Hindawi
Complexity
Volume 2021, Article ID 8826480, 13 pages
https://doi.org/10.1155/2021/8826480

Research Article

Robust Control Design for an Uncertain Macroeconomic Dynamical System with Unknown Characteristics and Inequality Control Constraint

Xiaorui Xie [1] and Ye-Hwa Chen [2,3]

[1]School of Accounting and Finance Anhui Finance & Trade Vocational College, Hefei, Anhui 230601, China
[2]The George W. Woodruff School of Mechanical Engineering Georgia Institute of Technology, Atlanta, GA 30332, USA
[3]Key Laboratory of Road Construction Technology and Equipment of MOE Chang'an University, Xi'an, Shaanxi 710064, China

Correspondence should be addressed to Ye-Hwa Chen; yehwa.chen@me.gatech.edu

Received 19 August 2020; Revised 9 October 2020; Accepted 21 December 2020; Published 6 January 2021

Academic Editor: Lucia Valentina Gambuzza

Copyright © 2021 Xiaorui Xie and Ye-Hwa Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The stabilization problem of a macroeconomic dynamical system is considered in this paper. The main features of this system are that the system uncertainties may be unknown functions of state and time but with known bounds. Furthermore, the control inputs are subject to constraints, which is a salient feature in an economic control problem. To ensure that the controls are within the specified boundaries, in our control design procedure, a creative diffeomorphism, which converts bounded controls into unbounded corresponding signals by choosing an appropriate transformation function, is proposed. For the uncertain system, a deterministic robust control is designed to render the practical stability: uniform boundedness and uniform ultimate boundedness. The range of the input bounds is related to the uncertainties and can be designed according to the actual situation. Numerical simulations are performed to verify the effectiveness of the stabilization policy.

1. Introduction

The issue of economic stability has always been a crucial topic in economics and academia. Over the past few decades, many important contributions have been made. Based on the information characteristics of the economic system, four control approaches are classified: (i) The information is deterministic, which means that all the information related to the model is known. (ii) The information is stochastic, which indicates that some statistical information (such as mean, variance, and standard deviation) about system parameters and disturbances is known. (iii) The information is adaptive, which is based on prior information and updated according to some kind of adaptive law. (iv) The information is with bounded uncertainty, which means that the system parameters and disturbances may be unknown functions of state and time but with known bounds.

In terms of the information in (i)–(iii), some outstanding research studies have been performed. To list just a few, Kendrick et al. summarized the past significant developments in the stochastic control of a dynamic economic system and made a proposal for a classification system according to a certain number of properties of stochastic control models in economics including stochastic elements, solution classification, and an estimation method [1]. Ashimov et al. discussed a macroeconomic analysis and economic policy based on parametric control of equilibrium states in national economic markets, cyclic dynamics of economic systems, and economic growth of a national economy [2]. Carraro et al. investigated different control theories and their relationships with economic policy analysis by presenting the main features of control and controllability theory, optimal decision making under certainty and uncertainty conditions, and software for optimal control models [3]. Murata studied the macroeconomic stabilization problem and presented the optimal control methods for linear discrete-time economic systems [4]. Otter proposed dynamic feature space modelling, filtering, and self-tuning control of a macroeconomic system [5]. Rao attempted to use stochastic control theory for macroeconomic regulation of the Indian economy with an annual nonlinear model [6]. Tintner et al. systematically summarized the stochastic view of economics, stochastic models of economic development, stochastic control theory, and programming methods with economic applications [7]. Day viewed the economic development as an adaptive process and proposed the adaptive economic theory, modelling, and dynamical analysis [8]. Hudgins and Na explored robust designs for an applied macroeconomic discrete-time LQ tracking model with perfect state measurements [9]. Berardi and Galimberti provided a critical review on several methods previously proposed in the literature of learning and expectations in macroeconomics in order to initialize its learning algorithms [10]. Jajarmi et al. designed a linear-state feedback controller together with an adaptive control technique to control the hyperchaos and realize the synchronization of a fractional economic model [11].

For bounded uncertainty, Chang and Steckler introduced the concept of fuzzy systems and fuzzy dynamic programming to economic systems with unknown but bounded distribution of the coefficients [12]. Leitmann and Wan proposed a stabilization policy for an economy with some unknown characteristics, with uniform asymptotic stability guaranteed [13]. For most of the control problems in the past, we paid more attention on the constraints of system outputs (e.g., robust control [14, 15], fuzzy control [16, 17], and sliding mode control [18–20]). In practical situation, the control inputs should be constrained as well [21–24]. Without these constraints, many factors can lead to an unexpected control input, such as uncertainties in the system, initial condition deviation, and excessive control cost. This problem is frequently seen in many practices, since control is mostly implemented via physical quantities such as energy, force, and torque, which are always finite.

In a macroeconomic system, there always exist control input constraints, which is inevitable. However, the values of control inputs are unlimited in many studies. Thus, this paper is motivated to design more practical control for the stabilization problem of a macroeconomic dynamical system with uncertainties, as well as constrained inputs. The main difficulty, in the past research, lied in a lack of a creative and practical new approach for control system analysis. Most of the past research followed the approach rooted in the traditional framework. In this paper, a new approach (i.e., *diffeomorphism-based approach*) to deal with input constrains for nonlinear uncertain systems is provided. The theoretical contributions could be summed up in two aspects. First, a creative bijective diffeomorphism is proposed to convert the two-side bounded constrained controls into unbounded corresponding signals. Through the diffeomorphism, as the burden of the design for the physical control is transformed to an auxiliary control, the inequality constraints of the control inputs can be satisfied. Second, a deterministic robust control scheme is designed. Under this control, the practical stability of the uncertain system regardless of the uncertainty can be guaranteed: uniform boundedness and uniform ultimate boundedness. The improvement, compared with previous related works, is to render a realistic constrained-control design which can stabilize a class of uncertain macroeconomic systems. Furthermore, the creative new approach should help to solve other constrained-control problems in engineering, social sciences, biological systems, and medical systems.

The organization of this paper is as follows: A deterministic robust control approach is presented to deal with nonlinear uncertain systems with bounded uncertainties and nonlinear inputs in Section 2, which could guarantee the practical stability. In Section 3, a macroeconomic system model with known bounded system noise and two-side control input constraints is introduced. In Section 4, a creative diffeomorphism is proposed to convert the bounded controls into the unbounded ones. In Section 5, we conduct some further analysis on the system model and discuss some control design details. In Section 6, numerical simulations are performed to verify the effectiveness of the deterministic robust control to solve the stabilization problem for the macroeconomic system. Section 7 gives the conclusion.

2. Uncertain Dynamical System

We consider the following uncertain dynamical system ([25]):

$$\dot{x}(t) = f(x(t), \sigma(t), t) + B(x(t), t) G(\omega(t), \sigma(t), t), \tag{1}$$

where $t \in R$ is the time, $x(t) \in R^n$ is the system state, $\omega(t) \in R^m$ is the control input, and $\sigma(t) \in \Sigma \subset R^p$ is the uncertain parameter. Here, $\Sigma \subset R^p$ is known and compact, which stands for the possible bound of σ. The functions $f(\cdot)$: $R^n \times \Sigma \times R \longrightarrow R^n$, $B(\cdot)$: $R^n \times R \longrightarrow R^{n\times m}$, and $G(\cdot)$: $R^m \times \Sigma \times R \longrightarrow R^m$ are all continuous. The uncertain parameter $\sigma \in R^p$ representation is in a generic form, which may imply multiple physical parameters being stacked up. It is possible that f may depend on certain entries of σ and G may depend on certain other entries.

We suppose, in practice, the input is required to be bounded: $\omega(t) \in W$, where $W \subset R^m$ is a bounded set. We suppose (this is an assumption) also that there is a smooth and bijective (meaning one-to-one) diffeomorphism $\Omega(\cdot)$: $W \longrightarrow R^m$, $\Omega(\omega) = u$, $\omega = \Omega^{-1}(u)$. This would allow us to rerepresent G as follows: $G(\omega, \sigma, t) = G(\Omega^{-1}(u), \sigma, t) =: g(u, \sigma, t)$. The choice of the diffeomorphism $\Omega(\cdot)$ is case dependent and may not even be unique. In Section 4, we demonstrate how to find such Ω in a macroeconomic dynamical system.

The system (1) can be rewritten as

$$\dot{x}(t) = f(x(t), \sigma(t), t) + B(x(t), t) g(u(t), \sigma(t), t), \tag{2}$$

where $u(t) \in R^m$ is regarded as the new control from now on. Obviously, a design of $u(t)$ corresponds to a design of $\omega(t)$.

$f(x, \sigma, t)$ is decomposed into two continuous portions as

$$f(x, \sigma, t) = \overline{f}(x, t) + \Delta f(x, \sigma, t), \tag{3}$$

where $\overline{f}(x, t)$ is the nominal portion and $\Delta f(x, \sigma, t)$ is the uncertain portion.

The nominal system, that is, the undisturbed and uncontrolled system, is

$$\dot{x}(t) = \overline{f}(x(t), t). \quad (4)$$

Definition 1. A continuous function $\overline{\lambda}(\cdot)$: $R_+ \longrightarrow R_+$ (where $R_+ = [0, \infty))$ is said to belong to class K_∞ [26] if (i) $\overline{\lambda}(0) = 0$, (ii) it is strictly increasing, and (iii) $\lim_{s\longrightarrow\infty}\overline{\lambda}(s) = \infty$.

Assumption 1. For the nominal system (4), the following hold:

(i) The origin $x = 0$ of the nominal system, with $\overline{f}(0, t) = 0$ for all $t \in R$, is a uniformly asymptotically stable equilibrium point

(ii) There exist a known C^1 function $V(\cdot)$: $R^n \times R \longrightarrow R_+$ and known functions $\overline{\lambda}_i(\cdot)$: $R_+ \longrightarrow R_+$, $i = 1, 2, 3$, belonging to class K_∞ such that, for all $(x, t) \in R^n \times R$,

$$\overline{\lambda}_1(\|x\|) \le V(x, t) \le \overline{\lambda}_2(\|x\|), \quad (5)$$

$$\frac{\partial V(x, t)}{\partial t} + \frac{\partial^T V(x, t)}{\partial x}\overline{f}(x, t) \le -\overline{\lambda}_3(\|x\|). \quad (6)$$

Remark 1. Assumption 1 assures that $x = 0$ of the nominal system (4) is globally asymptotically stable. Furthermore, the choice of the nominal system is not unique. The nominal portion $\overline{f}$, which is independent of σ, is selected at the designer's discretion: the designer simply chooses $\overline{f}$, which could satisfy the two conditions of Assumption 1, and lump the rest into Δf (so $\Delta f = f - \overline{f}$) [27], which meets the matching condition (7). Therefore, in practice, there is maybe some back-and-forth between choosing $\overline{f}$ and Δf.

Assumption 2. There exists a continuous $h(x, \sigma, t)$ such that

$$\Delta f(x, \sigma, t) = B(x, t)h(x, \sigma, t). \quad (7)$$

Choose a continuous function $\rho(\cdot)$: $R^n \times R \longrightarrow R_+$ such that, for all $(x, \sigma, t) \in R^n \times \Sigma \times R$,

$$\|h(x, \sigma, t)\| \le \rho(x, t). \quad (8)$$

Remark 2. Here, $h(x, \sigma, t)$ is determined by the system dynamics. The assumption is often called as the matching condition. The existence of a continuous $\rho(x, t)$ is assured based on the continuity of $\Delta f(x, \sigma, t)$ and the boundedness of σ (that is, $\sigma \in \Sigma$) [28]. The choice of $\rho(x, t)$ mainly depends on the characteristics of the function $h(x, \sigma, t)$ in different applications. This will be demonstrated in a macroeconomic system.

Assumption 3. There exist known continuous functions: (i) $\psi(\cdot)$: $R_+ \longrightarrow R_+$, $\psi(0) = 0$, $\psi(p) > 0$ for $p > 0$ and (ii) $\gamma(\cdot)$: $R_+ \longrightarrow R_+$, $\gamma(0) = 0$, $\gamma(p) > 0$ for $p > 0$ such that

$$\gamma(\|u\|) \le u^T g(u, \sigma, t), \quad (9)$$

for all $(u, \sigma, t) \in R^m \times \Sigma \times R$. Furthermore, $g(0, \sigma, t) = 0$.

$$\gamma(\psi(q)) \ge q\psi(q), \quad (10)$$

for all $q \in R_+$.

Remark 3. The choice of the function $\gamma(\cdot)$ is case dependent and may not even be unique. In Section 5, we demonstrate how to find such $\gamma(\cdot)$ in a macroeconomic dynamical system.

Our purpose is to design a control $u(t)$ to render the uncertain system (2) the following performance [29]:

(i) Uniform boundedness: if $x(\cdot)$: $[t_0, \infty) \longrightarrow R^n$, $x(t_0) = x_0$ is a solution of the controlled system, then for any $r > 0$ with $\|x_0\| \le r$, there is a $d(r)$ such that $\|x(t)\| \le d(r)$, $\forall t \ge t_0$

(ii) Uniform ultimate boundedness: if $x(\cdot)$: $[t_0, \infty) \longrightarrow R^n$, $x(t_0) = x_0$ is a solution of the controlled system with $\|x_0\| \le r$, then for any $\underline{d} > 0$, there is a $\overline{d} \ge \underline{d}$ and a finite time $T(\overline{d}, r)$ such that $\|x(t)\| \le \overline{d}$, $\forall t \ge t_0 + T(\overline{d}, r)$

Our task is to design a control so that, for given system dynamics, we are able to confine the system's state to be within a region and drive the system to be within a prescribed region in a finite time and stays there thereafter.

Subject to Assumptions 1–3, for a given ϵ (this is a design parameter), a robust control $u(t) = p(x(t), t)$ for the uncertain system (2) is proposed as

$$p(x, t) = -\frac{\mu(x, t)}{\|\mu(x, t)\|}\psi(\rho(x, t)), \quad \text{if } \|\mu(x, t)\| > \varepsilon,$$

$$p(x, t)\|\alpha(x, t)\| = -\|p(x, t)\|\alpha(x, t), \quad \text{if } \|\mu(x, t)\| \le \varepsilon, \quad (11)$$

with

$$\alpha(x, t) = B^T(x, t)\frac{\partial V(x, t)}{\partial x}, \quad (12)$$

$$\mu(x, t) = \alpha(x, t)\rho(x, t). \quad (13)$$

Theorem 1. *suppose the system (2) is subject to Assumptions 1–3; then, the control (11) renders the system's trajectory uniformly bounded and uniformly ultimately bounded.*

Proof. For a given admissible uncertainty $\sigma(\cdot)$, the derivative of the Lyapunov function $V(x, t)$ along the trajectory of the closed-loop system is given by

$$
\begin{aligned}
\dot{V}(x,t) &= \frac{\partial V(x,t)}{\partial t} + \frac{\partial^T V(x,t)}{\partial x}\dot{x}, \\
&= \frac{\partial V(x,t)}{\partial t} + \frac{\partial^T V(x,t)}{\partial x}[f(x,\sigma,t) + B(x,t)g(u,\sigma,t)], \\
&= \frac{\partial V(x,t)}{\partial t} + \frac{\partial^T V(x,t)}{\partial x}[\overline{f}(x,t) + \Delta f(x,\sigma,t)] + \frac{\partial^T V(x,t)}{\partial x}B(x,t)g(u,\sigma,t) \\
&\le -\overline{\lambda}_3(\|x\|) + \frac{\partial^T V(x,t)}{\partial x}\Delta f(x,\sigma,t) + \frac{\partial^T V(x,t)}{\partial x}B(x,t)g(u,\sigma,t), \\
&= -\overline{\lambda}_3(\|x\|) + \frac{\partial^T V(x,t)}{\partial x}B(x,t)h(x,\sigma,t) + \frac{\partial^T V(x,t)}{\partial x}B(x,t)g(u,\sigma,t), \\
&= -\overline{\lambda}_3(\|x\|) + \alpha^T h(x,\sigma,t) + \alpha^T g(u,\sigma,t) \\
&\le -\overline{\lambda}_3(\|x\|) + \|\mu\| + \alpha^T g(u,\sigma,t).
\end{aligned} \tag{14}
$$

Before proceeding, we note that, if $\|\mu\| > \epsilon$, then $u = -\eta\alpha$, where $\eta = \psi(\rho)\rho/\|\mu\|$. By using (9), we have

$$
\begin{aligned}
\alpha^T g(u,\sigma,t) &= \alpha^T g(-\eta\alpha,\sigma,t), \\
&= -\frac{1}{\eta}(-\eta\alpha)^T g(-\eta\alpha,\sigma,t) \\
&\le -\frac{1}{\eta}\gamma(\psi(\rho)).
\end{aligned} \tag{15}
$$

It is obvious to note that $\alpha \ne 0$ since $\|\mu\| > \epsilon$, $u \ne 0$ since $\|u\| = \psi(\rho) > 0$, and $\psi(\rho)\rho > 0$ whenever $\|\mu\| > \epsilon$.

If $\|\mu\| \le \epsilon$, we suppose $\alpha \ne 0$ and $u \ne 0$ and consider u and α such that $u\|\alpha\| = -\|u\|\alpha$. Then, $u = -\|u\|\alpha/\|\alpha\|$ and

$$
\begin{aligned}
\alpha^T g(u,\sigma,t) &= \alpha^T g\left(-\frac{\|u\|\alpha}{\|\alpha\|},\sigma,t\right), \\
&= -\frac{\|\alpha\|}{\|u\|}\left(-\frac{\|u\|\alpha}{\|\alpha\|}\right)^T g\left(-\frac{\|u\|\alpha}{\|\alpha\|},\sigma,t\right) \\
&\le -\frac{\|\alpha\|}{\|u\|}\gamma(\|u\|).
\end{aligned} \tag{16}
$$

Clearly, if $u = 0$, $\partial V(x,t)/\partial x B(x,t)g(u,\sigma,t) = 0$ since $g(0,\sigma,t) = 0$. In addition, if $\alpha = 0$, $\partial V(x,t)/\partial x B(x,t)$ $g(u,\sigma,t) = \partial V(x,t)/\partial x B(x,t)h(x,\sigma,t) = 0$. Thus, if $\|\mu\| > \varepsilon$, by (10)

$$
\begin{aligned}
\dot{V}(x,t) &\le -\overline{\lambda}_3(\|x\|) + \|\mu\| - \frac{1}{\eta}\gamma(\psi(\rho)), \\
&= -\overline{\lambda}_3(\|x\|) + \|\mu\| - \frac{\|\mu\|}{\psi(\rho)\rho}\gamma(\psi(\rho)) \\
&\le -\overline{\lambda}_3(\|x\|),
\end{aligned} \tag{17}
$$

and if $\|\mu\| \le \varepsilon$, $u \ne 0$, $\alpha \ne 0$,

$$
\begin{aligned}
\dot{V}(x,t) &\le -\overline{\lambda}_3(\|x\|) + \|\mu\| - \frac{\|\alpha\|}{\|u\|}\gamma(\|x\|) \\
&\le -\overline{\lambda}_3(\|x\|) + \varepsilon.
\end{aligned} \tag{18}
$$

Also, if $u = 0$ (this implies $\|\mu\| \le \varepsilon$),

$$
\begin{aligned}
\dot{V}(x,t) &\le -\overline{\lambda}_3(\|x\|) + \|\mu\| \\
&\le -\overline{\lambda}_3(\|x\|) + \varepsilon,
\end{aligned} \tag{19}
$$

and if $\alpha = 0$,

$$
\dot{V}(x,t) \le -\overline{\lambda}_3(\|x\|). \tag{20}
$$

Consequently, for all $(x,t) \in R^n \times R$,

$$
\dot{V}(x,t) \le -\overline{\lambda}_3(\|x\|) + \varepsilon. \tag{21}
$$

Therefore, $\dot{V}$ is negative definite for sufficiently large $\|x\|$. Upon invoking the standard arguments as in [30, 31], the control guarantees uniform boundedness and uniform ultimate boundedness. The uniform boundedness region is given by

$$
d(r) = \begin{cases} (\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2)(\widehat{R}), & \text{if } r \le \widehat{R}, \\ (\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2)(r), & \text{if } r > \widehat{R}, \end{cases} \tag{22}
$$

where $\widehat{R} = \overline{\lambda}_3^{-1}(\epsilon)$. It means the state will be bounded in a region with radius $d(r)$ if the initial state is within a region with radius r.

The lower bound $\underline{d}$ of the uniform ultimate boundedness region and the finite time $T(\overline{d}, tr)$ are given by

$$
\underline{d} = \left(\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2\right)(\widehat{R}), \tag{23}
$$

$$
T(\overline{d}, r) = \begin{cases} 0, & \text{if } r \le \overline{R}, \\ \dfrac{\overline{\lambda}_2(r) - \overline{\lambda}_1(\overline{R})}{\overline{\lambda}_3(\overline{R}) - \epsilon}, & \text{if } r > \overline{R}, \end{cases} \tag{24}
$$

where $\overline{R} = (\overline{\lambda}_2^{-1} \circ \overline{\lambda}_1)(\overline{\mathrm{d}})$. It means the state will eventually enter a region with radius $\overline{d}$ if the initial state is within a region with radius r QED.

Remark 4. The choice of the design parameter $\varepsilon > 0$ is at the designer's discretion. Notice that $\widehat{R}$ can be made arbitrary small since ε, which is at the designer's discretion, can be arbitrary small. The implication of this is that the region outside $\widehat{R}$ then expands, which is the region that is in proportional to the size of that of the initial condition r, i.e., $(\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2)(\mathrm{r})$. As a result, the smaller the initial condition region r, the smaller the uniform boundedness region $d(r)$ (that is, as $r \searrow$, $(\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2)(\mathrm{r}) \searrow$). Another way to put it is that a desirable uniform boundedness region, which is often of significance from system performance point of view, can be made possible if the initial condition can be located in an appropriate region. On the other hand, this trend does not hold when $r \le \widehat{R}$, in which case the uniform boundedness region (i.e., $(\overline{\lambda}_1^{-1} \circ \overline{\lambda}_2)(\widehat{\mathrm{R}})$) is fixed regardless of r.

Remark 5. Notice that since $\widehat{R}$ can be made arbitrary small (through an appropriate choice of ϵ), the lower bound $\underline{d}$ can be made arbitrary small. As a result, the uniform ultimate boundedness region $\overline{d}$ ($\ge \underline{d}$) can be selected to be arbitrary small. This means the state can be made to enter any small region in a finite time and remains there thereafter.

3. A Macroeconomic Dynamical System

Let $\widetilde{y}_t$ denote the "target aggregate demand," which represents the best compromise between current unemployment and inflation. The relationship between $\widetilde{y}_t$ and the time t is given as [13]

$$\widetilde{y}_t(t) = y^{\infty}\left(1 - e^{-at}\right), \tag{25}$$

where $y^{\infty} > 0$ is the limit constant output level and $a > 0$ is the constant relative rate. In (25), the term e^{-at}, to some extent, reflects a potential for expansion, and the term y^{∞} reflects a limit for growth.

For the determination of "actual aggregate demand" y_t, we postulate the dynamic relation

$$\dot{y}_t(t) = -a y_t(t) + k_\mu \mu_t(t) + r_g g_t(t) + r_y(t), \; y_0 = y_t(t_0), \tag{26}$$

where μ_t is the index of "smoothed" monetary control at time t, g_t is the index of fiscal control at time t, k_μ is the constant "policy multiplier" for μ_t, r_g is the "policy multiplier" for g_t, and r_y is a "forcing term." The multipliers are influenced by the market fluctuations and human subjective expectations.

Combining (25) and (26), one has

$$\frac{d}{dt}\left(y_t - \widetilde{y}_t\right) = -a\left(y_t - \widetilde{y}_t\right) + \left(r_y - a y^{\infty}\right) + k_\mu \mu_t + r_g g_t. \tag{27}$$

If there are no external disturbances $r_y - a y^{\infty}$ and no effects of monetary control $k_\mu \mu_t$ and fiscal control $r_g g_t$, the nominal system

$$\frac{d}{dt}\left(y_t - \widetilde{y}_t\right) = -a\left(y_t - \widetilde{y}_t\right) \tag{28}$$

describes the situation.

The term $r_y - a y^{\infty}$ may represent any stimulating or depressing force acting on the aggregate demand rate. For system (27), this term can be regarded as the system noise, which may be uncertain, nonlinear, and time varying. With regard to the system noise, we suppose only that this term has known bounds; that is, $\underline{r}_y \le r_y \le \overline{r}_y$, where $\underline{r}_y$ and $\overline{r}_y$ are the given constants.

Monetary control is usually carried out by a monetary authority either by open market operations in the bond market or by setting the rediscounting rate. The actual monetary control m_t is usually related to the term μ_t, via the following smooth relationship:

$$\dot{\mu}_t(t) = r_\mu\left(m_t(t) - \mu_t(t)\right), \; \mu_0 = \mu_t(t_0). \tag{29}$$

From the view of control, r_μ is regarded as a control parameter. This control parameter can be adjusted according to the control effect. However, in this macroeconomic system, the control parameter r_μ is bounded with $\underline{r}_\mu \le r_\mu \le \overline{r}_\mu$, where $\underline{r}_\mu$ and $\overline{r}_\mu$ are the given positive constants.

Fiscal control usually appears in the form of tax provisions and rates, as well as decisions about government spending. From the view of control, the coefficient r_g is also regarded as a control parameter for the fiscal control term. Similarly, this control parameter can be adjusted according to the control effect. However, in this macroeconomic system, the control parameter r_g is bounded with $\underline{r}_g \le r_g \le \overline{r}_g$, where $\underline{r}_g$ and $\overline{r}_g$ are the given positive constants.

We postulate the control bounds as follows:

$$\underline{g} < g_t < \overline{g}, \tag{30}$$

where $\underline{g}$ and $\overline{g}$ are given constants, $\underline{g} \le 0 \le \overline{g}$. This reflects the limits of fiscal control.

$$\underline{m}\left|\mu_t\right| < m_t < \overline{m}\left|\mu_t\right|, \tag{31}$$

where $\underline{m}$ and $\overline{m}$ are the prescribed constants with $\underline{m} \le 0 \le \overline{m}$. This reflects the fact that the extent at which the monetary authority can modify the accumulated force of past monetary control is limited. In this paper, we choose that $\overline{g} = -\underline{g}$ and $\overline{m} = -\underline{m}$.

Let $x_1 = \mu_t$, $x_2 = y_t - \tilde{y}_t$. The uncertain dynamical economic system becomes

$$\dot{x}_1 = -r_\mu x_1 + r_\mu m_t, \tag{32}$$

$$\dot{x}_2 = k_\mu x_1 - a x_2 + \left(r_y - a y^\infty\right) + r_g g_t. \tag{33}$$

Putting (32) and (33) into the matrix form yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -r_\mu x_1 \\ k_\mu x_1 - a x_2 + \left(r_y - a y^\infty\right) \end{bmatrix} + \begin{bmatrix} r_\mu & 0 \\ 0 & r_g \end{bmatrix} \begin{bmatrix} m_t \\ g_t \end{bmatrix}. \tag{34}$$

Thus, the uncertain macroeconomic model with known bounded system noise is established. However, the control inputs m_t and g_t are bounded. To apply the control method proposed in Section 2, we need to explore a diffeomorphism to convert these two bounded controls into unbounded ones and establish the system in the form of (2).

4. Control Variable Transformation

In practice, the control inputs, which are to be implemented physically, are often subject to constraints. This is a very important issue in an economic control problem. In the past, the major efforts were focused on one-sided control, which mainly due to the physical nature of the control [32, 33]. In this paper, we extend the constraints of control τ_i, $i = 1, 2, \ldots, l$ to two sides as

$$\underline{\tau}_i < \tau_i < \overline{\tau}_i, \tag{35}$$

where $\underline{\tau}_i$ and $\overline{\tau}_i$ are given constants.

In order to ensure that the bounded controls do not exceed the specified bounds, we propose a creative bijective (that is, one-to-one) transformation, which transforms bounded controls to unbounded signals by choosing an appropriate transformation function. As a result, the transformed system is not subject to control constraints.

Specifically, by recalling a property of the tangent function (that is, $\tan : (-\pi/2, \pi/2) \longrightarrow (-\infty, +\infty)$), we adopt the diffeomorphism (which we call Ω in Section 2) between the bounded control τ_i and the new signal z_i as follows:

$$z_i = \tan\left(k_{i0}\tau_i + k_{i1}\right) + k_{i2}, \tag{36}$$

where k_{i0}, k_{i1}, and k_{i2} are coefficients to be determined. By letting

$$k_{i0}\underline{\tau}_i + k_{i1} = -\frac{\pi}{2}, \tag{37}$$

$$k_{i0}\overline{\tau}_i + k_{i1} = \frac{\pi}{2}, \tag{38}$$

the range of the new control z_i is $-\infty \longrightarrow +\infty$. More specifically, $\tau_i \longrightarrow \underline{\tau}_i$ as $z_i \longrightarrow -\infty$ and $\tau_i \longrightarrow \overline{\tau}_i$ as $z_i \longrightarrow +\infty$.

By solving (37) and (38), we can obtain

$$k_{i0} = \frac{\pi}{\overline{\tau}_i - \underline{\tau}_i}, \tag{39}$$

$$k_{i1} = -\frac{\pi\left(\overline{\tau}_i + \underline{\tau}_i\right)}{2\left(\overline{\tau}_i - \underline{\tau}_i\right)}. \tag{40}$$

The coefficient k_{i2} is for the flexibility of the diffeomorphism, which will be determined according to the specifics of the control problem. For example, to ensure $\tau_i \longrightarrow 0$ as $z_i \longrightarrow 0$, k_{i2} can be chosen as

$$k_{i2} = \tan\frac{\pi\left(\overline{\tau}_i + \underline{\tau}_i\right)}{2\left(\overline{\tau}_i - \underline{\tau}_i\right)}. \tag{41}$$

A particular example of (35) is $\overline{\tau}_i = -\underline{\tau}_i$. Thus, the coefficients can be obtained as $k_{i0} = \pi/2\overline{\tau}_i$ and $k_{i1} = k_{i2} = 0$. The diffeomorphism Ω becomes

$$z_i = \tan\left(\frac{\pi}{2\overline{\tau}_i}\tau_i\right). \tag{42}$$

Figure 1 shows the mapping relationship for the diffeomorphism (36).

5. Further Analysis and Control for the Macroeconomic System

For the macroeconomic dynamical system, the control inputs g_t and m_t are bounded. We now apply the proposed diffeomorphism to obtain the new "control" z_1 and z_2 as

$$z_1 = \tan\left(\frac{\pi}{2\overline{m}|\mu_t|}m_t\right), \tag{43}$$

$$z_2 = \tan\left(\frac{\pi}{2\overline{g}}g_t\right). \tag{44}$$

Thus, the macroeconomic dynamical system (34) can be rewritten as

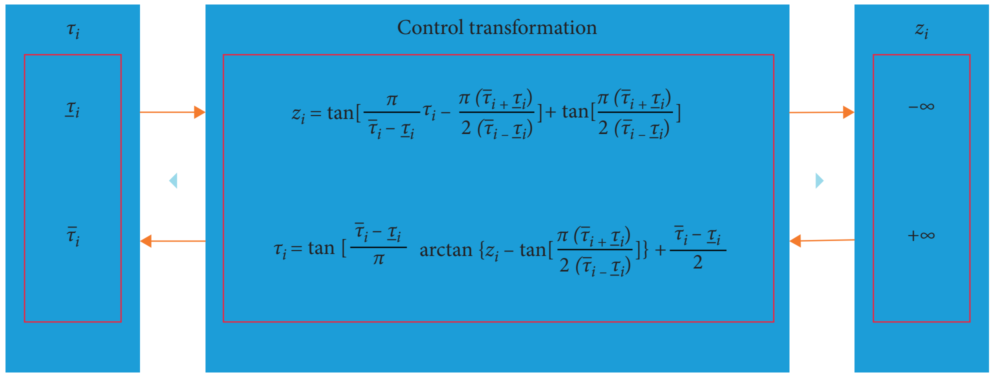

FIGURE 1: Mapping relationship of control variable transformation.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -r_\mu x_1 \\ k_\mu x_1 - a x_2 + \left(r_y - a y^\infty\right) \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} \frac{2\overline{m} r_\mu |x_1|}{\pi} & 0 \\ 0 & \frac{2\overline{g} r_g}{\pi} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \arctan(z_1) \\ \arctan(z_2) \end{bmatrix}}_{g},$$
$$= \underbrace{\begin{bmatrix} -r_\mu x_1 \\ -a x_2 \end{bmatrix}}_{\overline{f}} + \underbrace{\begin{bmatrix} 0 \\ k_\mu x_1 + \left(r_y - a y^\infty\right) \end{bmatrix}}_{\Delta f} + \begin{bmatrix} \frac{2\overline{m}|x_1| r_\mu}{\pi} & 0 \\ 0 & \frac{2\overline{g} r_g}{\pi} \end{bmatrix} \begin{bmatrix} \arctan(z_1) \\ \arctan(z_2) \end{bmatrix}. \tag{45}$$

Before proposing the control for the macroeconomic system (34), we first analyze the characteristics of the system. Let us consider the nominal system, which can be given as

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -r_\mu x_1 \\ -a x_2 \end{bmatrix}}_{\overline{f}}. \tag{46}$$

For Assumption 1,

(i) $\overline{f}(0, t) = 0$

(ii) Let $V(x, t) = (1/2)x_1^2 + (1/2)x_2^2$; then, we can choose $\overline{\lambda}_1(\|x\|) = \overline{\lambda}_2(\|x\|) = (1/2)\|x\|^2$. Furthermore,

$$\begin{aligned} \frac{\partial V(x,t)}{\partial t} + \frac{\partial^T V(x,t)}{\partial x}\overline{f}(x,t) &= -r_\mu x_1^2 - a x_2^2 \\ &\leq -\underline{r}_\mu x_1^2 - a x_2^2 \\ &\leq -\min\left\{\underline{r}_\mu, a\right\}\|x\|^2, \end{aligned} \tag{47}$$

and then, we can choose $\overline{\lambda}_3(\|x\|) = \min\left\{\underline{r}_\mu, a\right\}\|x\|^2$.

For Assumption 2, the uncertain portion Δf can be decomposed as

$$\underbrace{\begin{bmatrix} 0 \\ k_\mu x_1 + \left(r_y - a y^\infty\right) \end{bmatrix}}_{\Delta f} = \underbrace{\begin{bmatrix} \frac{2\overline{m}|x_1| r_\mu}{\pi} & 0 \\ 0 & \frac{2\overline{g} r_g}{\pi} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{h}, \tag{48}$$

with

$$h_1 = 0, \tag{49}$$

$$h_2 = \frac{\pi}{2\underline{g} r_g}\left[k_\mu x_1 + \left(r_y - a y^\infty\right)\right]. \tag{50}$$

Thus, we can choose

$$\rho(x, t) = \frac{\pi}{2\underline{g} r_g}\left[k_\mu |x_1| + \left(\overline{r}_y + a y^\infty\right)\right]. \tag{51}$$

For Assumption 3, with $z = [z_1\, z_2]^T$, we have

$$\begin{aligned} z^T g(z, \sigma, t) &= [z_1 z_2]^T \begin{bmatrix} \arctan(z_1) \\ \arctan(z_2) \end{bmatrix}, \\ &= z_1 \arctan(z_1) + z_2 \arctan(z_2). \end{aligned} \tag{52}$$

Now, we intend to find the function $\gamma(\cdot)$: $R_+ \longrightarrow R_+$ such that

$$\gamma(\|z\|) \leq z_1 \arctan(z_1) + z_2 \arctan(z_2), \tag{53}$$

where $\|z\| = (z_1^2 + z_2^2)^{1/2}$. This can be formulated as a constrained optimization problem as follows [34].

Let $\theta(z_1, z_2) = z_1 \arctan(z_1) + z_2 \arctan(z_2)$. For each $\xi \geq 0$, we find the minimum of the function $\theta(z_1, z_2)$ for all z_1 and z_2 such that $z_1^2 + z_2^2 - \xi^2 = 0$, i.e., $\min_{z_1, z_2} \theta(z_1, z_2)$, subject to $z_1^2 + z_2^2 - \xi^2 = 0$. The resulting minimum depends on ξ. Then, we formulate the function $\gamma(\xi)$.

In a compact form, the constrained optimization problem is to find $\gamma(\xi)$ such that, for each $z_1 \in R$, $z_2 \in R$,

$$\gamma(\xi) = \min\{\theta(z_1, z_2)\xi| = \|z\|\}. \tag{54}$$

We now proceed to solving this problem. Let $\Phi(z_1, z_2) = z_1^2 + z_2^2 - \xi^2$. The problem of finding $\gamma(\xi)$ can be solved by using the Lagrange method [34]. Let $L(z_1, z_2, \lambda) = \theta(z_1, z_2) + \lambda\Phi(z_1, z_2)$ where λ is the Lagrange multiplier. The stationary condition of $L(z_1, z_2, \lambda)$ is

$$\begin{cases} \dfrac{\partial}{\partial z_1}\theta(z_1, z_2) + \lambda \dfrac{\partial}{\partial z_1}\Phi(z_1, z_2) = 0, \\ \dfrac{\partial}{\partial z_2}\theta(z_1, z_2) + \lambda \dfrac{\partial}{\partial z_2}\Phi(z_1, z_2) = 0, \\ z_1^2 + z_2^2 = \xi^2. \end{cases} \tag{55}$$

The first two equations lead to

$$\begin{cases} \arctan(z_1) + \dfrac{z_1}{1 + z_1^2} + \lambda(2z_1) = 0, \\ \arctan(z_2) + \dfrac{z_2}{1 + z_2^2} + \lambda(2z_2) = 0, \end{cases} \tag{56}$$

which, in turn, mean the solution to this optimization problem is $z_1^* = z_2^* = \pm\xi/\sqrt{2}$ after imposing the third equation. Therefore, we have (both "+" and "−" yield the same result)

$$\begin{aligned} \theta(z_1^*, z_2^*) &= \left[\frac{\xi}{\sqrt{2}} \arctan\left(\frac{\xi}{\sqrt{2}}\right) + \frac{\xi}{\sqrt{2}} \arctan\left(\frac{\xi}{\sqrt{2}}\right)\right], \\ &= \frac{2\xi}{\sqrt{2}} \arctan\left(\frac{\xi}{\sqrt{2}}\right). \end{aligned} \tag{57}$$

Once we have established the dependence of $\theta(z_1^*, z_2^*)$ on ξ, we are able to establish γ. We simply select γ to be this minimum solution; that is,

$$\begin{aligned} \gamma(\xi) &= \frac{2\xi}{\sqrt{2}} \arctan\left(\frac{\xi}{\sqrt{2}}\right), \\ &= \sqrt{2}\xi \arctan\left(\frac{\xi}{\sqrt{2}}\right). \end{aligned} \tag{58}$$

Furthermore, we can choose $\psi(q) = \sqrt{2} \tan q/\sqrt{2}$.

Figure 2 simply shows the comparison of different results of $\theta(z_1, z_2) = z_1 * \arctan(z_1) + z_2 * \arctan(z_2)$ under different constraints.

With these chosen functions satisfying the three assumptions, the control (11) can be applied to the macroeconomic system. Here, we add two remarks to further substantiate the control design details.

Remark 6. When $\|\mu\| \leq \varepsilon$, the control (11) is presented in a symmetrical structure. Theoretically, there are infinite choices of $p(x, t)$. A particular example of (11) is

$$p(x, t) = \frac{-\mu(x, t)}{\varepsilon} \psi(\rho(x, t)). \tag{59}$$

Remark 7. There are two magnitudes to be focused on. One is the magnitude of the control function $\psi(\rho(x, t))$. The other one is the bound of the system noise $\rho(x, t)$. They are all related to $\rho(x, t)$, but not the same. The reason is that the input u does not directly act on the system, but plays its role through the bounding function $g(u, \sigma, t)$ which may reduce the influence of u. Therefore we need to design the control with the magnitude $\psi(\rho(x, t))$ to precompensate the influence due to $g(u, \sigma, t)$.

The control design procedure is shown in Figure 3.

6. Numerical Simulations

We perform numerical simulations to demonstrate that the proposed deterministic robust control design is able to address the stabilization problem of the macroeconomic system with known bounded system noise and two-side control input constraints. The numerical simulations throughout this paper are performed in the MATLAB environment, using a variable time step ode 15i integrator.

For simulations, the system parameters are selected as $a = 4$, $y^\infty = 8$, $k_\mu = 2$, $\bar{r}_y = 8$, $\bar{r}_m = 10$, and $\bar{r}_g = 10$. The control parameters are chosen as $r_\mu = 8$, $r_g = 10$, and $\varepsilon = 0.01$. For simulations, the system noise is chosen as $r_y = 6 + 2\sin(t)$.

Simulation results are shown in Figures 4–11. For comparison, we choose the classical LQR control [35–37]. Figures 4 and 5 display the time history of the two system states: x_1 (i.e., μ_t) and x_2 (*i.e.*, $y_t - \tilde{y}_t$), respectively. Under the diffeomorphism-based control, they all approach to a domain which is close to zero from the initial state deviations. This demonstrates that the macroeconomic dynamical system is well controlled with the "actual aggregate demand" approaching to the "target aggregate demand," while under the LQR control, the tracking trajectory of the system state x_2 varies with a certain degree of fluctuation. Figures 6 and 7 show the time history of actual controls m_t and g_t, respectively. It is shown that, under the diffeomorphism-based approach, these two actual control inputs both situate between the lower bound and the upper bound. In other words, the controls meet the prespecified constraints, while under

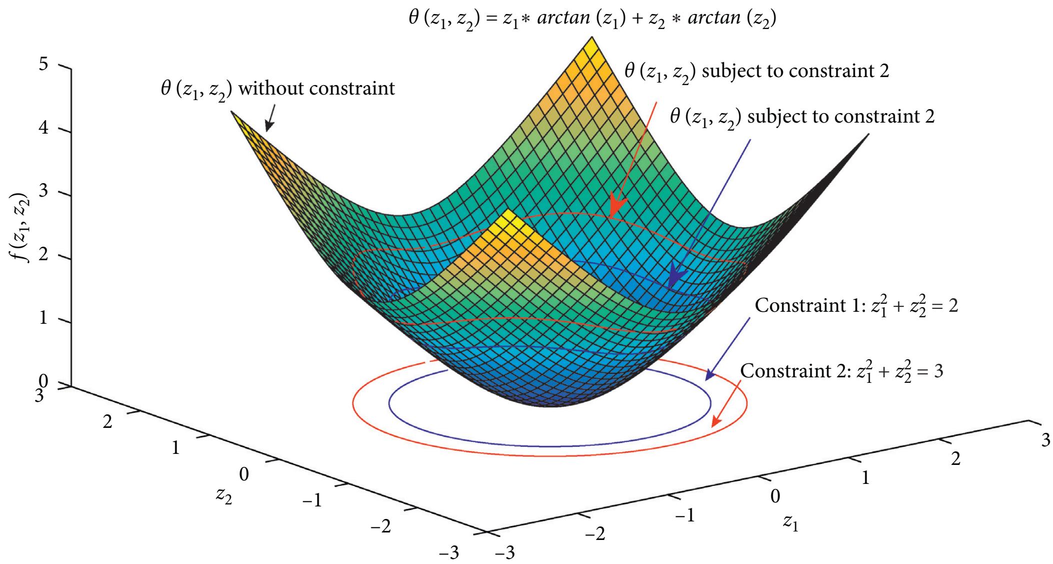

Figure 2: $\theta(z_1, z_2) = z_1 * \arctan(z_1) + z_2 * \arctan(z_2)$ under different constraints.

Figure 3: Control design procedure.

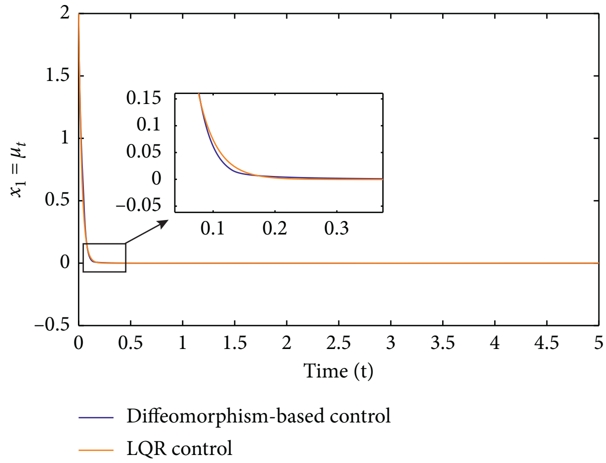

FIGURE 4: Time history of system state $x_1 = \mu_t$.

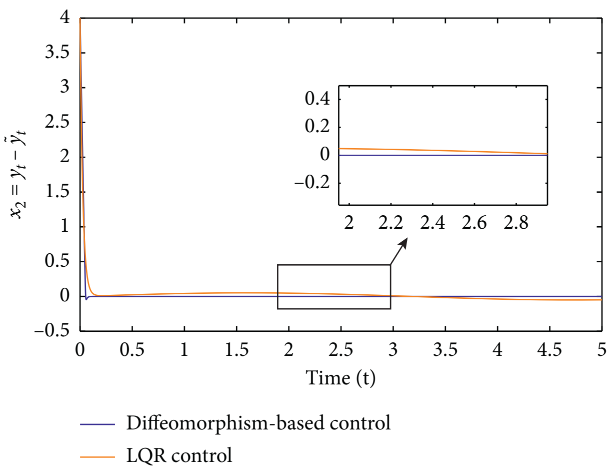

FIGURE 5: Time history of system state $x_2 = y_t - \tilde{y}_t$.

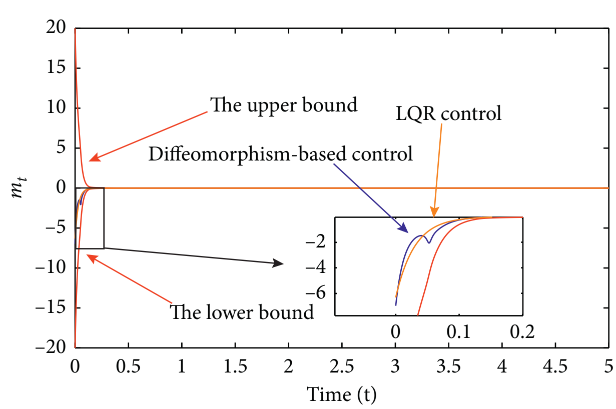

FIGURE 6: Time history of actual control m_t.

the LQR control, the minimum value of the control g_t is −11.57, which exceeds the lower bound of the prescribed control constraint. Thus, the proposed diffeomorphism-based control displays a significant superior performance over the LQR control. Figures 8 and 9 show the influence of

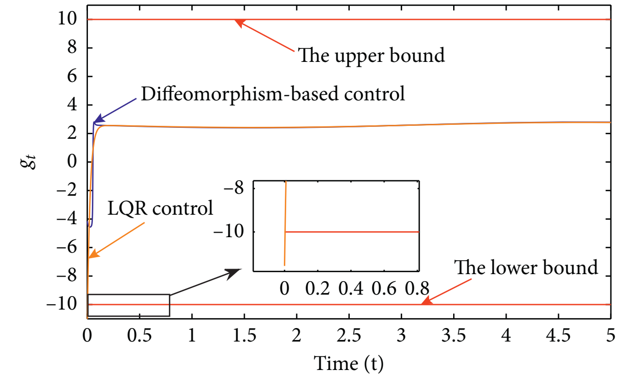

FIGURE 7: Time history of actual control g_t.

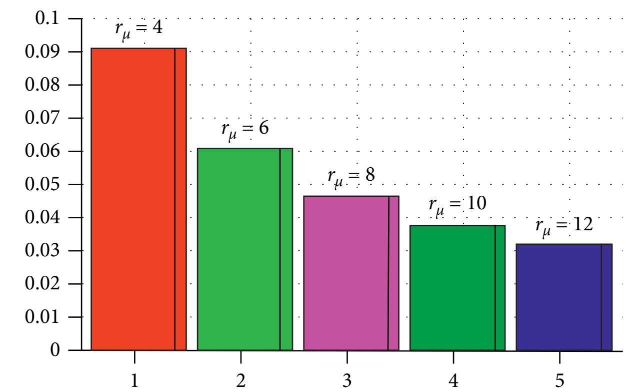

FIGURE 8: Comparison of the accumulative error for state x_1 via choosing different γ_μ.

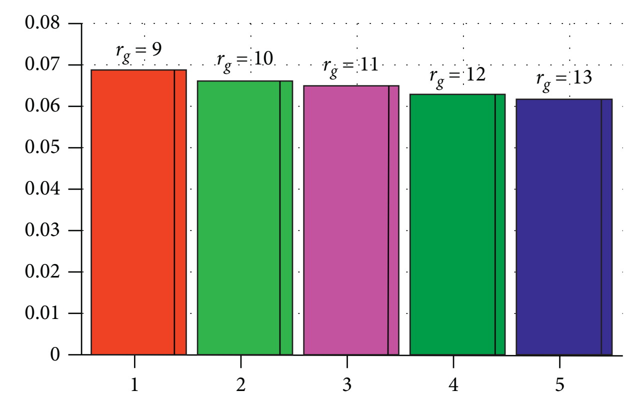

FIGURE 9: Comparison of the accumulative error for state x_2 via choosing different γ_g.

the different choices of control parameters r_μ and r_g on the accumulative error of system state x_1 and x_2, respectively. As the control parameter increases, the resulting accumulative error decreases. Figures 10 and 11 show the maximum values of different control inputs corresponding to different control bounds. By comparing these values, we notice that the inequality control constraints $\underline{m}\,|\mu| < m_t < \overline{m}|\mu|$ and $\underline{g} < g_t < \overline{g}$ are all met.

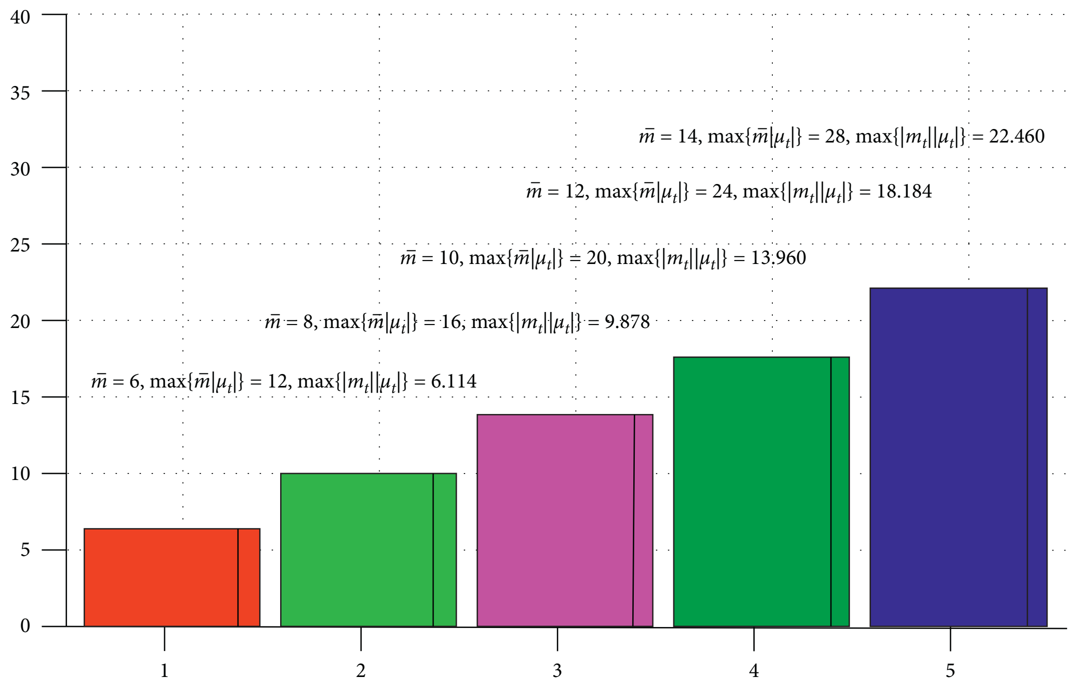

Figure 10: Comparison of the actual control m_t via choosing different $\overline{m}$.

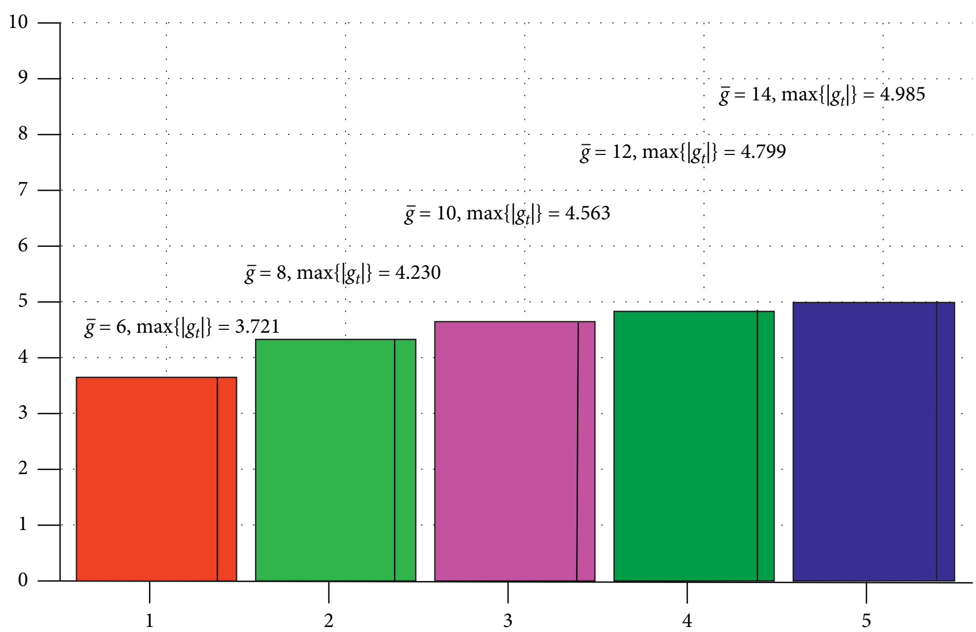

Figure 11: Comparison of the actual control g_t via choosing different $\overline{g}$.

7. Conclusions

A salient feature in an economic control problem is the control constraint, which is often not addressed in other areas such as engineering. In this regard, a robust control design for a class of uncertain dynamical systems is proposed. The uncertainties, including system parameter and external disturbances, may be time varying and are unknown functions of state and time. The control inputs are subject to two-side inequality constraints. To ensure that the bounded controls do not exceed the specified boundaries, we propose a creative diffeomorphism, which converts bounded controls into unbounded corresponding signals by choosing an appropriate transformation function. The magnitude of the control function which is related to the bound of the system noise is designed to precompensate the influence due to the transformation function. Finally, the system performance, including uniform boundedness and uniform ultimate boundedness, is guaranteed. Numerical simulations demonstrate that this bounded control design effectively solves the stabilization problem of the macroeconomic dynamic system. Our future work will mainly focus on two fronts. Firstly, a more practical and complex macroeconomic dynamic model, which contains a broader class of uncertainties, will be explored. Secondly, the optimal (rather than feasible) design issue of the control parameters will be studied.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the Fundamental Research Funds for the Central Universities of China (No. 300102258305).

References

[1] D. A. Kendrick and H. M. Amman, *Computational Economics*, Princeton University Press, Princeton, NJ, USA, 2006.
[2] A. A. Ashimov, B. T. Sultanov, Z. M. Adilov et al., *Macroeconomic Analysis and Economic Policy Based on Parametric Control*, Springer Science & Business Media, New York, NY, USA, 2011.
[3] C. Carraro and D. Sartore, *Developments of Control Theory for Economic Analysis*, Springer Science & Business Media, New York, NY, USA, 2012.
[4] Y. Murata, *Optimal Control Methods for Linear Discrete-Time Economic Systems*, Springer Science & Business Media, New York, NY, USA, 2012.
[5] P. W. Otter, *Dynamic Feature Space Modelling, Filtering and Self-Tuning Control of Stochastic Systems: A Systems Approach with Economic and Social Applications*, Springer Science & Business Media, New York, NY, USA, 2012.
[6] M. J. M. Rao, *Filtering and Control of Macroeconomic Systems: A Control System Incorporating the Kalman Filter for the Indian Economy*, Elsevier, Amsterdan, The Netherlands, 2013.
[7] G. Tintner and J. K. Sengupta, *Stochastic Economics: Stochastic Processes, Control, and Programming*, Academic Press, New York, NY, USA, 2014.
[8] R. H. Day, *The Divergent Dynamics of Economic Growth: Studies in Adaptive Economizing, Technological Change, and Economic Development*, Cambridge University Press, New York, NY, USA, 2003.
[9] D. Hudgins and J. Na, "Entering H∞ optimal control robustness into a macroeconomic LQ-tracking model," *Computational Economics*, vol. 47, no. 2, pp. 121–155, 2016.
[10] M. Berardi and J. K. Galimberti, "On the initialization of adaptive learning in macroeconomic models," *Journal of Economic Dynamics and Control*, vol. 78, pp. 26–53, 2017.
[11] A. Jajarmi, M. Hajipour, and D. Baleanu, "New aspects of the adaptive synchronization and hyperchaos suppression of a financial model," *Chaos, Solitons & Fractals*, vol. 99, pp. 285–296, 2017.
[12] S. S. Chang and H. O. Stekler, "Fuzziness in economic systems, its modeling and control," *Annals of Economic and Social Measurement*, vol. 6, no. 2, pp. 165–174, 1977.
[13] G. Leitmann and H. Y. Wan, "A stabilization policy for an economy with some unknown characteristics," *Journal of the Franklin Institute*, vol. 306, no. 1, pp. 23–33, 1978.
[14] L. Liu and X. Yang, "Robust adaptive state constraint control for uncertain switched high-order nonlinear systems," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 10, pp. 8108–8117, 2017.
[15] S. Mobayen and J. Ma, "Robust finite-time composite nonlinear feedback control for synchronization of uncertain chaotic systems with nonlinearity and time-delay," *Chaos, Solitons & Fractals*, vol. 114, pp. 46–54, 2018.
[16] J. Xia, J. Zhang, W. Sun, B. Zhang, and Z. Wang, "Finite-time adaptive fuzzy control for nonlinear systems with full state constraints," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 7, pp. 1541–1548, 2018.
[17] J. Wu, B. Su, J. Li, X. Zhang, X. Li, and W. Chen, "Adaptive fuzzy control for full states constrained systems with nonstrict-feedback form and unknown nonlinear dead zone," *Information Sciences*, vol. 376, pp. 233–247, 2017.
[18] S. Mobayen, M. J. Yazdanpanah, and V. J. Majd, "A finite-time tracker for nonholonomic systems using recursive singularity-free FTSM," 2011.
[19] S. Mobayen and F. Tchier, "Nonsingular fast terminal sliding-mode stabilizer for a class of uncertain nonlinear systems based on disturbance observer," *Scientia Iranica*, vol. 24, no. 3, pp. 1410–1418, 2017.
[20] S. Mobayen, "Design of LMI-based sliding mode controller with an exponential policy for a class of underactuated systems," *Complexity*, vol. 21, no. 5, pp. 117–124, 2016.
[21] L. C. Guo and X. K. Fang, "Bounded analysis and practical tracking control of complex stochastic nonlinear systems with unknown control coefficients," *Complexity*, vol. 2020, 2020.
[22] V. N. Phat and N. H. Sau, "Exponential stabilisation of positive singular linear discrete-time delay systems with bounded control," *IET Control Theory & Applications*, vol. 13, no. 7, pp. 905–911, 2018.
[23] Q. Yu, Z. Hou, and R. Chi, "Adaptive iterative learning control for nonlinear uncertain systems with both state and input constraints," *Journal of the Franklin Institute*, vol. 353, no. 15, pp. 3920–3943, 2016.
[24] Y. Fan, W. Wang, Y. Liu, and M. Wang, "Fuzzy adaptive switching control for an uncertain robot manipulators with time-varying output constraint," *Complexity*, vol. 10, 2018.
[25] Y. Chen, "Deterministic control for a new class of uncertain dynamical systems," *IEEE Transactions on Automatic Control*, vol. 32, no. 1, pp. 73-74, 1987.
[26] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, Upper Saddle River, NJ, USA, 3rd edition, 2002.
[27] J. Xu, Y. Du, Y.-H. Chen, and H. Guo, "Optimal robust control design for constrained uncertain systems: a fuzzy-set theoretic approach," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3494–3505, 2018.
[28] C. Berge, *Topological Spaces*, Oliver & Boyd, London, UK, 1963.
[29] M. Corless and G. Leitmann, "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 5, pp. 1139–1144, 1981.
[30] J. Xu, Y. Du, Y.-H. Chen, H. Guo, and X. Ding, "Guaranteeing uniform ultimate boundedness for uncertain systems free of

matching condition," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3479–3493, 2018.

[31] X. Chen, H. Zhao, H. Sun, S. Zhen, and K. Huang, "A novel adaptive robust control approach for underactuated mobile robot," *Journal of the Franklin Institute*, vol. 356, no. 5, pp. 2474–2490, 2019.

[32] Y. H. Chen and C. Y. Kuo, "Positive uncertain systems with one-sided robust control," *Journal of Dynamic Systems, Measurement, and Control*, vol. 119, no. 4, pp. 675–684, 1997.

[33] R. Zhao, Y.-H. Chen, and S. Jiao, "Optimal design of robust control for positive fuzzy dynamic systems with one-sided control constraint," *Journal of Intelligent & Fuzzy Systems*, vol. 32, no. 1, pp. 723–735, 2017.

[34] D. G. Luenberger, *Optimization by Vector Space Methods*, John Wiley & Sons, New York, NY, USA, 1969.

[35] R. Bhattacharya, "Robust LQR design for systems with probabilistic uncertainty," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 10, pp. 3217–3237, 2019.

[36] S. Janous, J. Talla, V. Smidl, and Z. Peroutka, "Constrained LQR control of dual induction motor single inverter drive," *IEEE Transactions on Industrial Electronics*, vol. 6, 2020.

[37] P. Xia, H. Shi, H. Wen, Q. Bu, Y. Hu, and Y. Yang, "Robust LMI-LQR control for dual-active-bridge DC-DC converters with high parameter uncertainties," *IEEE Transactions on Transportation Electrification*, vol. 6, no. 1, pp. 131–145, 2020.