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Research Article

An Iterative Algorithm for Solving *n*-Order Fractional Differential Equation with Mixed Integral and Multipoint Boundary Conditions

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In this paper, we consider the iterative algorithm for a boundary value problem of n-order fractional differential equation with mixed integral and multipoint boundary conditions. Using an iterative technique, we derive an existence result of the uniqueness of the positive solution, then construct the iterative scheme to approximate the positive solution of the equation, and further establish some numerical results on the estimation of the convergence rate and the approximation error.

1. Introduction

In this paper, we focus on the iterative algorithm for the following n-order fractional equation involving mixed integral and multipoint boundary conditions:

$$\begin{aligned}
&-D_{0+}^{\alpha}x(t) = f(t, x(t), x(t)), \quad 0 < t < 1,\\
&x^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, n-2,\\
&x(1) = \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}x(s)\mathrm{d}s + \sum_{i=1}^{m-2}\gamma_i x(\eta_i),
\end{aligned} \tag{1}$$

where D_{0+}^{α} is the standard fractional derivative of order α satisfying $n-1<\alpha\le n$ with $m, n \ge 3$ and $m, n \in N^+$, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$, $\beta_i, \gamma_i > 0$, $1 \le i \le m-2$, and $f(t, u, v)$ is may be singular at $v = 0$ and $t = 0, 1$.

Based on the wide range applications of calculus, in recent years, the study for various differential equations has become a frontier issue of nonlinear field and many mathematical methods and techniques, such as iterative techniques [1–17], dual approach and perturbed techniques [18–23], fixed-point theorems [24–50], lower-upper solution method [51–53], variational method [54–68], numerical methods and stability analysis [69–81], which were developed by many researchers to handle various nonlinear problems. In particular, in describing and modeling viscoelasticity and nonlocal problems in complex analysis, environmental issue, chemistry physics, and statistical physics, a large amount of work [18, 30, 64, 82–111] have shown that fractional differential equations possess greater advantage than classical integer differential equations. In recent work [95], Salm, by using the Hahn-Banach fixed point theorem, studied the following multipoint boundary value problem:

$$\begin{aligned}
&-D_{0+}^{\alpha}x(t) = q(t)f(t, x(t)), \quad 0 < t < 1, n-1 < \alpha \le n, n \ge 3,\\
&x^{k}(0) = 0, \quad 0 \le k \le n-2,\\
&x(1) = \sum_{i=1}^{m-2}\zeta_i x(\eta_i).
\end{aligned} \tag{2}$$

The weakly continuous solution for the above nonlinear boundary value problem of fractional type was derived. And then, Xie et al. [96] studied the following nonlinear fractional differential equation with a three-point nonlinear boundary condition:

$$\begin{aligned}&D_{0+}^{\alpha}x(t)+f(t,x(t))=0, \quad 0\le t\le 1,\\ &\qquad x(0)=0, g(x(1),x(\eta))=0, \quad 0\le \eta\le 1.\end{aligned} \tag{3}$$

By using the method of upper and lower solutions as well as the monotone iterative technique, the results about the existence of extremal solutions were obtained, where the iterative process can start from the fixed upper and lower solutions.

Among various techniques of dealing with differential equations, upper and lower solutions' method and fixed-point methods have been verified to be the most efficient approaches. However, the efficiency of those methods depends essentially on the monotonicity and compactness of the operator. So, how to overcome the requirement of the monotonicity and compactness is a huge challenge, since such qualities are not naturally available and difficult to prove which lead to the complexity to solve some BVP, especially for the fractional nonlinear boundary value problems.

Different from [96], in this paper, we develop the new iterative algorithm to overcome the requirement of the compactness for the nonlinear operator. Our work has three major features. Firstly, equation (1) possesses a more general nonlinear term which has two space variables and the boundary condition is a mixed integral and multipoint boundary condition. Secondly, the nonlinear term can be singular in the time variables and the second space variables. In the end, our results are more refined, that is, we not only construct a new iterative process which can perform from any initial value but also obtain the uniform convergence of the iterative sequences; at same time, the estimation of the convergence rate and the approximation error are also given, which imply that more strong results are established under the relatively weaker condition than that of [96].

The paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, the unique of positive solutions to BVP (1) are obtained. Finally, in Section 4, an illustrative example is also presented.

2. Preliminaries

In this section, we first list some notations and recall related definitions and lemmas to be used in our proofs later.

Definition 1 (see [5]). Let $p>0$, the Riemann–Liouville standard fractional integral derivative of order $p>0$ of a function $f\colon (0,\infty)\longrightarrow R$ is given by

$$\begin{aligned}I_{0+}^{p}f(x)&=\frac{1}{\Gamma(p)}\int_0^x\frac{f(t)}{(x-t)^{1-p}}\mathrm{d}t,\\ D_{0+}^{p}f(t)&=\frac{1}{\Gamma(n-p)}\left(\frac{d}{\mathrm{d}t}\right)^n\int_0^t\frac{f(s)}{(t-s)^{p-n+1}}\mathrm{d}s,\end{aligned} \tag{4}$$

where $n=[p]+1$, $[p]$ denotes the integer part of the real number p.

Lemma 1 (see [5]). *Suppose that $n-1<\alpha\le n$ with $n\ge 3$ and $h\in L^1[0,1]$; then, the boundary value problem*

$$\begin{aligned}&-D_{0+}^{\alpha}x(t)=h(t), \quad 0<t<1,\\ &x^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2,\\ &x(1)=\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}x(s)\mathrm{d}s+\sum_{i=1}^{m-2}\gamma_i x(\eta_i),\end{aligned} \tag{5}$$

is given by

$$\begin{aligned}x(t)&=\int_0^1 G(t,s)h(s)\mathrm{d}s+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1\\ &\cdot[\beta_i H(\eta_i,s)+\gamma_i G(\eta_i,s)]h(s)\mathrm{d}s,\end{aligned} \tag{6}$$

where $\xi=1-(1/\alpha)\sum_{i=1}^{m-2}\beta_i\eta_i^{\alpha}-\sum_{i=1}^{m-2}\gamma_i\eta_i^{\alpha-1}>0$, and

$$\begin{aligned}G(t,s)&=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1,\end{cases}\\ H(t,s)&=\frac{1}{\Gamma(\alpha+1)}\begin{cases}t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1.\end{cases}\end{aligned} \tag{7}$$

Lemma 2 (see [5]). *For all $t,s\in[0,1]$, the functions $G(t,s)$ and $H(t,s)$ in Lemma 1 satisfy the following properties:*

(1) $G(t,s)$ and $H(t,s)$ are continuous and nonnegative

(2) $((\alpha-1)/\Gamma(\alpha))t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s\le G(t,s)\le(1/\Gamma(\alpha))t^{\alpha-1}(1-s)^{\alpha-2}$

(3) $((\alpha-1)/\Gamma(\alpha+1))t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s\le H(t,s)\le(1/\Gamma(\alpha+1))t^{\alpha-1}(1-s)^{\alpha-2}$

In this paper, we will work in the space $E=C[0,1]$. Define a set P in E and an operator $T\colon E\times E\longrightarrow E$ as follows:

$$\begin{aligned}P&=\left\{\begin{array}{c}x\in E\mid \text{there exists a positive constant } l_x\in(0,1), \text{ such that}\\ l_x t^{\alpha-1}\le x(t)\le (l_x)^{-1}t^{\alpha-1}, t\in(0,1)\end{array}\right\},\\ T(x,y)(t)&=\int_0^1 G(t,s)f(s,x(s),y(s))\mathrm{d}s\\ &\quad+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1[\beta_i H(\eta_i,s)+\gamma_i G(\eta_i,s)]f(s,x(s),y(s))\mathrm{d}s.\end{aligned} \tag{8}$$

Obviously, $(t^{\alpha-1}, t^{\alpha-1}) \in P \times P$, *so* $P \times P$ *is not empty.*

3. Main Results

Before claiming our main results, we first introduce the following notations:

(i) $M = \xi\alpha + (1+\alpha)(m-2)$

(ii) $N = \sum_{i=1}^{m-2} (\beta_i + \alpha\gamma_i)\eta_i^{\alpha-1}(1-\eta_i)$

Theorem 1. *Assume that*

(H_1) $f(t,u,v) \in C[(0,1) \times [0,+\infty) \times (0,+\infty); (0,+\infty)]$ *and for* $(t,u,v) \in (0,1) \times [0,+\infty) \times (0,+\infty)$, f *is increasing with respect to* u, *decreasing with respect to* v

(H_2) For $(t,u,v) \in (0,1) \times [0,+\infty) \times (0,+\infty)$ *and* $k \in (0,1)$, *there exists a constant* $\mu \in (0,1)$ *such that* $f(t, ku, k^{-1}v) \geq k^{\mu} f(t,u,v)$

(H_3) $0 < \int_0^1 (1-s)^{\alpha-2} f(s, s^{\alpha-1}, s^{\alpha-1}) ds < \infty$

Then, the BVP (1) has unique positive solution $x^*(t) \in P$, *and there exists a constant* $0 < l < 1$ *satisfying*

$$l t^{\alpha-1} \leq x^*(t) \leq l^{-1} t^{\alpha-1}, \quad t \in [0,1]. \tag{9}$$

Proof. Firstly, it is easy to know that x^* is the solution of the BVP (1) if and only if x^* satisfies $T(x^*, x^*) = x^*$.

Next, it follows from (H_1) that the operator $T: P \times P \longrightarrow P$ is nondecreasing with respect to x and nonincreasing with respect to y; thus, by (H_1), (H_2), and (H_3), for any $(x,y) \in P \times P$ and $t \in (0,1)$, there exist two constants $0 < l_x < 1, 0 < l_y < 1$ such that

$$\begin{aligned} l_x t^{\alpha-1} &\leq x(t) \leq (l_x)^{-1} t^{\alpha-1}, \\ l_y t^{\alpha-1} &\leq y(t) \leq (l_y)^{-1} t^{\alpha-1}. \end{aligned} \tag{10}$$

Denote $l_x^* = \min\{l_x, l_y\}$; then, we have

$$\begin{aligned} l_x^* t^{\alpha-1} &\leq x(t) \leq (l_x^*)^{-1} t^{\alpha-1}, \\ l_x^* t^{\alpha-1} &\leq y(t) \leq (l_x^*)^{-1} t^{\alpha-1}. \end{aligned} \tag{11}$$

Consequently,

$$\begin{aligned} T(x,y)(t) &= \int_0^1 G(t,s) f(s, x(s), y(s)) ds \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 [\beta_i H(\eta_i, s) + \gamma_i G(\eta_i, s)] f(s, x(s), y(s)) ds \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f\left(s, (l_x^*)^{-1} s^{\alpha-1}, l_x^* s^{\alpha-1}\right) ds \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 \left[\frac{\beta_i \eta_i^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{\gamma_i \eta_i^{\alpha-1}}{\Gamma(\alpha)}\right] (1-s)^{\alpha-2} f\left(s, (l_x^*)^{-1} s^{\alpha-1}, l_x^* s^{\alpha-1}\right) ds \\ &\leq \frac{(l_x^*)^{-\mu} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds \\ &\quad + \frac{(l_x^*)^{-\mu} t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 \left[\frac{\beta_i \eta_i^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{\gamma_i \eta_i^{\alpha-1}}{\Gamma(\alpha)}\right] (1-s)^{\alpha-2} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds \\ &\leq (l_x^*)^{-\mu} t^{\alpha-1} \frac{\xi\alpha + (1+\alpha)(m-2)}{\xi\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds \\ &= t^{\alpha-1} \frac{M (l_x^*)^{-\mu}}{\xi\alpha\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds < +\infty, \end{aligned} \tag{12}$$

$$\begin{aligned} T(x,y)(t) &= \int_0^1 G(t,s) f(s, x(s), y(s)) ds \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 [\beta_i H(\eta_i, s) + \gamma_i G(\eta_i, s)] f(s, x(s), y(s)) ds \\ &\geq \frac{(\alpha-1) t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 \left[\frac{\beta_i \eta_i^{\alpha-1}(1-\eta_i)}{\Gamma(\alpha+1)} + \frac{\gamma_i \eta_i^{\alpha-1}(1-\eta_i)}{\Gamma(\alpha)}\right] s(1-s)^{\alpha-1} f\left(s, l_x^* s^{\alpha-1}, (l_x^*)^{-1} s^{\alpha-1}\right) ds \\ &\geq \frac{(l_x^*)^{\mu} t^{\alpha-1}(\alpha-1)}{\xi\alpha\Gamma(\alpha)} \sum_{i=1}^{m-2} (\beta_i + \alpha\gamma_i)\eta_i^{\alpha-1}(1-\eta_i) \int_0^1 s(1-s)^{\alpha-1} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds \\ &= t^{\alpha-1} \frac{(\alpha-1) N (l_x^*)^{\mu}}{\xi\alpha\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right) ds. \end{aligned} \tag{13}$$

Let

$$l_{\mathrm{Tx}} = \min\left\{\begin{array}{l} \dfrac{1}{2}, \dfrac{(\alpha-1)N\left(l_x^*\right)^{\mu}}{\xi\alpha\Gamma(\alpha)}\displaystyle\int_0^1 s(1-s)^{\alpha-1} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right)\mathrm{d}s, \\ \left[\dfrac{M\left(l_x^*\right)^{-\mu}}{\xi\alpha\Gamma(\alpha)}\displaystyle\int_0^1 (1-s)^{\alpha-2} f\left(s, s^{\alpha-1}, s^{\alpha-1}\right)\mathrm{d}s\right]^{-1} \end{array}\right\}. \tag{14}$$

Then, there exists a constant $0 < l_{\mathrm{Tx}} < 1$ such that

$$l_{\mathrm{Tx}} t^{\alpha-1} \leq (T(x,y))(t) \leq \left(l_{\mathrm{Tx}}\right)^{-1} t^{\alpha-1}, \quad t \in (0,1), \tag{15}$$

which implies that the operator $T\colon P \times P \longrightarrow P$ is well defined. According to the Arzela-Ascoli theorem, it is easy to know that $T\colon P \times P \longrightarrow P$ is completely continuous.

Now, take $h(t) = t^{\alpha-1}$, then $(h,h) \in P \times P$, it follows from (12) and (13) that $T(h,h) \in P$. Thus, by the definition of P, there exists a constant $0 < l_{\mathrm{Th}} < 1$ such that

$$l_{\mathrm{Th}} t^{\alpha-1} \leq T(h,h)(t) \leq \left(l_{\mathrm{Th}}\right)^{-1} t^{\alpha-1}. \tag{16}$$

Take

$$0 < \lambda \leq l_{\mathrm{Th}}^{(1/1-\mu)}, \tag{17}$$

and let

$$\begin{aligned} x_0 &= \lambda h(t), \\ y_0 &= \lambda^{-1} h(t). \end{aligned} \tag{18}$$

Now, construct the following iterative sequence:

$$\begin{aligned} x_n &= T\left(x_{n-1}, y_{n-1}\right), \\ y_n &= T\left(y_{n-1}, x_{n-1}\right), \quad n = 1, 2, \ldots \end{aligned} \tag{19}$$

We assert

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0. \tag{20}$$

In fact, it follows from $0 < \lambda < 1$ and (18) that $x_0, y_0 \in P$ and $x_0 \leq y_0$. Moreover,

$$\begin{aligned} x_1 = T\left(x_0, y_0\right)(t) = &\int_0^1 G(t,s) f\left(s, \lambda h(t), \lambda^{-1} h(t)\right)\mathrm{d}s \\ &+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1 \left[\beta_i H\left(\eta_i, s\right) + \gamma_i G\left(\eta_i, s\right)\right] f\left(s, \lambda h(t), \lambda^{-1} h(t)\right)\mathrm{d}s \\ \geq &\lambda^{\mu}\int_0^1 G(t,s) f(s, h(t), h(t))\mathrm{d}s \\ &+ \frac{\lambda^{\mu} t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1 \left[\beta_i H\left(\eta_i, s\right) + \gamma_i G\left(\eta_i, s\right)\right] f(s, h(t), h(t))\mathrm{d}s \\ = &\lambda^{\mu} T(h,h)(t) \geq \lambda^{\mu} l_{\mathrm{Th}} h(t) \geq \lambda^{\mu}\lambda^{1-\mu} h(t) = x_0, \\ y_1 = T\left(y_0, x_0\right)(t) = &\int_0^1 G(t,s) f\left(s, \lambda^{-1} h(t), \lambda h(t)\right)\mathrm{d}s \\ &+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1 \left[\beta_i H\left(\eta_i, s\right) + \gamma_i G\left(\eta_i, s\right)\right] f\left(s, \lambda^{-1} h(t), \lambda h(t)\right)\mathrm{d}s \\ \leq &\lambda^{-\mu}\int_0^1 G(t,s) f(s, h(t), h(t))\mathrm{d}s \\ &+ \frac{\lambda^{-\mu} t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\int_0^1 \left[\beta_i H\left(\eta_i, s\right) + \gamma_i G\left(\eta_i, s\right)\right] f(s, h(t), h(t))\mathrm{d}s \\ = &\lambda^{-\mu} T(h,h)(t) \leq \lambda^{-\mu}\left(l_{\mathrm{Th}}\right)^{-1} h(t) \leq \lambda^{-\mu}\lambda^{\mu-1} h(t) = y_0. \end{aligned} \tag{21}$$

On the other hand, it follows from $x_0 \le y_0$ and the fact of T being nondecreasing with respect to second variable and nonincreasing with respect to third variable that $x_1 \le y_1$. Thus, according to the above fact, we have (20) holds.

Notice that, for any nature number n,

$$\begin{aligned} x_n &= T(x_{n-1}, y_{n-1}) = T^n(y_0, x_0) \\ &= T^n\left(\lambda h(t), \lambda^{-1} h(t)\right) \\ &= T^n\left(\lambda^2 \lambda^{-1} h(t), \lambda^{-2} \lambda h(t)\right) \\ &\ge \left(\lambda^2\right)^{\mu^n} T^n\left(\lambda^{-1} h(t), \lambda h(t)\right) \\ &= c^{\mu^n} y_n, \end{aligned} \tag{22}$$

where $c = \lambda^2$. So, for any nature numbers n and n^*, we have

$$\begin{aligned} 0 \le x_{n+n^*} - x_n \le y_n - x_n \le \left(1 - c^{\mu^n}\right) y_n \le \left(1 - c^{\mu^n}\right)\lambda^{-1} \\ h(t) \longrightarrow 0, \quad n \longrightarrow +\infty, \end{aligned} \tag{23}$$

which implies that there exists $x^* \in P$ such that

$$x_n(t) \longrightarrow x^*(t), \tag{24}$$

uniformly on $(0,1)$. By the same method, we can also prove that

$$y_n(t) \longrightarrow x^*(t), \tag{25}$$

uniformly on $(0,1)$. In view of the continuous of T, take the limits in $x_n = T(x_n, y_n)$, we have $x^* = T(x^*, x^*)$. So, x^* is a positive solution of BVP (1). Since $x^* \in P$, for any $t \in (0,1)$, there exists a constant $l \in (0,1)$ such that

$$l t^{\alpha-1} \le x^*(t) \le l^{-1} t^{\alpha-1} \tag{26}$$

holds.

Finally, we show that the uniqueness of the positive solution. Let $y^*(t)$ be another positive solution of BVP (1); then, for any $t \in (0,1)$, there exists a constant $m \in (0,1)$ such that

$$m t^{\alpha-1} \le y^*(t) \le m^{-1} t^{\alpha-1}. \tag{27}$$

Taking λ defined in (17) be small enough such that $\lambda < m$. So,

$$x_0(t) \le y^*(t) \le y_0(t), \quad t \in (0,1). \tag{28}$$

According to $T(y^*, y^*) = y^*$, using the nondecreasing of T, we can show that

$$x_n(t) \le y^*(t) \le y_n(t), \quad t \in (0,1). \tag{29}$$

Taking limits to the both sides of (29), we have $x^* = y^*$. It follows that the solution of BVP (1) is unique. The proof of Theorem 1 is completed.

In the following, we consider the error estimation between unique solution and iterative value. □

Theorem 2. *Let conditions* (H_1), (H_2), *and* (H_3) *be satisfied. Then, for any initial value* $z_0 \in P$, *there exists a sequence* $z_n(t)$ *that uniformly converges to the unique positive solution* $x^*(t)$ *with the following error estimation:*

$$\max\left\{\left|z_n(t) - x^*(t)\right|\right\} = o\left(1 - c^{\mu^n}\right), \tag{30}$$

where $c \in (0,1)$ *is determined by* z_0.

Proof. By Theorem 1, we know that the positive solution x^* is unique. For any $z_0 \in P$, there exists a constant $l_{z_0} \in (0,1)$, such that

$$l_{z_0} t^{\alpha-1} \le z_0(t) \le l_{z_0}^{-1} t^{\alpha-1}. \tag{31}$$

Similar to Theorem 1, we can take $\lambda < \min\left\{l_{z_0}, l_{\mathrm{Th}}^{(1/1-\mu)}\right\}$ as a fixed number. It follows that

$$x_0(t) \le z_0(t) \le y_0(t), \quad t \in (0,1). \tag{32}$$

Let $z_n = T(z_{n-1}, z_{n-1})$, $n = 1, 2, \ldots$; then, by monotonicity of the operator T, we have

$$x_1(t) \le z_1(t) \le y_1(t), \quad t \in (0,1). \tag{33}$$

Thus, it follows from mathematical induction that

$$x_n(t) \le z_n(t) \le y_n(t), \quad t \in (0,1). \tag{34}$$

Take limits for the above inequality, we get that $\{z_n(t)\}$ uniformly converges to the unique positive solution x^* of BVP (1). Using (23), we can now derive the error estimation (30), which implies that the error estimation is the same order infinitesimal of $(1 - c^{\mu^n})$, where $c = \lambda^2$ and determined by z_0. This completes the proof of Theorem 2. □

4. Example

Let us illustrate the main results with an example.

Example 1. Let $\alpha = (7/2)$, $m = 4$, $\eta_1 = (1/3)$, $\eta_2 = (2/3)$, $\beta_1 = (3/2)$, $\beta_2 = 4$, $\gamma_1 = (5/2)$, and $\gamma_2 = 2$. We consider the following BVP:

$$\begin{aligned} D_{0+}^{(7/2)} x(t) + a(t) x^{(1/8)} + b(t) y^{-(1/5)} = 0, \quad 0 < t < 1, \\ x'(0) = x''(0) = 0, \\ x(1) = \frac{3}{2}\int_0^{(1/3)} x(s)\mathrm{d}s \\ + 4\int_0^{(2/3)} x(s)\mathrm{d}s \\ + \frac{5}{2}x\left(\frac{1}{3}\right) + 2x\left(\frac{2}{3}\right), \end{aligned} \tag{35}$$

where $f(t,x,y) = a(t)x^{(1/8)} + b(t)y^{-(1/5)}$. For any $k \in (0,1)$, take $\mu = (1/4)$, and it is easy to verify that

$$f\left(t, kx, k^{-1}y\right) \ge k^{\mu} f(t,x,y). \tag{36}$$

According to the expression of f and the above inequality, it follows that (H_1) and (H_2) are held. In addition,

$$0 < \int_0^1 (1-t)^{(3/2)} f\left(t, t^{(5/2)}, t^{(5/2)}\right)\mathrm{d}t < \infty. \tag{37}$$

So, all of the assumptions of Theorem 1 are satisfied. As a result, BVP (35) has a unique positive solution x^* and for any initial value $x_0 \in P$, and the successive iterative sequence $\{x_n(t)\}$ is generated by

$$\begin{aligned} x_n(t) &= \int_0^1 G(t,s) f\left(s, x_{n-1}(s), x_{n-1}(s)\right)\mathrm{d}s \\ &\quad + \frac{t^{\alpha-1}}{\xi} \sum_{i=1}^{m-2} \int_0^1 \left[\beta_i H(\eta_i, s) + \gamma_i G(\eta_i, s)\right] f \\ &\quad \cdot \left(s, x_{n-1}(s), x_{n-1}(s)\right)\mathrm{d}s, \quad n = 1, 2, \ldots, \end{aligned} \tag{38}$$

and uniformly converges to the unique positive solution x^* on $(0,1)$. We also obtain the error estimation

$$\max\left\{\left|x_n(t) - x^*(t)\right|\right\} = o\left(1 - c^{(1/4)^n}\right), \tag{39}$$

where $c \in (0,1)$ is a constant and determined by the initial value x_0. Moreover, for any $t \in (0,1)$, there exists a constant $l \in (0,1)$ which satisfies that

$$l t^{(5/2)} \le x^*(t) \le l^{-1} t^{(5/2)}. \tag{40}$$

Remark 1. According to the definition of $f(t,x,y)$, let $f(t,x,x) = a(t)x^{(1/8)} + b(t)x^{-(1/5)}$; then, f does not have monotonicity with respect to x. Thus, the iterative process in some previous work such as [31, 33, 35] cannot be performed, which implies our developed iterative technique in this paper can be suitable for a wider range of functions; in particular, even if $f(t,x,y)$ is reduced to only have one space variable $f(t,x)$, our results is more general than those of [96].

5. Result and Discussion

In this paper, we obtain the results of the existence solutions of the n-order fractional equation involving mixed integral and multipoint boundary conditions by using a new iterative algorithm. The efficiency of those methods depends essentially on the monotonicity and compactness of the operator. Different from [96], the iterative process does not need to start with the fixed upper and lower solution. We can not only construct a new iterative process which can perform from any initial value but also obtain the uniform convergence of the iterative sequences; at the same time, the estimation of the convergence rate and the approximation error are also given, which imply that the more strong results are established under the relatively weaker condition than that of [96].

Data Availability

The calculating data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration with all authors. All authors read and approved the final manuscript.

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