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Research Article

Fractional Hermite–Jensen–Mercer Integral Inequalities with respect to Another Function and Application

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In this paper, authors prove new variants of Hermite–Jensen–Mercer type inequalities using ψ–Riemann–Liouville fractional integrals with respect to another function via convexity. We establish generalized identities involving ψ–Riemann–Liouville fractional integral pertaining first and twice differentiable convex function λ, and these will be used to derive novel estimates for some fractional Hermite–Jensen–Mercer type inequalities. Some known results are recaptured from our results as special cases. Finally, an application from our results using the modified Bessel function of the first kind is established as well.

1. Introduction and Preliminaries

The theory of fractional integrals and derivatives has occurred in many fields and directions such as partial differential equations, difference equations, probability, and stochastic processes (see [1–6]). Behind it, the theory of convex functions with integral inequalities is also useful.

Definition 1. A function $\lambda\colon J \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be convex on J if

$$\lambda((1-\zeta)y_1 + \zeta y_2) \leq (1-\zeta)\lambda(y_1) + \zeta\lambda(y_2), \tag{1}$$

holds for every $y_1, y_2 \in J$ and $\zeta \in [0,1]$.

One of the best-known inequalities for convex functions is the following Hermite–Hadamard's inequality: if $\lambda\colon J \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ is a convex function in J, where $y_1, y_2 \in J$ and $y_1 < y_2$, then

$$\lambda\left(\frac{y_1+y_2}{2}\right) \leq \frac{1}{y_2-y_1}\int_{y_1}^{y_2}\lambda(\zeta)d\zeta \leq \frac{\lambda(y_1)+\lambda(y_2)}{2} \quad (\mathrm{H-H}). \tag{2}$$

It is also known as classical $(\mathrm{H-H})$ inequality. A number of mathematicians in the field of applied and pure mathematics have dedicated their efforts to extend, generalize, counterpart, and refine Hermite–Hadamard's inequality $(\mathrm{H-H})$ for different classes of convex functions. For more recent results obtained on inequality $(\mathrm{H-H})$, we refer the reader to references [7–10].

Let $\sigma_i \in [0,1]$ be nonnegative weights such that $\sum_{i=1}^{n}\sigma_i = 1$. The Jensen inequality states that, if λ is convex function on $[y_1, y_2]$, then

$$\lambda\left(\sum_{i=1}^{n}\sigma_i\ell_i\right) \leq \sum_{i=1}^{n}\sigma_i\lambda(\ell_i), \tag{3}$$

holds for all $\ell_i \in [y_1, y_2]$ and all $i = 1, 2, \ldots, n$, see [11].

In the literature, Jensen's inequality and Hermite–Hadamard's inequality are highly familiar results pertaining convex functions. One of the well-known and most significant inequalities in mathematical analysis is Jensen's and related inequalities. Jensen's inequality for differentiable convex functions plays a significant role in the field of inequalities as several other inequalities can be seen as special cases of it. It is used in order to make claims regarding the function while just a little is known or is needed to be known about the distribution. Furthermore, this inequality has been used in various areas of sciences and technology to solve several problems, such as engineering, mathematical statistics, financial economics, and computer science. Some recent results can be seen in [12–14].

Jensen's inequality has a following variant gave by Mercer (see [15]).

Theorem 1. *Let* λ *be a convex function on* $[y_1, y_2]$, *then*

$$\lambda\left(y_1 + y_2 - \sum_{i=1}^{n} \sigma_i \ell_i\right) \leq \lambda(y_1) + \lambda(y_2) - \sum_{i=1}^{n} \sigma_i \lambda(\ell_i), \quad (4)$$

holds for all $\ell_i \in [y_1, y_2]$ *and all* $\sigma_i \in [0,1]$, $(i = 1, 2, \ldots, n)$.

Jensen–Mercer's type inequality is a topic of supreme interest as it gives more information with explicit boundary conditions. It is quite effective for applications in operator analysis in higher dimensions [16–18]. Moradi et al. established some new improvements and generalization of Jensen–Mercer's type inequalities [19]. Recently, in [20], Adil et al. gave applications of Jensen–Mercer's inequality in information theory. They computed new estimates for Csiszár and related divergences. Taking into consideration the wonderful packages of Jensen's and associated inequalities in diverse fields of mathematics and engineering sciences, their generalizations and upgrades were a subject of an excellent hobby for the researchers in the last few years as obvious from a massive variety of investigation on it (see [21–24]).

In [25], Vanterler da Costa Sousa and Capelas de Oliveira introduced ψ–fractional integrals and ψ–Hilfer fractional derivative with respect to another function. They also studied Gronwall inequalities using ψ–Hilfer operator (see [26]).

Definition 2 (see [23]). Suppose that (y_1, y_2) $(-\infty \leq y_1 < y_2 \leq \infty)$ and $\alpha > 0$. Also let ψ be an increasing and positive monotone function on $(y_1, y_2]$, having a continuous derivative ψ' on (y_1, y_2). Then, the left-sided and right-sided ψ–Riemann–Liouville fractional integrals of a function λ with respect to another function ψ on $[y_1, y_2]$ are defined as follows:

$$\begin{aligned}
\left(I_{y_1+}^{\alpha:\psi}\right)\lambda(\ell) &= \frac{1}{\Gamma(\alpha)} \int_{y_1}^{\ell} \psi'(\zeta)(\psi(\ell) - \psi(\zeta))^{\alpha-1} \lambda(\zeta)\mathrm{d}\zeta, \quad y_1 < \ell, \\
\left(I_{y_2-}^{\alpha:\psi}\right)\lambda(\ell) &= \frac{1}{\Gamma(\alpha)} \int_{\ell}^{y_2} \psi'(\zeta)(\psi(\zeta) - \psi(\ell))^{\alpha-1} \lambda(\zeta)\mathrm{d}\zeta, \quad \ell < y_2,
\end{aligned} \quad (5)$$

respectively.

If we choose $\psi(\zeta) = \zeta$ and $\psi(\zeta) = \ln\zeta$, then we get, respectively, Riemann–Liouville and Hadamard fractional integrals.

Motivated by previous results, we will establish several new Hermite–Hadamard–Mercer type inequalities involving ψ–Riemann–Liouville fractional integrals (i.e., Riemann–Liouville fractional integral of any function with respect to another function). Moreover, our results recover several known results. Finally, an application using the modified Bessel function of the first kind will be established as well.

2. Hermite–Jensen–Mercer Type Inequalities

Throughout the paper, the following assumption will be used in the sequel.

(A_1): Let $0 \leq y_1 < y_2$, λ: $[y_1, y_2] \longrightarrow \mathfrak{R}$ be a positive function and $\lambda \in L_1[y_1, y_2]$. Also suppose, $\psi(\cdot)$ is an increasing and positive monotone function on $(y_1, y_2]$, having a continuous derivative ψ' on (y_1, y_2) and $\alpha > 0$.

Theorem 2. *If* (A_1) *is satisfied and* λ *is a convex function on* $[y_1, y_2]$, *then*

$$\begin{aligned}
\lambda\left(y_1 + y_2 - \frac{\ell_1}{+}\ell_2 2\right) &\leq [\lambda(y_1) + \lambda(y_2)] - \frac{\Gamma(\alpha+1)}{2(\ell_2 - \ell_1)^{\alpha}} \\
&\quad \times \left\{\left(I_{\psi^{-1}(\ell_1)^{+}}^{\alpha:\psi}\right)(\lambda \circ \psi)\left(\psi^{-1}(\ell_2)\right) + \left(I_{\psi^{-1}(\ell_2)^{-}}^{\alpha:\psi}\right)(\lambda \circ \psi)\left(\psi^{-1}(\ell_1)\right)\right\} \\
&\leq [\lambda(y_1) + \lambda(y_2)] - \lambda\left(\frac{\ell_1 + \ell_2}{2}\right),
\end{aligned} \quad (6)$$

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\leq\lambda(y_1)+\lambda(y_2)-\left(\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}\right), \tag{7}$$

for all $\ell_1,\ell_2\in[y_1,y_2]$, *where* $\Gamma(\cdot)$ *is the gamma function.*

Proof. Using Jensen–Mercer's inequality, we have

$$\lambda\left(y_1+y_2-\frac{x+z}{2}\right)\leq\lambda(y_1)+\lambda(y_2)-\frac{\lambda(x)+\lambda(z)}{2}, \tag{8}$$

for all $x,z\in[y_1,y_2]$.

Now, by change of variables $x=\zeta\ell_1+(1-\zeta)\ell_2$ and $z=(1-\zeta)\ell_1+\zeta\ell_2$, for all $\ell_1,\ell_2\in[y_1,y_2]$ and $\zeta\in[0,1]$ in (8), we get

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\lambda(y_1)+\lambda(y_2)-\frac{\lambda(\zeta\ell_1+(1-\zeta)\ell_2)+\lambda((1-\zeta)\ell_1+\zeta\ell_2)}{2}. \tag{9}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ on $[0,1]$, we obtain

$$\frac{1}{\alpha}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{1}{\alpha}\{\lambda(y_1)+\lambda(y_2)\}-\frac{1}{2}\left\{\int_0^1\zeta^{\alpha-1}(\lambda(\zeta\ell_1+(1-\zeta)\ell_2)+\lambda((1-\zeta)\ell_1+\zeta\ell_2))\mathrm{d}\zeta\right\}, \tag{10}$$

where

$$\frac{\alpha}{2}\left\{\int_0^1\zeta^{\alpha-1}(\lambda(\zeta\ell_1+(1-\zeta)y_2)+\lambda((1-\zeta)y_1+\zeta y_2))\mathrm{d}\zeta\right\}=\frac{\alpha}{2}\int_0^1\zeta^{\alpha-1}\lambda(\zeta\ell_1+(1-\zeta)\ell_2)\mathrm{d}\zeta+\frac{\alpha}{2}\int_0^1\zeta^{\alpha-1}\lambda((1-\zeta)\ell_1+\zeta\ell_2)\mathrm{d}\zeta. \tag{11}$$

Now, let $\zeta=(\psi(\gamma)-\ell_1)/(\ell_2-\ell_1)$, then $\mathrm{d}\zeta=(\psi'(\gamma)\mathrm{d}\gamma)/(\ell_2-\ell_1)$. Using the above equality, we obtain

$$
\begin{aligned}
&= \frac{\alpha}{2}\int_{\psi^{-1}(\ell_1)}^{\psi^{-1}(\ell_2)}\left(\frac{\ell_2-\psi(\gamma)}{\ell_2-\ell_1}\right)^{\alpha-1}\lambda(\psi(\gamma))\frac{\psi'(\gamma)}{\ell_2-\ell_1}\mathrm{d}\gamma \\
&+\frac{\alpha}{2}\int_{\psi^{-1}(\ell_1)}^{\Psi^{-1}(\ell_2)}\left(\frac{\psi(\gamma)-\ell_1}{\ell_2-\ell_1}\right)^{\alpha-1}\lambda(\psi(\gamma))\frac{\psi'(\gamma)}{\ell_2-\ell_1}\mathrm{d}\gamma \\
&= \frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\frac{1}{\Gamma(\alpha)}\left\{\int_{\psi^{-1}(\ell_1)}^{\psi^{-1}(\ell_2)}\psi'(\gamma)(\ell_2-\psi(\gamma))^{\alpha-1}(\lambda\circ\psi)(\gamma)\mathrm{d}\gamma\right\} \\
&+\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\frac{1}{\Gamma(\alpha)}\left\{\int_{\psi^{-1}(\ell_1)}^{\psi^{-1}(\ell_2)}\psi'(\gamma)(\psi(\gamma)-\ell_1)^{\alpha-1}(\lambda\circ\psi)(\gamma)\mathrm{d}\gamma\right\}.
\end{aligned}
\tag{12}
$$

So, the final form will be of this type as follows:

$$
\begin{aligned}
&= \frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}} \\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_1)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_2)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_2)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_1)\right)\right\},
\end{aligned}
\tag{13}
$$

and so the first inequality of (6) is proved.

Regarding the second inequality of (6), since λ is convex function, then for $\zeta\in[0,1]$, we have

$$
\begin{aligned}
\lambda\left(\frac{\ell_1+\ell_2}{2}\right) &= \lambda\left(\frac{\zeta\ell_1+(1-\zeta)\ell_2+(1-\zeta)\ell_1+\zeta\ell_2}{2}\right) \\
&\leq \frac{\lambda(\zeta\ell_1+(1-\zeta)\ell_2)+\lambda((1-\zeta)\ell_1+\zeta\ell_2)}{2}.
\end{aligned}
\tag{14}
$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ on $[0,1]$, we get

$$
\frac{1}{\alpha}\lambda\left(\frac{\ell_1+\ell_2}{2}\right)\leq\frac{1}{2}\left\{\int_0^1\zeta^{\alpha-1}\left(\lambda(\zeta\ell_1+(1-\zeta)\ell_2)+\lambda((1-\zeta)\ell_1+\zeta\ell_2)\right)\right\}.
\tag{15}
$$

Let $\psi(\gamma)=\zeta\ell_1+(1-\zeta)\ell_2$ and $\psi(\beta)=(1-\zeta)\ell_1+\zeta\ell_2$. Then, we have

$$
\lambda\left(\frac{\ell_1+\ell_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_1)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_2)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_2)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_1)\right)\right\}.
\tag{16}
$$

Multiplying by (-1), we will get

$$
-\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_1)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_2)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(\ell_2)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(\ell_1)\right)\right\}\leq-\lambda\left(\frac{\ell_1+\ell_2}{2}\right).
\tag{17}
$$

Adding $\lambda(y_1)+\lambda(y_2)$ both sides in (17), we obtain our second inequality of (6).

To prove the first inequality of (7) by using the convexity of λ, we have

$$2\lambda\left(y_1 + y_2 - \frac{x+z}{2}\right) \leq \lambda(y_1 + y_2 - x) + \lambda(y_1 + y_2 - z), \tag{18}$$

for all $x, z \in [y_1, y_2]$. By change of variables $x = \zeta\ell_1 + (1-\zeta)\ell_2$ and $z = (1-\zeta)\ell_1 + \zeta\ell_2$, $\zeta \in [0,1]$, we get

$$2\lambda\left(y_1 + y_2 - \frac{\ell_1+\ell_2}{2}\right) \leq \lambda(y_1 + y_2 - (\zeta\ell_1 + (1-\zeta)\ell_2)) + \lambda(y_1 + y_2 - ((1-\zeta)\ell_1 + \zeta\ell_2)). \tag{19}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ over $[0,1]$, we get

$$\frac{2}{\alpha}\lambda\left(y_1 + y_2 - \frac{\ell_1+\ell_2}{2}\right) \leq \int_0^1 \zeta^{\alpha-1}(\lambda(y_1 + y_2 - (\zeta\ell_1 + (1-\zeta)\ell_2)) + \lambda(y_1 + y_2 - ((1-\zeta)\ell_1 + \zeta\ell_2)))\mathrm{d}\zeta. \tag{20}$$

Hence, by change of variables, we obtain

$$\lambda\left(y_1 + y_2 - \frac{\ell_1+\ell_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha} \times \left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\Psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right) + \left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}, \tag{21}$$

and so the first inequality of (7) is proved.

About the second inequality of (7), since λ is convex function, then for $\zeta \in [0,1]$, we obtain

$$\lambda(y_1 + y_2 - (\zeta\ell_1 + (1-\zeta)\ell_2)) \leq \lambda(y_1) + \lambda(y_2) - [\zeta\lambda(\ell_1) + (1-\zeta)\lambda(\ell_2)], \tag{22}$$

$$\lambda(y_1 + y_2 - ((1-\zeta)\ell_1 + \zeta\ell_2)) \leq \lambda(y_1) + \lambda(y_2) - [(1-\zeta)\lambda(\ell_1) + \zeta\lambda(\ell_2)]. \tag{23}$$

By adding inequalities (22) and (23), we have

$$\lambda(y_1 + y_2 - (\zeta\ell_1 + (1-\zeta)\ell_2)) + \lambda(y_1 + y_2 - ((1-\zeta)\ell_1 + \zeta\ell_2)) \leq 2(\lambda(y_1) + \lambda(y_2)) - (\lambda(\ell_1) + \lambda(\ell_2)). \tag{24}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ on $[0,1]$, we get

$$\frac{2^\alpha\Gamma(\alpha)}{(\ell_2-\ell_1)^\alpha}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right) + \left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\} \leq \{2(\lambda(y_1) + \lambda(y_2)) - (\lambda(\ell_1) + \lambda(\ell_2))\} \cdot \frac{1}{\alpha}. \tag{25}$$

Multiplying by $\alpha/2$, we will get

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}$$
$$\leq(\lambda(y_1)+\lambda(y_2))-\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}. \tag{26}$$

From inequalities (21) and (26), we get the desired double inequality (7). □

Remark 1. Taking $\psi(\gamma)=\gamma$ in Theorem 2, we will get Theorem 2 proved in [27].

Remark 2. Taking $\psi(\gamma)=\gamma$ and $\alpha=1$ in Theorem 2, we will obtain Theorem 2 proved by Kian and Moslehian in [28].

Theorem 3. *If (A_1) is satisfied and λ is a convex function on $[y_1,y_2]$, then*

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}$$
$$\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.$$
$$\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}$$
$$\leq\lambda(y_1)+\lambda(y_2)-\left(\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}\right), \tag{27}$$

for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. About the first inequality (27) by using the convexity of λ, we have

$$2\lambda\left(y_1+y_2-\frac{x+z}{2}\right)\leq\lambda(y_1+y_2-x)+\lambda(y_1+y_2-z), \tag{28}$$

for all $x,z\in[y_1,y_2]$. By change of variables $x=(\zeta/2)\ell_1+((2-\zeta)/2)\ell_2$ and $z=((2-\zeta)/2)\ell_1+(\zeta/2)\ell_2$, $\zeta\in[0,1]$, we get

$$2\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)$$
$$+\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right). \tag{29}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ over $[0,1]$, we have

$$\frac{2}{\alpha}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$\leq\int_0^1\zeta^{\alpha-1}\left(\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\right)d\zeta. \tag{30}$$

Hence, by change of variables, we obtain

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}$$
$$\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\Psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}, \tag{31}$$

which proved the first inequality of (27).

Regarding the second inequality of (27), since λ is convex function, then for $\zeta\in[0,1]$, we have

$$\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)\leq\lambda(y_1)+\lambda(y_2)-\left[\frac{\zeta}{2}\lambda(\ell_1)+\frac{2-\zeta}{2}\lambda(\ell_2)\right], \tag{32}$$

$$\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\leq\lambda(y_1)+\lambda(y_2)-\left[\frac{2-\zeta}{2}\lambda(\ell_1)+\frac{\zeta}{2}\lambda(\ell_2)\right]. \tag{33}$$

By adding inequalities (32) and (33), we get

$$\begin{aligned}&\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\\&\leq 2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)).\end{aligned} \tag{34}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ on $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\zeta^{\alpha-1}\left(\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\right)\mathrm{d}\zeta\\&\leq(2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)))\int_0^1\zeta^{\alpha-1}\mathrm{d}\zeta.\end{aligned} \tag{35}$$

Then, we have the following inequality:

$$\begin{aligned}&\frac{2^{\alpha}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\\&\leq(2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)))\cdot\frac{1}{\alpha}.\end{aligned} \tag{36}$$

Multiplying by $\alpha/2$, we will get

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\\&\leq(\lambda(y_1)+\lambda(y_2))-\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}.\end{aligned} \tag{37}$$

From inequalities (31) and (37), we get the desired double inequality (27). □

Remark 3. Taking $\psi(\gamma)=\gamma$ in Theorem 3, we will get Theorem 3 proved in [27].

Remark 4. Taking $\psi(\gamma)=\gamma$ and $\alpha=1$ in Theorem 3, we will obtain Theorem 2.1 proved by Kian and Moslehian in [28].

Theorem 4. *If (A_1) is satisfied and λ is a convex function on $[y_1,y_2]$, then*

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right\}\leq\lambda(y_1)+\lambda(y_2)-\left(\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}\right), \tag{38}$$

for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. Regarding the first part of inequality (38) by using the convexity of λ, we have

$$2\lambda\left(y_1+y_2-\frac{x+z}{2}\right)\leq\lambda(y_1+y_2-x)+\lambda(y_1+y_2-z), \tag{39}$$

for all $x,z\in[y_1,y_2]$. By change of variables $x=((1+\zeta)/2)\ell_1+((1-\zeta)/2)\ell_2$ and $z=((1-\zeta)/2)\ell_1+((1+\zeta)/2)\ell_2$, $\zeta\in[0,1]$, we get

$$2\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right). \tag{40}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to $\zeta\in[0,1]$, we obtain

$$\frac{2}{\alpha}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\int_0^1\zeta^{\alpha-1}\left(\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right)\mathrm{d}\zeta. \tag{41}$$

Hence, by change of variables, we have

$$\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right\}, \tag{42}$$

which concludes the first inequality of (38).

About the second inequality of (38), since λ is convex function, then for $\zeta\in[0,1]$, we get

$$\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\leq\lambda(y_1)+\lambda(y_2)-\left[\frac{1+\zeta}{2}\lambda(\ell_1)+\frac{1-\zeta}{2}\lambda(\ell_2)\right], \tag{43}$$

$$\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\leq\lambda(y_1)+\lambda(y_2)-\left[\frac{1-\zeta}{2}\lambda(\ell_1)+\frac{1+\zeta}{2}\lambda(\ell_2)\right]. \tag{44}$$

By adding inequalities (43) and (44), we have

$$\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)$$
$$\leq 2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)). \tag{45}$$

Multiplying the above inequality by $\zeta^{\alpha-1}$ on both sides and integrating with respect to ζ over $[0,1]$, we obtain

$$\int_0^1 \zeta^{\alpha-1}\left(\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)+\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right)d\zeta$$
$$\leq (2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)))\int_0^1 \zeta^{\alpha-1}d\zeta. \tag{46}$$

Then, we have the following inequality:

$$\frac{2^{\alpha}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha}}$$
$$\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right\}$$
$$\leq (2(\lambda(y_1)+\lambda(y_2))-(\lambda(\ell_1)+\lambda(\ell_2)))\cdot\frac{1}{\alpha}. \tag{47}$$

Multiplying by $\alpha/2$, we will get

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^+}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^-}\right)(\lambda\circ\psi)\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right\}$$
$$\leq (\lambda(y_1)+\lambda(y_2))-\frac{\lambda(\ell_1)+\lambda(\ell_2)}{2}. \tag{48}$$

So, the second inequality of (38) holds. □

Remark 5. Taking $\psi(\gamma)=\gamma$ in Theorem 4, we will get Theorem 2 proved in [29].

Remark 6. Taking $\psi(\gamma)=\gamma$ and $\alpha=1$ in Theorem 4, we will obtain Theorem 2.1 proved by Kian and Moslehian in [28].

3. New Generalized Identities and Their Integral Inequalities

In this section, the following lemmas will play a basic role in our next results.

Lemma 1. *If* (A_1) *is satisfied and* $\lambda\colon [y_1,y_2]\longrightarrow\mathfrak{R}$ *is a differentiable function on* $L_1[y_1,y_2]$, *then*

$$
\begin{aligned}
&\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\\
&=\frac{1}{2(\ell_2-\ell_1)^{\alpha}}\\
&\times\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\left((\psi(\gamma)-(y_1+y_2-\ell_2))^{\alpha}-((y_1+y_2-\ell_1)-\psi(\gamma))^{\alpha}\right)\times(\lambda'\circ\psi)(\gamma)\psi'(\gamma)\mathrm{d}\gamma,
\end{aligned}
\tag{49}
$$

for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. It suffices to note that

$$
I=\frac{\lambda(y_1+y_2-\ell_1)-\lambda(y_1+y_2-\ell_2)}{2}-\{I_1+I_2\},\tag{50}
$$

where

$$
\begin{aligned}
I_1&=\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left[I^{\alpha:\psi}_{\Psi^{-1}(y_1+y_2-\ell_2)^{+}}(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right]\\
&=\frac{\alpha}{2(\ell_2-\ell_1)^{\alpha}}\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\psi'(\gamma)((y_1+y_2-\ell_1)-\psi(\gamma))^{\alpha-1}(\lambda\circ\psi)(\gamma)\mathrm{d}\gamma\\
&=\frac{-1}{2(\ell_2-\ell_1)^{\alpha}}\int_{\psi^{-1}(y_1+y_2-\ell_1)}^{\psi^{-1}(y_1+y_2-\ell_2)}\mathrm{d}((y_1+y_2-\ell_1)-\psi(\gamma))^{\alpha}(\lambda\circ\psi)(\gamma)\\
&=\frac{1}{2(\ell_2-\ell_1)^{\alpha}}[\lambda(y_1+y_2-\ell_2)(\ell_2-\ell_1)^{\alpha}]\\
&\quad+\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\psi'(\gamma)((y_1+y_2-\ell_1)-\psi(\gamma))^{\alpha}(\lambda'\circ\psi)(\gamma)\mathrm{d}\gamma,
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
I_2&=\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left[I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right]\\
&=\frac{\alpha}{2(\ell_2-\ell_1)^{\alpha}}\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\psi'(\gamma)(-(y_1+y_2-\ell_2)+\psi(\gamma))^{\alpha-1}(\lambda\circ\psi)(\gamma)\mathrm{d}\gamma\\
&=\frac{1}{2(\ell_2-\ell_1)^{\alpha}}\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\mathrm{d}(-(y_1+y_2-\ell_2)+\psi(\gamma))^{\alpha}(\lambda\circ\psi)(\gamma)\\
&=\frac{1}{2(\ell_2-\ell_1)^{\alpha}}[\lambda(y_1+y_2-\ell_1)(\ell_2-\ell_1)^{\alpha}]\\
&\quad-\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\psi'(\gamma)(-(y_1+y_2-\ell_2)+\psi(\gamma))^{\alpha}(\lambda'\circ\psi)(\gamma)\mathrm{d}\gamma.
\end{aligned}
\tag{52}
$$

Substituting (51) and (52) in (50), we get the desired equality (49). □

Remark 7. For $\ell_1=y_1$ and $\ell_2=y_2$ in Lemma 1, we will get Lemma 3.1 proved in [30].

Theorem 5. *If* (A_1) *is satisfied and* $|\lambda'|$ *is a convex function on* $[y_1, y_2]$*, then*

$$\left|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left(\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\lambda(y_1+y_2-\ell_1)+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\lambda(y_1+y_2-\ell_2)\right)\right| \leq \frac{(\ell_2-\ell_1)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{|\lambda'(\ell_1)|+|\lambda'(\ell_2)|}{2}\right)\right\}. \tag{53}$$

for all $\ell_1, \ell_2 \in [y_1, y_2]$.

Proof. Here, we will use Lemma 1, properties of modulus, and Jensen–Mercer's inequality.

For every $\gamma \in (\psi^{-1}(y_1+y_2-\ell_2), \psi^{-1}(y_1+y_2-\ell_1))$, we have $(y_1+y_2-\ell_2) < \psi(\gamma) < (y_1+y_2-\ell_1)$.

Let $\zeta = ((y_1+y_2-\ell_1)-\psi(\gamma))/(\ell_2-\ell_1)$, and then $\psi(\gamma) = y_1+y_2-(\zeta\ell_1+(1-\zeta)\ell_2)$. So, we get

$$\begin{aligned}&\left|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\left(\left(I^{\alpha:\psi}_{(y_1+y_2-\ell_2)^{+}}\right)\lambda(y_1+y_2-\ell_1)+\left(I^{\alpha:\psi}_{(y_1+y_2-\ell_1)^{-}}\right)\lambda(y_1+y_2-\ell_2)\right)\right|\\ &\leq \frac{1}{2(\ell_2-\ell_1)^{\alpha}}\int_{\psi^{-1}(y_1+y_2-\ell_2)}^{\psi^{-1}(y_1+y_2-\ell_1)}\left|(\psi(\gamma)-(y_1+y_2-\ell_2))^{\alpha}-((y_1+y_2-\ell_1)-\psi(\gamma))^{\alpha}\right|\times\left|(\lambda'\circ\psi)(\gamma)\right|\psi'(\gamma)\mathrm{d}\gamma\\ &=\frac{(\ell_2-\ell_1)}{2}\int_0^1\left|\zeta^{\alpha}-(1-\zeta)^{\alpha}\right|\left|\lambda'(y_1+y_2-(\zeta\ell_1+(1-\zeta)\ell_2))\right|\mathrm{d}\zeta\\ &\leq\frac{(\ell_2-\ell_1)}{2}\int_0^1\left|\zeta^{\alpha}-(1-\zeta)^{\alpha}\right|\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\zeta|\lambda'(\ell_1)|+(1-\zeta)|\lambda'(\ell_2)|\right)\right\}\mathrm{d}\zeta\\ &=\frac{(\ell_2-\ell_1)}{2}[I_1+I_2],\end{aligned} \tag{54}$$

where

$$\begin{aligned}I_1&=\int_0^{1/2}\left((1-\zeta)^{\alpha}-\zeta^{\alpha}\right)\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\zeta|\lambda'(\ell_1)|+(1-\zeta)|\lambda'(\ell_2)|\right)\right\}\mathrm{d}\zeta\\ &=\left(|\lambda'(y_1)|+|\lambda'(y_2)|\right)\left(\frac{1}{(\alpha+1)}-\frac{2^{-\alpha}}{(\alpha+1)}\right)\\ &\quad-\left\{|\lambda'(\ell_1)|\left(\frac{1}{(\alpha+1)(\alpha+2)}-\frac{2^{-\alpha-1}}{(\alpha+1)}\right)+|\lambda'(\ell_2)|\left(\frac{1}{(\alpha+2)}-\frac{2^{-\alpha-1}}{(\alpha+1)}\right)\right\},\end{aligned} \tag{55}$$

$$\begin{aligned}I_2&=\int_{1/2}^{1}\left(\zeta^{\alpha}-(1-\zeta)^{\alpha}\right)\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\zeta|\lambda'(\ell_1)|+(1-\zeta)|\lambda'(\ell_2)|\right)\right\}\mathrm{d}\zeta\\ &=\left(|\lambda'(y_1)|+|\lambda'(y_2)|\right)\left(\frac{1}{(\alpha+1)}-\frac{2^{-\alpha}}{(\alpha+1)}\right)\\ &\quad-\left\{|\lambda'(\ell_1)|\left(\frac{1}{(\alpha+2)}-\frac{2^{-\alpha-1}}{(\alpha+1)}\right)+|\lambda'(\ell_2)|\left(\frac{1}{(\alpha+1)(\alpha+2)}-\frac{2^{-\alpha-1}}{(\alpha+1)}\right)\right\}.\end{aligned} \tag{56}$$

Substituting (55) and (56) in (54), we get (53). □

Remark 8. For $\ell_1 = y_1$ and $\ell_2 = y_2$ in Theorem 5, we will get Theorem 3.4 proved in [30].

Remark 9. Taking $\psi(\gamma) = \gamma$ in Theorem 5, we will get Theorem 4 proved in [27].

Lemma 2. *If (A_1) is satisfied and λ: $[y_1, y_2] \longrightarrow \mathfrak{R}$ is a differentiable function on $L_1[y_1, y_2]$, then*

$$\begin{aligned}
&\frac{\lambda(y_1 + y_2 - \ell_1) + \lambda(y_1 + y_2 - \ell_2)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2 - \ell_1)^{\alpha}} \\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda \circ \psi\left(\psi^{-1}\left(y_1 + y_2 - \frac{\ell_1 + \ell_2}{2}\right)\right)\right]\right. \\
&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda \circ \psi\left(\psi^{-1}\left(y_1 + y_2 - \frac{\ell_1 + \ell_2}{2}\right)\right)\right]\right\} \\
&= \frac{(\ell_2 - \ell_1)}{4}\left[\int_0^1 \zeta^{\alpha}\lambda'\left(y_1 + y_2 - \left(\frac{1+\zeta}{2}\ell_1 + \frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right. \\
&\left.- \int_0^1 \zeta^{\alpha}\lambda'\left(y_1 + y_2 - \left(\frac{1-\zeta}{2}\ell_1 + \frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right].
\end{aligned} \tag{57}$$

Proof. It suffices to note that

$$I = \frac{(\ell_2 - \ell_1)}{4}\{I_1 - I_2\}, \tag{58}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \zeta^{\alpha}\lambda'\left(y_1 + y_2 - \left(\frac{1+\zeta}{2}\ell_1 + \frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta \\
&= \frac{2}{(\ell_2 - \ell_1)}\lambda(y_1 + y_2 - \ell_1) - \frac{2^{\alpha}}{\ell_2 - \ell_1}\int_0^1 \zeta^{\alpha-1}\lambda\left(y_1 + y_2 - \left(\frac{1+\zeta}{2}\ell_1 + \frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta \\
&= \frac{2}{(\ell_2 - \ell_1)}\lambda(y_1 + y_2 - \ell_1) \\
&\quad - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\ell_2 - \ell_1)^{\alpha+1}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda \circ \psi\left(\psi^{-1}\left(y_1 + y_2 - \frac{\ell_1 + \ell_2}{2}\right)\right)\right],
\end{aligned} \tag{59}$$

$$\begin{aligned}
I_2 &= \int_0^1 \zeta^{\alpha}\lambda\left(y_1 + y_2 - \left(\frac{1-\zeta}{2}\ell_1 + \frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta \\
&= -\frac{2}{(\ell_2 - \ell_1)}\lambda(y_1 + y_2 - \ell_2) - \frac{2^{\alpha}}{\ell_2 - \ell_1}\int_0^1 \zeta^{\alpha-1}\lambda\left(y_1 + y_2 - \left(\frac{1-\zeta}{2}\ell_1 + \frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta \\
&= -\frac{2}{(\ell_2 - \ell_1)}\lambda(y_1 + y_2 - \ell_2) \\
&\quad - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\ell_2 - \ell_1)^{\alpha+1}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda \circ \psi\left(\psi^{-1}\left(y_1 + y_2 - \frac{\ell_1 + \ell_2}{2}\right)\right)\right].
\end{aligned} \tag{60}$$

Substituting (59) and (60) in (58), we get (57). □

Remark 10. Taking $\psi(\gamma) = \gamma$ in Lemma 2, we will get Lemma 1 proved in [29].

Theorem 6. *If (A_1) is satisfied and λ' is a convex function on $[y_1, y_2]$, then*

$$\begin{aligned}&\Bigg|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\\&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right\}\Bigg|\\&\leq\frac{(\ell_2-\ell_1)}{4(\alpha+2)}\sup_{\xi\in[y_1,y_2]}\left|\lambda''(\xi)\right|,\end{aligned}\tag{61}$$

for all $\ell_1, \ell_2 \in [y_1, y_2]$.

Proof. From Lemma 2 and using mean value theorem for λ', we have

$$\begin{aligned}&\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\\&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right\}\\&=\frac{(\ell_2-\ell_1)^2}{4}\int_0^1\zeta^{\alpha+1}\lambda''(\xi)\mathrm{d}\zeta,\end{aligned}\tag{62}$$

where $\xi \in [y_1, y_2]$. This leads us to

$$\begin{aligned}&\Bigg|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\\&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right\}\Bigg|\\&\leq\frac{(\ell_2-\ell_1)^2}{4}\int_0^1\left|\zeta^{\alpha+1}\right|\left|\lambda''(\xi)\right|\mathrm{d}\zeta\\&\leq\frac{(\ell_2-\ell_1)^2}{4}\sup_{\xi\in[y_1,y_2]}\left|\lambda''(\xi)\right|\left\{\int_0^1\zeta^{\alpha+1}\mathrm{d}\zeta\right\}\\&=\frac{(\ell_2-\ell_1)^2}{4(\alpha+2)}\sup_{\xi\in[y_1,y_2]}\left|\lambda''(\xi)\right|.\end{aligned}\tag{63}$$

□

Remark 11. For $\psi(\gamma)=\gamma$ in Theorem 6, we will get Theorem 3 proved in [29].

Theorem 7. *If* (A_1) *is satisfied and* $|\lambda'|$ *is a convex function on* $[y_1,y_2]$, *then*

$$\begin{aligned}&\Bigg|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\\&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right\}\Bigg|\\&\leq\frac{(\ell_2-\ell_1)}{2(\alpha+1)}\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{|\lambda'(\ell_1)|+|\lambda'(\ell_2)|}{2}\right)\right\},\end{aligned}\tag{64}$$

for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. By using Lemma 2, properties of modulus, and Jensen–Mercer inequality, we have

$$\begin{aligned}&\Bigg|\frac{\lambda(y_1+y_2-\ell_1)+\lambda(y_1+y_2-\ell_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\\&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_2)^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-\ell_1)^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right)\right]\right\}\Bigg|\\&\leq\frac{(\ell_2-\ell_1)}{4}\left[\int_0^1\zeta^{\alpha}\left|\lambda'\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right.\\&\left.+\int_0^1\zeta^{\alpha}\left|\lambda'\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right]\\&\leq\frac{(\ell_2-\ell_1)}{4}\left[\int_0^1\zeta^{\alpha}\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{(1+\zeta)}{2}|\lambda'(\ell_1)|+\frac{(1-\zeta)}{2}|\lambda'(\ell_2)|\right)\right\}\mathrm{d}\zeta\right.\\&\left.+\int_0^1\zeta^{\alpha}\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{(1-\zeta)}{2}|\lambda'(\ell_1)|+\frac{(1+\zeta)}{2}|\lambda'(\ell_2)|\right)\right\}\mathrm{d}\zeta\right],\end{aligned}\tag{65}$$

and after integration, we get required result. □

Remark 12. For $\psi(\gamma)=\gamma$ in Theorem 7, we will get Theorem 4 proved in [29].

Lemma 3. *If (A_1) is satisfied and λ: $[y_1, y_2] \longrightarrow \Re$ is a differentiable function on $L_1[y_1, y_2]$, then*

$$\begin{aligned}
&\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha}\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.\\
&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\\
&=\frac{(\ell_2-\ell_1)}{4}\left[\int_0^1\zeta^\alpha\lambda'\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)d\zeta\right.\\
&\left.-\int_0^1\zeta^\alpha\lambda'\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)d\zeta\right].
\end{aligned}\tag{66}$$

Proof. See the proof of Lemma 2. □

Remark 13. For $\psi(\gamma)=\gamma$, $\ell_1=y_1$, and $\ell_2=y_1$ in Lemma 3, we will get Lemma 3 proved in [31].

Remark 14. For $\psi(\gamma)=\gamma$ in Lemma 3, we will get Lemma 2 proved in [27].

Theorem 8. *If (A_1) is satisfied and $|\lambda'|$ is a convex function on $[y_1, y_2]$, then*

$$\begin{aligned}
&\left|\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha}\right.\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.\\
&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\right|\\
&\leq\frac{(\ell_2-\ell_1)}{2(\alpha+1)}\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{|\lambda'(\ell_1)|+|\lambda'(\ell_2)|}{2}\right)\right\},
\end{aligned}\tag{67}$$

for all $\ell_1, \ell_2 \in [y_1, y_2]$.

Proof. By using Lemma 3, properties of modulus, and Jensen–Mercer inequality, we have

$$\begin{aligned}
&\left|\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha}\right.\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.\\
&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\right|\\
&\leq\frac{(\ell_2-\ell_1)}{4}\left[\int_0^1\zeta^\alpha\left|\lambda'\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)\right|d\zeta\right.\\
&\left.+\int_0^1\zeta^\alpha\left|\lambda'\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\right|d\zeta\right]\\
&\leq\frac{(\ell_2-\ell_1)}{4}\left[\int_0^1\zeta^\alpha\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{\zeta}{2}|\lambda'(\ell_1)|+\frac{(2-\zeta)}{2}|\lambda'(\ell_2)|\right)\right\}d\zeta\right.\\
&\left.+\int_0^1\zeta^\alpha\left\{|\lambda'(y_1)|+|\lambda'(y_2)|-\left(\frac{(2-\zeta)}{2}|\lambda'(\ell_1)|+\frac{\zeta}{2}|\lambda'(\ell_2)|\right)\right\}d\zeta\right].
\end{aligned}\tag{68}$$

After integration, we get required result. □

Remark 15. Taking $\psi(\gamma)=\gamma$ in Theorem 8, we will get Theorem 5 proved in [27].

Theorem 9. *If (A_1) is satisfied and $|\lambda'|^q$ is convex function, then*

$$\begin{aligned}
&\left|\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha}\right.\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.\\
&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\right|\\
&\leq\frac{(\ell_2-\ell_1)}{4}\left(\frac{1}{p\alpha+1}\right)^{1/p}\left[\left(|\lambda'(y_1)|^q+|\lambda'(y_2)|^q-\left(\frac{1}{4}|\lambda'(\ell_1)|^q+\frac{3}{4}|\lambda'(\ell_2)|^q\right)\right)^{1/q}\right.\\
&\left.+\left(|\lambda'(y_1)|^q+|\lambda'(y_2)|^q-\left(\frac{3}{4}|\lambda'(\ell_1)|^q+\frac{1}{4}|\lambda'(\ell_2)|^q\right)\right)^{1/q}\right],
\end{aligned}\tag{69}$$

where $q>1$ and $(1/p)+(1/q)=1$ for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. Applying Lemma 3, Hölder and Jensen–Mercer inequalities, the fact that $|\lambda'|^q$ is convex function, and properties of modulus, we have

$$\begin{aligned}
&\left|\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^\alpha}\right.\\
&\times\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right.\\
&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda^\circ\psi)\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right\}\right|\\
&\leq\frac{(\ell_2-\ell_1)}{4}\int_0^1\zeta^\alpha\left|\lambda'\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{(2-\zeta)}{2}\ell_2\right)\right)\right|d\zeta\\
&+\frac{(\ell_2-\ell_1)}{4}\int_0^1\zeta^\alpha\left|\lambda'\left(y_1+y_2-\left(\frac{(2-\zeta)}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\right|d\zeta\\
&\leq\frac{(\ell_2-\ell_1)}{4}\left(\int_0^1\zeta^{p\alpha}d\zeta\right)^{1/p}\left(\int_0^1\left|\lambda'\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{(2-\zeta)}{2}\ell_2\right)\right)\right|^q d\zeta\right)^{1/q}\\
&+\frac{(\ell_2-\ell_1)}{4}\left(\int_0^1\zeta^{p\alpha}d\zeta\right)^{1/p}\left(\int_0^1\left|\lambda'\left(y_1+y_2-\left(\frac{(2-\zeta)}{2}\ell_1+\frac{\zeta}{2}\lambda'\ell_2\right)\right)\right|^q d\zeta\right)^{1/q}\\
&\leq\frac{(\ell_2-\ell_1)}{4}\left(\frac{1}{p\alpha+1}\right)^{1/p}\left(|\lambda'(y_1)|^q+|\lambda'(y_2)|^q-\left(\frac{1}{4}|\lambda'(\ell_1)|^q+\frac{3}{4}|\lambda'(\ell_2)|^q\right)\right)^{1/q}\\
&+\frac{(\ell_2-\ell_1)}{4}\left(\frac{1}{p\alpha+1}\right)^{1/p}\left(|\lambda'(y_1)|^q+|\lambda'(y_2)|^q-\left(\frac{3}{4}|\lambda'(\ell_1)|^q+\frac{1}{4}|\lambda'(\ell_2)|^q\right)\right)^{1/q}.
\end{aligned}\tag{70}$$

After further simplifications, we get required result. □

Remark 16. For $\psi(\gamma)=\gamma$ in Theorem 9, we will get Theorem 6 proved in [27].

Lemma 4. *If* (A_1) *is satisfied and* $\lambda\colon [y_1, y_2] \longrightarrow \mathfrak{R}$ *is a twice differentiable function on* $L_1[y_1, y_2]$, *then*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1(1-\zeta)^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right.\\&\left.+\int_0^1(1-\zeta)^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right].\end{aligned}\tag{71}$$

Proof. It suffices to note that

$$I=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\{I_1+I_2\},\tag{72}$$

where

$$\begin{aligned}I_1&=\int_0^1(1-\zeta)^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\\&=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{2(\alpha+1)}{\ell_2-\ell_1}\int_0^1(1-\zeta)^{\alpha}\lambda'\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\\&=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{4\alpha(\alpha+1)}{(\ell_2-\ell_1)^2}\int_0^1(1-\zeta)^{\alpha-1}\lambda\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\\&=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{2^{\alpha+2}\Gamma(\alpha+2)}{(\ell_2-\ell_1)^{\alpha+2}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right],\\I_2&=\int_0^1(1-\zeta)^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\\&=\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad-\frac{2(\alpha+1)}{\ell_2-\ell_1}\int_0^1(1-\zeta)^{\alpha}\lambda'\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\end{aligned}\tag{73}$$

$$\begin{aligned}
&= \frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right) - \frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\
&\quad + \frac{4\alpha(\alpha+1)}{(\ell_2-\ell_1)^2}\int_0^1 (1-\zeta)^{\alpha-1}\lambda\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\\
&= \frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right) - \frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\
&\quad + \frac{2^{\alpha+2}\Gamma(\alpha+2)}{(\ell_2-\ell_1)^{\alpha+2}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right].
\end{aligned} \tag{74}$$

Substituting (73) and (74) in (72), we get (71). □

Corollary 1. *If we set $\ell_1 = y_1$ and $\ell_2 = y_2$, we get*

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_2-y_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right.\\
&\quad\left.+\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)\right\} - \lambda\left(\frac{y_1+y_2}{2}\right)\\
&= \frac{(y_2-y_1)^2}{8(\alpha+1)}\left[\int_0^1 (1-\zeta)^{\alpha+1}\lambda''\left(\frac{1+\zeta}{2}y_1+\frac{1-\zeta}{2}y_2\right)\mathrm{d}\zeta\right.\\
&\quad\left.+\int_0^1 (1-\zeta)^{\alpha+1}\lambda''\left(\frac{1-\zeta}{2}y_1+\frac{1+\zeta}{2}y_2\right)\mathrm{d}\zeta\right].
\end{aligned} \tag{75}$$

Remark 17. If we set $\psi(\gamma) = \gamma$ in Lemma 4, we get Lemma 2 of [29].

Moreover, if we set $\ell_1 = y_1$ and $\ell_2 = y_2$, we obtain Lemma 1 of [32].

Remark 18. For $\psi(\gamma) = \gamma$, $\alpha = 1$, $\ell_1 = y_1$, and $\ell_2 = y_2$ in Lemma 4, it reduces to Lemma 2 proved in [32].

Theorem 10. *If (A_1) is satisfied and $|\lambda''|$ is a convex function on $[y_1, y_2]$, then*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\
&\quad\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\} - \lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\le \frac{(\ell_2-\ell_1)^2}{4(\alpha+1)(\alpha+2)}\left\{|\lambda''(y_1)|+|\lambda''(y_2)|-\left(\frac{|\lambda''(\ell_1)|+|\lambda''(\ell_2)|}{2}\right)\right\}.
\end{aligned} \tag{76}$$

Proof. By using Lemma 4, properties of modulus, and Jensen–Mercer inequality, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1(1-\zeta)^{\alpha+1}\left|\lambda''\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right.\\&\left.+\int_0^1(1-\zeta)^{\alpha+1}\left|\lambda''\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right]\\&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1(1-\zeta)^{\alpha+1}\left\{|\lambda''(y_1)|+|\lambda''(y_2)|-\left(\frac{1+\zeta}{2}|\lambda''(\ell_1)|+\frac{1-\zeta}{2}|\lambda''(\ell_2)|\right)\right\}\mathrm{d}\zeta\right.\\&\left.+\int_0^1(1-\zeta)^{\alpha+1}\left\{|\lambda''(y_1)|+|\lambda''(y_2)|-\left(\frac{1-\zeta}{2}|\lambda''(\ell_1)|+\frac{1+\zeta}{2}|\lambda''(\ell_2)|\right)\right\}\mathrm{d}\zeta\right],\end{aligned}\tag{77}$$

and after integration, we get required result. □

Corollary 2. *If we set $\ell_1 = y_1$ and $\ell_2 = y_2$ in Theorem 10, we get*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_2-y_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right.\right.\\&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)\right\}-\lambda\left(\frac{y_1+y_2}{2}\right)\right|\\&\leq\frac{(y_2-y_1)^2}{4(\alpha+1)(\alpha+2)}\left\{\left(\frac{|\lambda''(y_1)|+|\lambda''(y_2)|}{2}\right)\right\}.\end{aligned}\tag{78}$$

Remark 19. If we set $\psi(\gamma)=\gamma$ in Theorem 10, we obtain Theorem 5 of [29].

Moreover, if we set $\ell_1 = y_1$ and $\ell_2 = y_2$, we get Theorem 5 of [32].

Corollary 3. *If we set $\psi(\gamma)=\gamma$, $\ell_1 = y_1$, $\ell_2 = y_2$, and $\alpha = 1$ in Theorem 10, we get Proposition 1 of [33]:*

$$\left|\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\lambda(x)\mathrm{d}x-\lambda\left(\frac{y_1+y_2}{2}\right)\right|\leq\frac{(y_2-y_1)^2}{24}\left\{\left(\frac{|\lambda''(y_1)|+|\lambda''(y_2)|}{2}\right)\right\}.\tag{79}$$

Theorem 11. *If (A_1) is satisfied and $|\lambda''|^q$ is convex function, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{1/p}\left[\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{|3\lambda''(\ell_1)|^q+|\lambda''(\ell_2)|^q}{4}\right)^{1/q}\right.\\&\left.+\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{|\lambda''(\ell_1)|^q+3|\lambda''(\ell_2)|^q}{4}\right)^{1/q}\right],\end{aligned}\tag{80}$$

where $q > 1$ *and* $(1/p) + (1/q) = 1$ *for all* $\ell_1, \ell_2 \in [y_1, y_2]$.

Proof. From Lemma 4, Hölder and Jensen–Mercer inequalities, the fact that $|\lambda''|^q$ is convex function, and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\
&\left.\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1(1-\zeta)^{\alpha+1}\left|\lambda''\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right.\\
&\left.+\int_0^1(1-\zeta)^{\alpha+1}\left|\lambda''\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right|\mathrm{d}\zeta\right]\\
&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\left(\int_0^1(1-\zeta)^{p(\alpha+1)}\mathrm{d}\zeta\right)^{1/p}\left(\int_0^1\left|\lambda''\left(y_1+y_2-\left(\frac{1+\zeta}{2}\ell_1+\frac{1-\zeta}{2}\ell_2\right)\right)\right|^q\mathrm{d}\zeta\right)^{1/q}\right.\\
&\left.+\left(\int_0^1(1-\zeta)^{p(\alpha+1)}\mathrm{d}\zeta\right)^{1/p}\left(\int_0^1\left|\lambda''\left(y_1+y_2-\left(\frac{1-\zeta}{2}\ell_1+\frac{1+\zeta}{2}\ell_2\right)\right)\right|^q\mathrm{d}\zeta\right)^{1/q}\right]\\
&\leq\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left(\int_0^1(1-\zeta)^{p(\alpha+1)}\mathrm{d}\zeta\right)^{1/p}\\
&\times\left[\left(\int_0^1\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{1+\zeta}{2}|\lambda''(\ell_1)|^q-\frac{1-\zeta}{2}|\lambda''(\ell_2)|^q\right)\mathrm{d}\zeta\right)^{1/q}\right.\\
&\left.+\left(\int_0^1\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{1-\zeta}{2}|\lambda''(\ell_1)|^q-\frac{1+\zeta}{2}|\lambda''(\ell_2)|^q\right)\mathrm{d}\zeta\right)^{1/q}\right]\\
&=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{1/p}\left[\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{|3\lambda''(\ell_1)|^q+|\lambda''(\ell_2)|^q}{4}\right)^{1/q}\right.\\
&\left.+\left(\left||\lambda''(y_1)|^q+|\lambda''(y_2)|^q-\frac{|\lambda''(\ell_1)|^q+3|\lambda''(\ell_2)|^q}{4}\right|^q\right)\right)^{1/q}\Bigg].
\end{aligned}\tag{81}$$

□

Remark 20. If we set $\psi(\gamma) = \gamma$ in Theorem 11, we get Theorem 6 of [29].

Lemma 5. *If* (A_1) *satisfied and* $\lambda\colon [y_1, y_2] \longrightarrow \mathfrak{R}$ *is a twice differentiable function on* $L_1[y_1, y_2]$, *then*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.$$
$$\left.+\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)d\zeta\right.$$
$$\left.+\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)d\zeta\right]. \tag{82}$$

Proof. It suffices to note that

$$I=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\{I_1+I_2\}, \tag{83}$$

where

$$I_1=\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$+\frac{2(\alpha+1)}{\ell_2-\ell_1}\int_0^1\zeta^{\alpha}\lambda'\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$+\frac{4\alpha(\alpha+1)}{(\ell_2-\ell_1)^2}\int_0^1\zeta^{\alpha-1}\lambda\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=-\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$+\frac{2^{\alpha+2}\Gamma(\alpha+2)}{(\ell_2-\ell_1)^{\alpha+2}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right], \tag{84}$$

$$I_2=\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$-\frac{2(\alpha+1)}{\ell_2-\ell_1}\int_0^1\zeta^{\alpha}\lambda'\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$+\frac{4\alpha(\alpha+1)}{(\ell_2-\ell_1)^2}\int_0^1\zeta^{\alpha-1}\lambda\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)d\zeta$$
$$=\frac{2}{(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{4(\alpha+1)}{(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)$$
$$+\frac{2^{\alpha+2}\Gamma(\alpha+2)}{(\ell_2-\ell_1)^{\alpha+2}}\left(I^{\alpha:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right]. \tag{85}$$

Substituting (84) and (85) in (83), we get (82). □

Corollary 4. *If we set $\ell_1 = y_1$ and $\ell_2 = y_2$, we get*

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_2-y_1)^{\alpha}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right.\\
&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)\right\}-\lambda\left(\frac{y_1+y_2}{2}\right)\\
&=\frac{(y_2-y_1)^2}{8(\alpha+1)}\left[\int_0^1\zeta^{\alpha+1}\lambda''\left(\frac{2-\zeta}{2}y_1+\frac{\zeta}{2}y_2\right)\mathrm{d}\zeta+\int_0^1\zeta^{\alpha+1}\lambda''\left(\frac{\zeta}{2}y_1+\frac{2-\zeta}{2}y_2\right)\mathrm{d}\zeta\right].
\end{aligned}\tag{86}$$

Corollary 5. *If we set $\psi(\gamma) = \gamma$, we get*

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha}}\left\{\left(I^{\alpha}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)(\lambda(y_1+y_2-\ell_1))\right.\\
&\left.+\left(I^{\alpha}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)(\lambda(y_1+y_2-\ell_2))\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\
&=\frac{(\ell_2-\ell_1)^2}{8(\alpha+1)}\left[\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{2-\zeta}{2}\ell_1+\frac{\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right.\\
&\left.+\int_0^1\zeta^{\alpha+1}\lambda''\left(y_1+y_2-\left(\frac{\zeta}{2}\ell_1+\frac{2-\zeta}{2}\ell_2\right)\right)\mathrm{d}\zeta\right].
\end{aligned}\tag{87}$$

Moreover, if we set $\ell_1 = y_1$ and $\ell_2 = y_2$, we get

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_2-y_1)^{\alpha}}\left\{I^{\alpha}_{((y_1+y_2)/2)^{+}}\lambda(y_2)+I^{\alpha}_{((y_1+y_2)/2)^{-}}\lambda(y_1)\right\}-\lambda\left(\frac{y_1+y_2}{2}\right)\\
&=\frac{(y_2-y_1)^2}{8(\alpha+1)}\left[\int_0^1\zeta^{\alpha+1}\lambda''\left(\frac{2-\zeta}{2}y_1+\frac{\zeta}{2}y_2\right)\mathrm{d}\zeta+\int_0^1\zeta^{\alpha+1}\lambda''\left(\frac{\zeta}{2}y_1+\frac{2-\zeta}{2}y_2\right)\mathrm{d}\zeta\right].
\end{aligned}\tag{88}$$

Remark 21. By using Lemma 5, we can get the same results of Theorems 10 and 11, so we omit their proof here.

Lemma 6. *If (A_1) is satisfied and λ: $[y_1, y_2] \longrightarrow \Re$ is a twice differentiable function on $L_1[y_1, y_2]$, then*

$$\begin{aligned}
&\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left\{\left(I^{\alpha-1:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\\
&\left.+\left(I^{\alpha-1:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\
&=\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\right.\\
&\left.+\int_{1/2}^1(1-\zeta)^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\right].
\end{aligned}\tag{89}$$

Proof. It suffices to note that

$$I=\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{1-\alpha}}\{I_1+I_2\}, \tag{90}$$

where

$$\begin{aligned}I_1&=\int_0^{1/2}\zeta^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\\&=-\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{\alpha}{\ell_2-\ell_1}\int_0^{1/2}\zeta^{\alpha-1}\lambda'(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\mathrm{d}\zeta\\&=-\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{\alpha}{2^{\alpha-1}(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{\alpha(\alpha-1)}{(\ell_2-\ell_1)^2}\int_0^{1/2}\zeta^{\alpha-2}\lambda(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\mathrm{d}\zeta\\&=-\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{\alpha}{2^{\alpha-1}(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha+1}}\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right],\end{aligned} \tag{91}$$

$$\begin{aligned}I_2&=\int_{1/2}^{1}(1-\zeta)^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\\&=\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad-\frac{\alpha}{\ell_2-\ell_1}\int_{1/2}^{1}(1-\zeta)^{\alpha-1}\lambda'(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\mathrm{d}\zeta\\&=\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)+\frac{\alpha}{2^{\alpha-1}(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{\alpha(\alpha-1)}{(\ell_2-\ell_1)^2}\int_{1/2}^{1}(1-\zeta)^{\alpha-2}\lambda(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\mathrm{d}\zeta\\&=\frac{1}{2^{\alpha}(\ell_2-\ell_1)}\lambda'\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)-\frac{\alpha}{2^{\alpha-1}(\ell_2-\ell_1)^2}\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&\quad+\frac{\Gamma(\alpha+1)}{(\ell_2-\ell_1)^{\alpha+1}}\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left[\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right].\end{aligned} \tag{92}$$

Substituting (91) and (92) in (90), we get (89). □

Corollary 6. *If we set $\ell_1=y_1$ and $\ell_2=y_2$ in Lemma 6, we get*

$$\begin{aligned}&\frac{2^{\alpha-2}\Gamma(\alpha)}{(y_2-y_1)^{\alpha-1}}\left\{\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right.\\&\left.+\left(I^{\alpha:\psi}_{\psi^{-1}((y_1+y_2)/2)^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)\right\}-\lambda\left(\frac{y_1+y_2}{2}\right)\\&=\frac{(y_2-y_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^{\alpha}\lambda''(\zeta y_1+(1-\zeta)y_2)\mathrm{d}\zeta+\int_{1/2}^{1}(1-\zeta)^{\alpha}\lambda''(\zeta y_1+(1-\zeta)y_2)\mathrm{d}\zeta\right].\end{aligned}\tag{93}$$

Corollary 7. *If we set $\psi(\gamma)=\gamma$ in Lemma 6, we get*

$$\begin{aligned}&\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left\{\left(J^{\alpha}_{(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)(\lambda(y_1+y_2-\ell_1))\right.\\&\left.+\left(J^{\alpha}_{(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)(\lambda(y_1+y_2-\ell_2))\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\\&=\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\right.\\&\left.+\int_{1/2}^{1}(1-\zeta)^{\alpha}\lambda''(y_1+y_2-(\zeta\ell_2+(1-\zeta)\ell_1))\mathrm{d}\zeta\right].\end{aligned}\tag{94}$$

Moreover, if we set $\ell_1=y_1$ and $\ell_2=y_2$, we obtain Lemma 2.1 of [34] for $m=1$.

Theorem 12. *If (A_1) is satisfied and $|\lambda''|$ is a convex function on $[y_1,y_2]$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left\{\left(I^{\alpha-1:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\&\left.\left.+\left(I^{\alpha-1:\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\&\leq\frac{(\ell_2-\ell_1)^2}{4\alpha(\alpha+1)}\left\{|\lambda''(y_1)|+|\lambda''(y_2)|-\frac{|\lambda''(\ell_1)|+|\lambda''(\ell_2)|}{2}\right\}.\end{aligned}\tag{95}$$

Proof. By using Lemma 6, properties of modulus, and Jensen–Mercer inequality, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left\{\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right.\right.\\
&\left.\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)\right\}-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^{\alpha}\left|\lambda''(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\right|\mathrm{d}\zeta\right.\\
&\left.+\int_{1/2}^{1}(1-\zeta)^{\alpha}\left|\lambda''(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\right|\mathrm{d}\zeta\right]\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^{\alpha}\left\{\left|\lambda''(y_1)\right|+\left|\lambda''(y_2)\right|-\left((1-\zeta)\left|\lambda''(\ell_1)\right|+\zeta\left|\lambda''(\ell_2)\right|\right)\right\}\mathrm{d}\zeta\right.\\
&\left.+\int_{1/2}^{1}(1-\zeta)^{\alpha}\left\{\left|\lambda''(y_1)\right|+\left|\lambda''(y_2)\right|-\left((1-\zeta)\left|\lambda''(\ell_1)\right|+\zeta\left|\lambda''(\ell_2)\right|\right)\right\}\mathrm{d}\zeta\right],
\end{aligned}\tag{96}$$

and after integration, we get required result. □

Corollary 8. *If we set $\ell_1 = y_1$ and $\ell_2 = y_2$ in Theorem 12, we get*

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(y_2-y_1)^{\alpha-1}}\left\{\left(I^{\alpha-1:\,\psi}_{\psi^{-1}((y_1+y_2)/2)^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right.\right.\\
&\left.\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}((y_1+y_2)/2)^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)-\lambda\left(\frac{y_1+y_2}{2}\right)\right.\right|\\
&\leq\frac{(y_2-y_1)^2}{8\alpha(\alpha+1)}\left(\left|\lambda''(y_1)\right|+\left|\lambda''(y_2)\right|\right).
\end{aligned}\tag{97}$$

Corollary 9. *If we set $\psi(\gamma) = \gamma$ in Theorem 12, we get*

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left\{\left(J^{\alpha-1}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)(\lambda(y_1+y_2-\ell_1))\right.\right.\\
&\left.\left.+\left(J^{\alpha-1}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)(\lambda(y_1+y_2-\ell_2))-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right.\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{4\alpha(\alpha+1)}\left\{\left|\lambda''(y_1)\right|+\left|\lambda''(y_2)\right|-\left(\frac{\left|\lambda''(\ell_1)\right|+\left|\lambda''(\ell_2)\right|}{2}\right)\right\}.
\end{aligned}\tag{98}$$

Remark 22. If we set $\psi(\gamma) = \gamma$, $\ell_1 = y_1$, and $\ell_2 = y_2$ in Theorem 12, we get Theorem 2.1 of [34].

Moreover, if we set $\alpha = 2$, we obtain Proposition 1 of [33].

Theorem 13. *If (A_1) is satisfied and $|\lambda''|^q$ is convex function, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left(\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right)\right.\\&\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot2^{2-\alpha}}\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{1/p}\left[\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{3|\lambda''(\ell_1)|^q+|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right.\\&\left.+\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{|\lambda''(\ell_1)|^q+3|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right],\end{aligned}\tag{99}$$

where $q>1$ and $(1/p)+(1/q)=1$ for all $\ell_1,\ell_2\in[y_1,y_2]$.

Proof. Applying Lemma 6, Hölder and Jensen–Mercer inequalities, the fact that $|\lambda''|^q$ is convex function, and properties of modulus, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left(\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right)\right.\\&\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot2^{2-\alpha}}\left[\int_0^{1/2}\zeta^\alpha\left|\lambda''\left(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2)\right)\right|d\zeta\right.\\&\left.+\int_{1/2}^1(1-\zeta)^\alpha\left|\lambda''\left(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2)\right)\right|d\zeta\right]\\&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot2^{2-\alpha}}\left[\left(\int_0^{1/2}\zeta^{p\alpha}d\zeta\right)^{1/p}\left(\int_0^{1/2}\left|\lambda''\left(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2)\right)\right|^q d\zeta\right)^{1/q}\right.\\&\left.+\left(\int_{1/2}^1(1-\zeta)^{p\alpha}d\zeta\right)^{1/p}\left(\int_{1/2}^1\left|\lambda''\left(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2)\right)\right|^q d\zeta\right)^{1/q}\right]\\&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot2^{2-\alpha}}\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{1/p}\\&\times\left[\left(\int_0^{1/2}\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-(1-\zeta)|\lambda''(\ell_1)|^q-\zeta|\lambda''(\ell_2)|^q\right)d\zeta\right)^{1/q}\right.\\&\left.+\left(\int_{1/2}^1\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q-(1-\zeta)|\lambda''(\ell_1)|^q-\zeta|\lambda''(\ell_2)|^q\right)d\zeta\right)^{1/q}\right]\\&=\frac{(\ell_2-\ell_1)^2}{\alpha\cdot2^{2-\alpha}}\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{1/p}\left[\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{3|\lambda''(\ell_1)|^q+|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right.\\&\left.+\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{|\lambda''(\ell_1)|^q+3|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right].\end{aligned}\tag{100}$$

□

Corollary 10. *If we set $\ell_1 = y_1$ and $\ell_2 = y_2$ in Theorem 12, we get*

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(y_2 - y_1)^{\alpha-1}}\left(\left(I^{\alpha-1:\,\psi}_{\psi^{-1}((y_1+y_2)/2)^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_2)\right)\right)\right)\right.\\
&\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}((y_1+y_2)/2)^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1)\right)\right)-\lambda\left(\frac{y_1+y_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{1/p}\left[\left(\frac{|\lambda''(y_1)|^q+3|\lambda''(y_2)|^q}{8}\right)^{1/q}\right.\\
&\left.+\left(\frac{3|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{8}\right)^{1/q}\right].
\end{aligned}\tag{101}$$

Corollary 11. *If we set $\psi(\gamma) = \gamma$ in Theorem 12, we get*

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2 - \ell_1)^{\alpha-1}}\left(\left(J^{\alpha-1}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda(y_1+y_2-\ell_1)\right)\right)\right.\\
&+\left(J^{\alpha-1}_{(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda(y_1+y_2-\ell_2)\right)\\
&\left.-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{1/p}\left[\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{3|\lambda''(\ell_1)|^q+|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right.\\
&\left.+\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2}-\frac{|\lambda''(\ell_1)|^q+3|\lambda''(\ell_2)|^q}{8}\right)^{1/q}\right].
\end{aligned}\tag{102}$$

Theorem 14. *If (A_1) is satisfied and $|\lambda''|^q$ is convex function, then*

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2 - \ell_1)^{\alpha-1}}\left(\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{+}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right)\right.\\
&\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^{-}}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-(1/q)}\\
&\times\left[\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2^{\alpha+1}(\alpha+1)}-\frac{(\alpha+3)|\lambda''(\ell_1)|^q}{2^{\alpha+2}(\alpha+1)(\alpha+2)}-\frac{|\lambda''(\ell_2)|^q}{2^{\alpha+2}(\alpha+2)}\right)^{1/q}\right.\\
&\left.+\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2^{\alpha+1}(\alpha+1)}-\frac{|\lambda''(\ell_1)|^q}{2^{\alpha+2}(\alpha+2)}-\frac{(\alpha+3)|\lambda''(\ell_2)|^q}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)^{1/q}\right],
\end{aligned}\tag{103}$$

where $q \geq 1$ *for all* $\ell_1, \ell_2 \in [y_1, y_2]$.

Proof. From Lemma 2, power-mean and Jensen–Mercer inequalities, the fact that $|\lambda''|^q$ is convex function, and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(\ell_2-\ell_1)^{\alpha-1}}\left(\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^+}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_1)\right)\right)\right)\right.\\
&\left.+\left(I^{\alpha-1:\,\psi}_{\psi^{-1}(y_1+y_2-((\ell_1+\ell_2)/2))^-}\right)\left(\lambda\circ\psi\left(\psi^{-1}(y_1+y_2-\ell_2)\right)\right)-\lambda\left(y_1+y_2-\frac{\ell_1+\ell_2}{2}\right)\right|\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\int_0^{1/2}\zeta^\alpha\left|\lambda''(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\right|\mathrm{d}\zeta\right.\\
&+\int_{1/2}^1(1-\zeta)^\alpha\left|\lambda''(y_1+y_2-((1-\zeta)\ell_1+\zeta\ell_2))\right|\mathrm{d}\zeta\\
&\leq\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left[\left(\int_0^{1/2}\zeta^\alpha\mathrm{d}\zeta\right)^{1-(1/q)}\right.\\
&\times\left(\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q\right)\int_0^{1/2}\zeta^\alpha\mathrm{d}\zeta-|\lambda''(\ell_1)|^q\int_0^{1/2}\zeta^\alpha(1-\zeta)\mathrm{d}\zeta-|\lambda''(\ell_2)|^q\int_0^{1/2}\zeta^{\alpha+1}\mathrm{d}\zeta\right)^{1/q}\\
&+\left(\int_{1/2}^1(1-\zeta)^\alpha\mathrm{d}\zeta\right)^{1-(1/q)}\\
&\times\left(\left(|\lambda''(y_1)|^q+|\lambda''(y_2)|^q\right)\int_{1/2}^1(1-\zeta)^\alpha\mathrm{d}\zeta-|\lambda''(\ell_1)|^q\int_{1/2}^1\zeta(1-\zeta)^\alpha\mathrm{d}\zeta-|\lambda''(\ell_2)|^q\int_{1/2}^1(1-\zeta)^{\alpha+1}\mathrm{d}\zeta\right)^{1/q}\\
&=\frac{(\ell_2-\ell_1)^2}{\alpha\cdot 2^{2-\alpha}}\left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-(1/q)}\\
&\times\left[\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2^{\alpha+1}(\alpha+1)}-\frac{(\alpha+3)|\lambda''(\ell_1)|^q}{2^{\alpha+2}(\alpha+1)(\alpha+2)}-\frac{|\lambda''(\ell_2)|^q}{2^{\alpha+2}(\alpha+2)}\right)^{1/q}\right.\\
&\left.+\left(\frac{|\lambda''(y_1)|^q+|\lambda''(y_2)|^q}{2^{\alpha+1}(\alpha+1)}-\frac{|\lambda''(\ell_1)|^q}{2^{\alpha+2}(\alpha+2)}-\frac{(\alpha+3)|\lambda''(\ell_2)|^q}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)^{1/q}\right].
\end{aligned}\tag{104}$$

□

Remark 23. If we set $\psi(\gamma)=\gamma$, $\ell_1=y_1$, and $\ell_2=y_2$ in Theorem 14, we get Theorem 2.2 of [34] for $m=s=1$.

Moreover, if we set $\alpha=2$, we obtain Proposition 5 of [33].

4. Application

In this last section, we will give an application of our results using modified Bessel function of the first kind.

Let the function $\mathfrak{I}_p\colon \mathfrak{R}\longrightarrow[1,+\infty)$ be defined by

$$\mathfrak{I}_p(\ell_1)=2^p\Gamma(p+1)\ell_1^{-v}I_p(\ell_1),\quad p>1,\ell_1\in\mathfrak{R}. \tag{105}$$

For this, we recall the modified Bessel function of the first kind $\mathfrak{I}_p$ which is defined as follows [35]:

$$\mathfrak{I}_p(\ell_1)=\sum_{n\geq 0}\frac{(\ell_1/2)^{p+2n}}{n!\Gamma(p+n+1)}. \tag{106}$$

The first and the n th order derivative formula of $\mathfrak{I}_p$ is, respectively, given by the following [36]:

$$\mathfrak{I}_p'(\ell_1) = \frac{\ell_1}{2(p+1)}\mathfrak{I}_{p+1}(\ell_1), \tag{107}$$

$$\begin{aligned}\frac{d^n\mathfrak{I}_p(\ell_1)}{d\ell_1^n} &= 2^{n-2p}\sqrt{\pi}\ell_1^{p-n}\Gamma(p+1)\\ &\times {}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p+1-n}{2},\frac{p+2-n}{2},p+1;\frac{\ell_1^2}{4}\right),\end{aligned} \tag{108}$$

where ${}_2F_3(\cdot,\cdot;\cdot,\cdot,\cdot;\cdot)$ is the hypergeometric function defined by the following [36]:

$$\begin{aligned}&{}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p+1-n}{2},\frac{p+2-n}{2},p+1;\frac{\ell_1^2}{4}\right)\\ &= \sum_{k=0}^{\infty}\frac{((p+1)/2)_k((p+2)/2)_k}{((p-1)/2)_k((p-2)/2)_k(p+1)_k}\cdot\frac{\ell_1^{2k}}{4^k\cdot k!},\end{aligned} \tag{109}$$

and for some parameter v, the Pochhammer symbol $(v)_k$ is defined as

$$\begin{aligned}&(v)_0 = 1,\\ &(v)_k = v(v+1)\cdots(v+k+1), \quad k = 1,2,3,\ldots.\end{aligned} \tag{110}$$

Proposition 1. *Let $0 < y_1 < y_2$ be real numbers and $p > -1$, then*

$$\begin{aligned}&\left|\frac{\mathfrak{I}_p(y_2)-\mathfrak{I}_p(y_1)}{y_2-y_1} - \frac{y_1+y_2}{4(p+1)}\mathfrak{I}_{p+1}\left(\frac{y_1+y_2}{2}\right)\right|\\ &\leq \frac{(y_2-y_1)^2}{48}2^{3-2p}\sqrt{\pi}\Gamma(p+1)\\ &\times\left\{y_1^{p-3}\cdot{}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p-2}{2},\frac{p-1}{2},p+1;\frac{y_1^2}{4}\right)\right.\\ &\left.+ y_2^{p-3}\cdot{}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p-2}{2},\frac{p-1}{2},p+1;\frac{y_2^2}{4}\right)\right\}.\\ &\leq \frac{(y_2-y_1)^2}{48}2^{3-2p}\sqrt{\pi}\Gamma(p+1)\\ &\times\left\{y_1^{p-3}\cdot{}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p-2}{2},\frac{p-1}{2},p+1;\frac{y_1^2}{4}\right)\right.\\ &\left.+ y_2^{p-3}\cdot{}_2F_3\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p-2}{2},\frac{p-1}{2},p+1;\frac{y_2^2}{4}\right)\right\}.\end{aligned} \tag{111}$$

Proof. Let $\lambda(\ell_1) = \mathfrak{I}_p'(\ell_1)$. Note that the function $\ell_1 \longrightarrow \mathfrak{I}'''(\ell_1)$ is convex on the interval $[0,+\infty)$ for each $p > -1$. Using Corollary 3 and relations (107) and (108), we obtain the desired inequality (111). □

Remark 24. Using the same technique like Proposition 1, we can obtain some new interesting inequalities pertaining modified Bessel function of the first kind or the well-known q–digamma function for $q \in (0,1)$ from our generic results. We omit here their proofs, and the details are left to the interested reader.

5. Conclusion

In this article, some new Hermite–Jensen–Mercer type inequalities involving ψ–Riemann–Liouville fractional integrals are found. Several ψ–Riemann–Liouville fractional integral inequalities using identities as auxiliary results are provided, and the known results are recaptured as special cases as well. Finally, the efficiency of our results is showed with an application via modified Bessel function of the first kind. We hope that current work using our idea and technique will attract the attention of researchers working in mathematical analysis and other related fields in pure and applied sciences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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