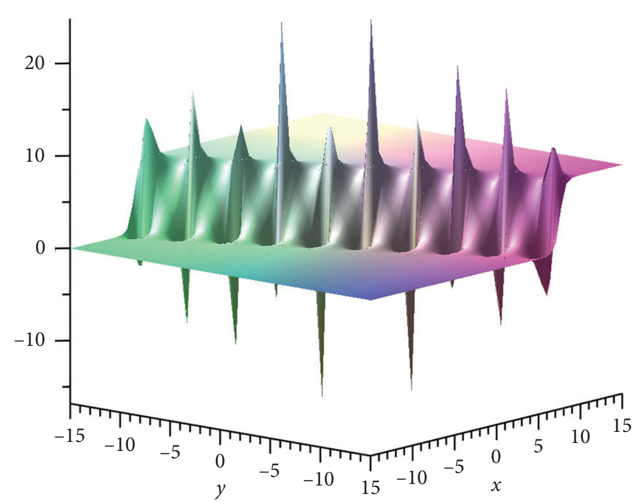


Figure 6: The breather solution v of equation (1) with the parameter selections $m = 2$, $p = 1$, $k = 2$, $\delta_1 = 1$, and $\delta_2 = -1$.

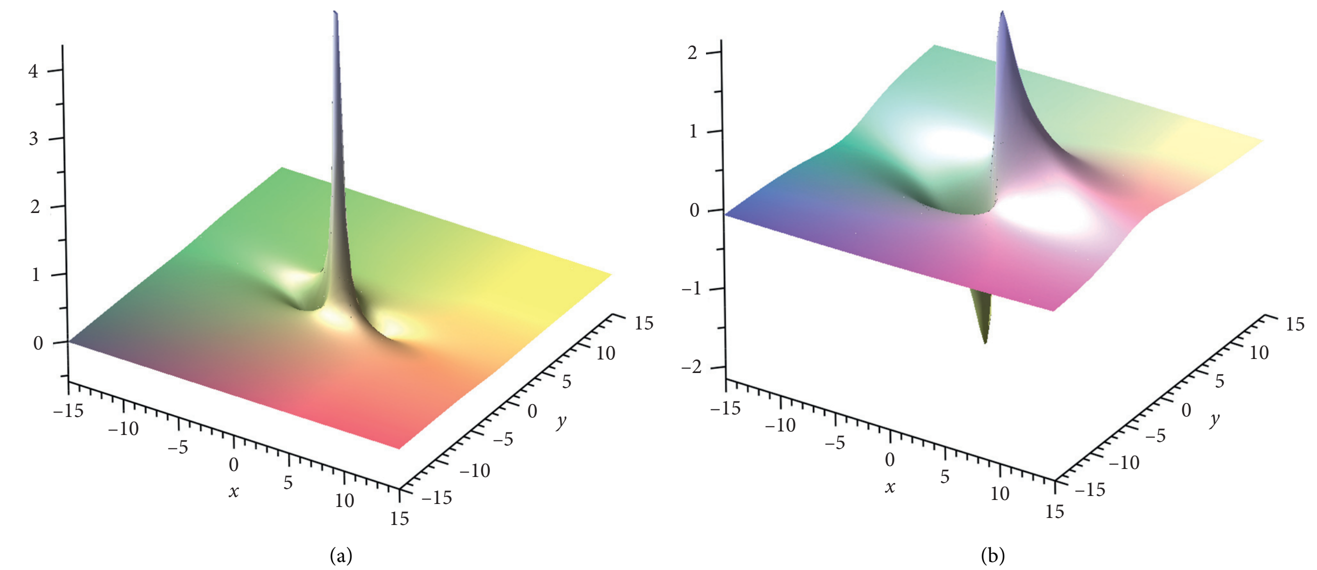


Figure 7: The lump solution u and v of equation (1) with the parameter selections $a = 1$, $b = 2$, $\delta_1 = 1$, and $\delta_2 = -1$, at $t = 0$. (a) u. (b) v.

Figure 7(a). The lump solutions v have one global maximum point and one global minimum point in Figure 7(b).

5. Conclusions

In summary, we have investigated the 1-soliton, 2-soliton, and localized nonlinear wave solutions of the generalized Calogero–Bogoyavlenskii–Schiff equation. Through the Hirota bilinear method, 1-soliton and 2-soliton solutions have been shown in Figures 1 and 2. Breathers are derived via choosing appropriate parameters on 2-soliton solutions, while lumps solution are obtained through the long wave limit on the soliton solutions. Some obtained results are shown in Figures 3–7. We analysed their dynamic behavior and vividly demonstrated their evolution process. Meanwhile, these methods used in this paper are powerful and absolutely reliable to search the exact local wave solutions of other nonlinear models. And it is helpful for us to find the soliton molecules [60–63] in future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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