Research Article

An Integral Boundary Value Problem of Fractional Differential Equations with a Sign-Changed Parameter in Banach Spaces

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This paper is to investigate the existence and uniqueness of solutions for an integral boundary value problem of new fractional differential equations with a sign-changed parameter in Banach spaces. The main used approach is a recent fixed point theorem of increasing $\Psi - (h, r)$ -concave operators defined on ordered sets. In addition, we can present a monotone iterative scheme to approximate the unique solution. In the end, two simple examples are given to illustrate our main results.

1. Introduction

With the intensive development of theory and applications of fractional calculus, fractional differential equations have been paid great interest in many fields and thus different boundary conditions of fractional differential equations have also attracted much attention, see [1–42]. However, we can see that there are very few results reported on solutions for boundary value problems of fractional differential equations in Banach spaces.

In 2010, by using contraction mapping principle, fixed point index theory of completely continuous operator, and so on, Bai [2] obtained the existence results of positive solutions for the following three-point problem involving with fractional derivative:

$$\begin{cases} D_{0+}^{\beta} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & \gamma u(\eta) = u(1), \end{cases} \quad (1)$$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $0 < \gamma\eta^{\beta-1} < 1$, D_{0+}^{β} is the Riemann–Liouville (**R-L** for short) fractional derivative, and the function $f(t, u)$ is continuous on $[0, 1] \times t[0, +\infty)$.

In 2014, by using Guo–Krasnoselskii fixed point theorem, Cabada and Hamdi [4] obtained the existence of

positive solutions for the following fractional differential equations with an integral condition:

$$\begin{cases} D_{0+}^{\beta} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (2)$$

where $2 < \beta \leq 3$, $0 < \rho < \beta$, D_{0+}^{β} is the **R-L** fractional derivative, and $f(t, u)$ is a continuous function.

As we know, most of the existing results of solutions for fractional differential equation boundary value problems have been obtained in real space **R**. There exists very few papers studied in abstract spaces, and we can find [3, 9] and others.

In 2018, Chen and Gao [3] discussed the following problem of fractional differential equations in a Banach space E :

$$\begin{cases} -D_{0+}^{\beta} u(t) = f(t, u(t)), & t \in J, \\ u(0) = u'(0) = \theta, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (3)$$

where $2 < \beta \leq 3$, $0 < \rho < \beta$, $J = [0, 1]$, D_{0+}^{β} is the **R-L** fractional derivative, $f: J \times K \rightarrow K$ is continuous, here K is a normal cone in E , and θ denotes the zero element of E . They

obtained the existence results of positive solutions by using fixed point index theory of condensing mapping.

Motivated by these works, we consider the following new form of fractional differential equation with an integral boundary condition:

$$\begin{cases} -D_{0+}^{\beta} u(t) = f(t, u(t)) + \gamma g(t), & t \in J, \\ u(0) = u'(0) = \theta, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (4)$$

where $2 < \beta \leq 3$, $0 < \rho < \beta$, and β and ρ are real numbers, $\gamma \in \mathbf{R}$ is a sign-changed parameter, $J = [0, 1]$, D_{0+}^{β} is the $\mathbf{R}$ -L fractional derivative, $f: J \times K \rightarrow K$ is continuous, $g: J \rightarrow K$ is continuous, here K is a normal cone in E , and θ is the zero element of E . From the literature, we know that problem (4) is a new form of fractional differential equations. Different from the previous works, we use a recent fixed point theorem for $\Psi - (h, r)$ -concave operators

$$P_{h,r} = \{x \in E \mid \text{there exist } \mu = \mu(h, r, x) > 0, \nu = \nu(h, r, x) > 0 \text{ such that } \mu h \leq x + r \leq \nu h\}. \quad (5)$$

Evidently, $P_h = P_{h,\theta}$.

Next, we list the definition of $\Psi - (h, r)$ -concave operator and fixed point theorems that will play a key role in this paper.

Definition 1 (see [43]). Let $T: P_{h,r} \rightarrow E$ be a given operator which satisfies, for $x \in P_{h,r}$, $\lambda \in (0, 1)$, and there is $\Psi(\lambda) > \lambda$ such that $T(\lambda x + (\lambda - 1)r) \geq \Psi(\lambda)Tx + (\Psi(\lambda) - 1)r$. Then, we call T a $\Psi - (h, r)$ -concave operator.

Lemma 1 (see [43]). Let T be an increasing $\Psi - (h, r)$ -concave operator and P be normal, $Th \in P_{h,r}$. Then, T has a unique fixed point x^* in $P_{h,r}$. Take any $v_0 \in P_{h,r}$, putting a sequence $v_n = Tv_{n-1}$, $n = 1, 2, \dots$; then, $\|v_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2 (see [44]). Let T be an increasing $\Psi - (h, \theta)$ -concave operator and P be normal, $Th \in P_h$. Then, T has a unique fixed point x^* in P_h . Take any $v_0 \in P_h$, making a sequence $v_n = Tv_{n-1}$, $n = 1, 2, \dots$; then, $\|v_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

defined in ordered set $P_{h,r}$ (see [43]) to study (4), and we obtain the existence and uniqueness of solution, while the uniqueness is not treated in the literature [3]. It should be pointed out that this method is interesting for solving nonlinear differential boundary value problems. In particular, we do not need the existence of upper-lower solutions, which is a critical condition in many articles.

2. Preliminaries

In this section, we state some known facts on concave operators; one can see [43–45].

Let E be a Banach space, which has a partial order induced by a cone $K \subset E$, i.e., $x \leq y$ if and only if $y - x \in K$. For $x, y \in E$, the notation $x \sim y$ means that there are $\mu > 0$ and $\nu > 0$ such that $\mu x \leq y \leq \nu x$. Clearly, $\sim$ is an equivalence relation. For $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), define $P_h = \{x \in E \mid x \sim h\}$. Clearly, $P_h \subset K$ is convex. Let $r \in K$ with $\theta \leq r \leq h$, and define $P_{h,r} = \{x \in E \mid x + r \in P_h\}$, that is,

3. Main Results

Let E be an ordered Banach space, positive cone $K \subset E$ is normal, θ is the zero element of E , and E has an identity element e , i.e., $ex = x$, $\forall x \in E$. In this section, let $J = [0, 1]$, and we apply Lemmas 1 and 2 to study problem (4) in abstract Banach space $\mathcal{B} = C(J, E)$; the space of all continuous E -value functions on interval J with the norm $\|u\|_c = \max\{\|u(t)\| : t \in J\}$. Then, $C(J, E)$ is an ordered Banach space induced by the cone $P = \{u \in C(J, E) \mid u(t) \in K, t \in J\}$, and P is also a normal cone in $C(J, E)$.

Lemma 3 (see [3, 4]). Assume $2 < \beta \leq 3$ and $0 < \rho < \beta$. Then, for every $y \in C(J, E)$, the problem of fractional linear differential equation in Banach spaces

$$\begin{cases} -D_{0+}^{\beta} u(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = \theta, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (6)$$

has a unique solution $u(t) = \int_0^1 G(t, s)y(s)ds$, where

$$G(t, s) = \begin{cases} \frac{t^{\beta-1}(1-s)^{\beta-1}(\beta-\rho+\rho s) - (\beta-\rho)(t-s)^{\beta-1}}{(\beta-\rho)\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1}(\beta-\rho+\rho s)}{(\beta-\rho)\Gamma(\beta)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (7)$$

Lemma 4 (see [3, 4]). Let $2 < \beta \leq 3$ and $\rho \geq 0$. The function $G(t, s)$ given by (7) has the following properties:

- (i) $G(t, s) > 0$ for $t, s \in (0, 1)$ and $\rho \in [0, \beta)$
- (ii) $G(t, s) \leq (\beta / ((\beta - \rho)\Gamma(\beta)))$ for $t, s \in (0, 1)$ and $\rho \in [0, \beta)$
- (iii) $((1-s)^{\beta-1}\rho s) / ((\beta - \rho)\Gamma(\beta)) t^{\beta-1} \leq G(t, s) \leq ((1-s)^{\beta-1}(\beta - \rho + \rho s)) / ((\beta - \rho)\Gamma(\beta)) t^{\beta-1}$, for $t, s \in (0, 1)$ and $\rho \in [0, \beta)$

Next, we intend to give some existence and uniqueness results for problem (4). We break our discussion up into two sections. Two cases of interest are given in Section 3.1 for $\gamma < 0$ and in Section 3.2 for $\gamma \geq 0$, respectively. In Section 3.3, we discuss the existence and uniqueness corollaries for problem (4) in real space $C[J, \mathbf{R}]$.

3.1. Investigation for the Case $\gamma < 0$. Let

$$r(t) = -\gamma \int_0^1 G(t, s)g(s)ds, \quad t \in [0, 1],$$

$$\hat{r} = -\frac{\gamma\beta}{(\beta - \rho)\Gamma(\beta)} \int_0^1 g(s)ds, \quad (8)$$

$$h(t) = Ht^{\beta-1} \int_0^1 g(s)ds, \quad \text{with } H \geq -\frac{\gamma\beta}{(\beta - \rho)\Gamma(\beta)}. \quad (9)$$

To prove the main results, we need the following assumptions:

- (H1) $f: [0, 1] \times [-\hat{r}, \theta] \cup K \rightarrow E$ is continuous and $g: [0, 1] \rightarrow K$ is continuous, and $f(t, u) \leq f(t, v)$ for $-\hat{r} \leq u \leq v$.
- (H2) For $\lambda \in (0, 1)$ and $y \in [\theta, \hat{r}]$, there is $1 > \Psi(\lambda) > \lambda$ such that $f(t, \lambda x + (\lambda - 1)y) \geq \Psi(\lambda)f(t, x)$, $t \in [0, 1]$, $x \in E$.
- (H3) $f(t, \theta) \geq \theta$ with $f(t, \theta) \equiv \theta$ for $t \in [0, 1]$, and there are two constants $a_2 \geq a_1 > 0$ such that

$$\int_0^1 t(1-t)^{\beta-1} f(t, \theta)dt \geq a_1 \int_0^1 g(t)dt, \quad f\left(t, H \int_0^1 g(s)ds\right) \leq a_2 g(t), \quad t \in [0, 1]. \quad (10)$$

Theorem 1. Assume $e \in K$ and (H1)–(H3) are satisfied; then, problem (4) has a unique solution u^* in $P_{h,r}$, where r and h are given in (8) and (9). Furthermore, set a sequence by

$$v_n(t) = \int_0^1 G(t, s)f(s, v_{n-1}(s))ds + \gamma \int_0^1 G(t, s)g(s)ds,$$

$$n = 1, 2, \dots, \quad (11)$$

for any taken $v_0 \in P_{h,r}$, and one has $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$.

Proof. For $t \in [0, 1]$, by Lemma 4, one has

$$r(t) = -\gamma \int_0^1 G(t, s)g(s)ds \leq -\gamma \int_0^1 \frac{(1-s)^{\beta-1}(\beta - \rho + \rho s)}{(\beta - \rho)\Gamma(\beta)} t^{\beta-1} g(s)ds,$$

$$\leq -\frac{\gamma\beta}{(\beta - \rho)\Gamma(\beta)} \int_0^1 g(s)ds \cdot t^{\beta-1} \leq Ht^{\beta-1} \int_0^1 g(s)ds = h(t). \quad (12)$$

Obviously, it follows $r(t) \geq \theta$, from $\gamma < 0$ and Lemma 4.

As $\theta \leq r(t) \leq h(t)$, we have $r \in P$. In the following, we will consider a set $P_{h,r} = \{u \in C(J, E) | u + r \in P_h\}$. By means of Lemma 3, if problem (4) has a solution u , then

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) [f(s, u(s)) + \gamma g(s)] ds \\ &= \int_0^1 G(t, s) f(s, u(s)) ds + \gamma \int_0^1 G(t, s) g(s) ds \quad (13) \\ &= \int_0^1 G(t, s) f(s, u(s)) ds - r(t). \end{aligned}$$

So, for any $u \in P_{h,r}$ and $t \in [0, 1]$, we can define an operator by $Tu(t) = \int_0^1 G(t, s) f(s, u(s)) ds - r(t)$. Hence, $u(t)$ is a solution of problem (4) if and only if u is a fixed point of T .

Now, we show that $T: P_{h,r} \longrightarrow E$ satisfied the definition of $\Psi - (h, r)$ -concave operator. For every $\lambda \in (0, 1)$, $u \in P_{h,r}$, and from condition (H2), one has

$$\begin{aligned} T(\lambda u + (\lambda - 1)r)(t) &= \int_0^1 G(t, s) f(s, \lambda u(s) + (\lambda - 1)r(s)) ds - r(t) \\ &\geq \Psi(\lambda) \int_0^1 G(t, s) f(s, u(s)) ds - \Psi(\lambda)r(t) + (\Psi(\lambda) - 1)r(t) \quad (14) \\ &= \Psi(\lambda)Tu(t) + (\Psi(\lambda) - 1)r(t). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} T(\lambda u + (\lambda - 1)r) &\geq \Psi(\lambda)Tu + (\Psi(\lambda) - 1)r, \\ \lambda &\in (0, 1), u \in P_{h,r}. \end{aligned} \quad (15)$$

In view of Definition 1, T is a $\Psi - (h, r)$ -concave operator.

Next, we prove that $T: P_{h,r} \longrightarrow E$ is increasing. Since $u \in P_{h,r}$, we get $u + r \in P_h$, so there is $\tau > 0$ such that $u(t) + r(t) \geq \tau h(t)$; thus, we obtain

$$u(t) \geq \tau h(t) - r(t) \geq -r(t) \geq -\hat{r}. \quad (16)$$

By condition (H1), $T: P_{h,r} \longrightarrow E$ is increasing.

The main step is to prove that $Th \in P_{h,r}$; in other words, we prove $Th + r \in P_h$. By (H1), (H3), and Lemma 4, one has

$$\begin{aligned} Th(t) + r(t) &= \int_0^1 G(t, s) f(s, h(s)) ds - r(t) + r(t) = \int_0^1 G(t, s) f(s, h(s)) ds \\ &= \int_0^1 G(t, s) f\left(s, Hs^{\beta-1} \int_0^1 g(z) dz\right) ds \\ &\leq \int_0^1 \frac{(1-s)^{\beta-1}(\beta-\rho+\rho s)}{(\beta-\rho)\Gamma(\beta)} t^{\beta-1} f\left(s, H \int_0^1 g(z) dz\right) ds \\ &\leq \frac{t^{\beta-1}}{(\beta-\rho)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (\beta-\rho+\rho s) a_2 g(s) ds \leq \frac{a_2 \beta}{(\beta-\rho)\Gamma(\beta)} t^{\beta-1} \int_0^1 g(s) ds \\ &= \frac{a_2 \beta}{H(\beta-\rho)\Gamma(\beta)} H t^{\beta-1} \int_0^1 g(s) ds = \frac{a_2 \beta}{H(\beta-\rho)\Gamma(\beta)} h(t), \\ Th(t) + r(t) &= \int_0^1 G(t, s) f(s, h(s)) ds = \int_0^1 G(t, s) f\left(s, Hs^{\beta-1} \int_0^1 g(z) dz\right) ds \\ &\geq \int_0^1 \frac{(1-s)^{\beta-1} \rho s}{(\beta-\rho)\Gamma(\beta)} t^{\beta-1} f(s, \theta) ds = \frac{\rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} f(s, \theta) ds \cdot t^{\beta-1} \geq \frac{a_1 \rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 g(s) ds \cdot t^{\beta-1} \\ &= \frac{a_1 \rho}{H(\beta-\rho)\Gamma(\beta)} H \int_0^1 g(s) ds \cdot t^{\beta-1} = \frac{a_1 \rho}{H(\beta-\rho)\Gamma(\beta)} h(t). \end{aligned} \quad (17)$$

Let

$$\begin{aligned} l_1 &= \frac{a_1 \rho}{H(\beta - \rho)\Gamma(\beta)}, \\ l_2 &= \frac{a_2 \beta}{H(\beta - \rho)\Gamma(\beta)}. \end{aligned} \quad (18)$$

Since $\Gamma(\beta) > 0$, $H > 0$, and from (H1) and (H3), we can easily get $l_2 \geq l_1 > 0$. It follows that $Th + r \in P_h$.

Finally, by using Lemma 1, operator T has a unique fixed point u^* in $P_{h,r}$, and then,

$$u^*(t) = \int_0^1 G(t, s) f(s, u^*(s)) ds - r(t), \quad t \in [0, 1]. \quad (19)$$

Consequently, $u^*(t)$ is the unique solution of problem (4) in $P_{h,r}$. Taking any $v_0 \in P_{h,r}$, the sequence $v_n = Tv_{n-1}$, $n = 1, 2, \dots$, satisfies $v_n \rightarrow u^*$ as $n \rightarrow \infty$. That is,

$$\begin{aligned} v_n(t) &= \int_0^1 G(t, s) f(s, v_{n-1}(s)) ds + \gamma \int_0^1 G(t, s) g(s) ds, \\ &\quad n = 1, 2, \dots, \end{aligned} \quad (20)$$

and $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$. $\square$

3.2. Investigation for the Case $\gamma \geq 0$. In this section, we can get the unique result by using Lemma 2. And, some other assumptions are listed in the following:

(H4) $f: [0, 1] \times K \rightarrow K$ is continuous with $f(t, \theta) \equiv \theta$ for $t \in [0, 1]$, $g: [0, 1] \rightarrow K$ is continuous.

(H5) For $t \in [0, 1]$, $f(t, x)$ is increasing with respect to the second variable, and there are $l_1, l_2 > 0$ such that

$$\begin{aligned} l_1 e &= \frac{\rho}{(\beta - \rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, \theta) + \gamma g(s)] ds, \\ l_2 e &= \frac{1}{(\beta - \rho)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (\beta - \rho + \rho s) \\ &\quad \cdot [f(s, e) + \gamma g(s)] ds. \end{aligned} \quad (21)$$

(H6) For $\lambda \in (0, 1)$, there exists $1 > \Psi(\lambda) > \lambda$ such that $f(t, \lambda x) \geq \Psi(\lambda) f(t, x)$, $\forall t \in [0, 1], x \in K$.

Theorem 2. Assume (H4)–(H6) are satisfied; then, for any taken $\gamma \geq 0$, problem (4) has a unique positive solution u^* in P_h , where $h(t) = t^{\beta-1}e$, $t \in [0, 1]$. Moreover, taking any initial value $v_0 \in P_h$, the sequence

$$v_n(t) = \int_0^1 G(t, s) [f(s, v_{n-1}(s)) + \gamma g(s)] ds, \quad n = 1, 2, \dots, \quad (22)$$

satisfies $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$.

Proof. For $u \in P_h$, we define an operator T by

$$Tu(t) = \int_0^1 G(t, s) [f(s, u(s)) + \gamma g(s)] ds, \quad t \in [0, 1], \quad (23)$$

with $G(t, s)$ given as in Lemma 3.

From Lemma 3, $u(t)$ is the solution of problem (4) if and only if u is the fixed point of T . Noting that $f(t, x) \geq \theta$ and $g(t) \geq \theta$, it is easy to check that $T: P \rightarrow P$. From conditions (H4) and (H5), it is easy to prove that $T: P \rightarrow P$ is increasing.

Next, we show that $T: P \rightarrow P$ satisfied the definition of $\Psi - (h, \theta)$ -concave operator. From (H6), for $\lambda \in (0, 1)$ and $u \in P$, we obtain

$$\begin{aligned} T(\lambda u)(t) &= \int_0^1 G(t, s) [f(s, \lambda u(s)) + \gamma g(s)] ds \\ &\geq \Psi(\lambda) \int_0^1 G(t, s) f(s, u(s)) ds + \gamma \int_0^1 G(t, s) g(s) ds \\ &\geq \Psi(\lambda) Tu(t), \quad t \in [0, 1]. \end{aligned} \quad (24)$$

Hence, $T(\lambda u) \geq \Psi(\lambda) Tu$, $\forall u \in P$, and $\lambda \in (0, 1)$.

In the following, we prove that $Th \in P_h$. Firstly, note that $h(t) = t^{\beta-1}e$, and from (H4), (H5), and Lemma 4, it follows that

$$\begin{aligned} Th(t) &= \int_0^1 G(t, s) [f(s, h(s)) + \gamma g(s)] ds = \int_0^1 G(t, s) [f(s, s^{\beta-1}e) + \gamma g(s)] ds \\ &\geq \int_0^1 \frac{(1-s)^{\beta-1} \rho s}{(\beta - \rho)\Gamma(\beta)} t^{\beta-1} [f(s, \theta) + \gamma g(s)] ds \\ &= \frac{\rho}{(\beta - \rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, \theta) + \gamma g(s)] ds \cdot t^{\beta-1} \\ &= l_1 e \cdot t^{\beta-1} = l_1 h(t). \end{aligned} \quad (25)$$

Secondly, for $t \in [0, 1]$, one has

$$\begin{aligned}
 Th(t) &= \int_0^1 G(t, s) [f(s, s^{\beta-1}e) + \gamma g(s)] ds \\
 &\leq \int_0^1 \frac{(1-s)^{\beta-1}(\beta-\rho+\rho s)}{(\beta-\rho)\Gamma(\beta)} t^{\beta-1} [f(s, e) + \gamma g(s)] ds \\
 &= \frac{t^{\beta-1}}{(\beta-\rho)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (\beta-\rho+\rho s) [f(s, e) + \gamma g(s)] ds \\
 &= l_2 e \cdot t^{\beta-1} = l_2 h(t).
 \end{aligned} \tag{26}$$

$$(l_2 - l_1)e = l_2 e - l_1 e,$$

$$\begin{aligned}
 &= \frac{1}{(\beta-\rho)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (\beta-\rho+\rho s) [f(s, e) + \gamma g(s)] ds - \frac{\rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, \theta) + \gamma g(s)] ds \\
 &= \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} [f(s, e) + \gamma g(s)] ds + \frac{\rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, e) + \gamma g(s)] ds \\
 &\quad - \frac{\rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, \theta) + \gamma g(s)] ds \\
 &= \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} [f(s, e) + \gamma g(s)] ds + \frac{\rho}{(\beta-\rho)\Gamma(\beta)} \int_0^1 s(1-s)^{\beta-1} [f(s, e) - f(s, \theta)] ds \geq \theta.
 \end{aligned} \tag{27}$$

Hence $l_2 \geq l_1 > 0$. It follows that $Th \in P_h$. Finally, by using Lemma 2, the operator T has a unique fixed point u^* in P_h , and

$$u^*(t) = \int_0^1 G(t, s) [f(s, u^*(s)) + \gamma g(s)] ds, \quad t \in [0, 1]. \tag{28}$$

Consequently, $u^*(t)$ is the unique solution of problem (4) in P_h . For $v_0 \in P_h$, the sequence $v_n = Tv_{n-1}$, $n = 1, 2, \dots$, satisfies $v_n \rightarrow u^*$ as $n \rightarrow \infty$. That is,

$$v_n(t) = \int_0^1 G(t, s) [f(s, v_{n-1}(s)) + \gamma g(s)] ds, \quad n = 1, 2, \dots, \tag{29}$$

satisfies $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$. $\square$

Remark 1. The form of problem (4) is more general. Our method is new to study fractional differential equations in Banach spaces, which guarantees the existence and uniqueness of solutions or positive solutions. We also can make an iteration to approximate the unique solution.

Since $e \in P$, from (H5), it is easy to prove that

3.3. Direct Corollaries. Let $E = \mathbf{R}$, $\mathcal{B} = C[0, 1]$, and $P = \{u \in C[0, 1], u(t) \geq 0, t \in [0, 1]\}$. We can get some obvious results from Theorems 1 and 2. Set

$$r(t) = -\gamma \int_0^1 G(t, s) g(s) ds, \quad \forall t \in [0, 1], \tag{30}$$

$$\begin{aligned}
 \hat{r} &= -\frac{\gamma\beta}{(\beta-\rho)\Gamma(\beta)} \int_0^1 g(s) ds, \\
 h(t) &= Ht^{\beta-1} \int_0^1 g(s) ds, \text{ with } H \geq -\frac{\gamma\beta}{(\beta-\rho)\Gamma(\beta)}.
 \end{aligned} \tag{31}$$

(H1') $f: [0, 1] \times [-\hat{r}, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, $f(t, u) \leq f(t, v)$ for $-\hat{r} \leq u \leq v < +\infty$, and $g: [0, 1] \rightarrow [0, +\infty)$ is continuous.

(H2') For $\lambda \in (0, 1)$ and $\gamma \in [0, \hat{r}]$, there is $\Psi(\lambda) > \lambda$ such that $f(t, \lambda x + (\lambda - 1)\gamma) \geq \Psi(\lambda)f(t, x)$, $\forall t \in [0, 1]$, $x \in (-\infty, +\infty)$.

(H3') $f(t, 0) \geq 0$, $f(t, 0) \equiv 0$ for $t \in [0, 1]$, and there are two constants $a_2 \geq a_1 > 0$ such that

$$\int_0^1 t(1-t)^{\beta-1} f(t, 0) dt \geq a_1 \int_0^1 g(t) dt, \quad f\left(t, H \int_0^1 g(s) ds\right) \leq a_2 g(t), \quad \forall t \in [0, 1]. \quad (32)$$

Corollary 1. Let $\beta \in (2, 3]$ and $\rho \in (0, \beta)$. Assume that $(H1')$ – $(H3')$ hold; then, for $\gamma < 0$, the boundary value problem

$$\begin{cases} -D_{0+}^{\beta} u(t) = f(t, u(t)) + \gamma g(t), & 0 \leq t \leq 1, \\ u(0) = u'(0) = 0, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (33)$$

has a unique solution u^* in $P_{h,r}$, where r and h are given in (30 and 31). Furthermore, set a sequence

$$v_n(t) = \int_0^1 G(t, s) f(s, v_{n-1}(s)) ds + \gamma \int_0^1 G(t, s) g(s) ds, \quad n = 1, 2, \dots, \quad (34)$$

for any taken $v_0 \in P_{h,r}$, and one has $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$.

for any taken $v_0 \in P_h$, and one has $v_n(t) \rightarrow u^*(t)$ as $n \rightarrow \infty$.

Corollary 2. Let $\beta \in (2, 3]$ and $\rho \in (0, \beta)$. Assume that

$(H4')$ $f: [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $f(t, 0) \equiv 0$ for $t \in [0, 1]$ and $g: [0, 1] \rightarrow [0, +\infty)$ is continuous

$(H5')$ For $t \in [0, 1]$, $f(t, x)$ is increasing in the second variable

$(H6')$ For $\lambda \in (0, 1)$, there exists $\Psi(\lambda) > \lambda$ such that $f(t, \lambda x) \geq \Psi(\lambda) f(t, x)$, $\forall t \in [0, 1], x \in [0, +\infty)$; then, for $\gamma \geq 0$, the boundary value problem

$$\begin{cases} -D_{0+}^{\beta} u(t) = f(t, u(t)) + \gamma g(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \rho \int_0^1 u(t) dt, \end{cases} \quad (35)$$

has a unique solution u^* in P_h , where $h(t) = t^{\beta-1}$. Furthermore, put a sequence by

$$v_n(t) = \int_0^1 G(t, s) [f(s, v_{n-1}(s)) + \gamma g(s)] ds, \quad n = 1, 2, \dots, \quad (36)$$

4. Examples

To illustrate our main results, we present two examples.

Example 1. Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{(5/2)} u(t) = f(t, u(t)) - 2t, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \frac{1}{2} \int_0^1 u(t) dt, \end{cases} \quad (37)$$

where

$$f(t, u) = \left\{ \left(\frac{8}{35} u + \frac{15}{7\Gamma(7/2)} \right) t^{(5/2)} - \left(\frac{16}{25} u + \frac{2}{\Gamma(7/2)} \right) t^{(7/2)} \right\}^{(1/3)}, \quad (38)$$

and $\beta = (5/2), \rho = (1/2), \gamma = -2, g(t) = t$, and $t \in [0, 1]$. After a simple calculation,

$$r(t) = -\gamma \int_0^1 G(t, s) g(s) ds = \frac{2t^{(5/2)}(1-t)}{\Gamma(7/2)} + \frac{t^{(5/2)}}{2\Gamma(9/2)} = \left[\frac{2}{\Gamma(7/2)} + \frac{1}{7\Gamma(7/2)} \right] t^{(5/2)} - \frac{2}{\Gamma(7/2)} t^{(7/2)}, \quad t \in [0, 1], \quad (39)$$

where $G(t, s)$ is given as Lemma 1. Moreover, $h(t) = Ht^{(3/2)}$ with $H \geq (5/(4\Gamma(5/2)))$, $\forall t \in [0, 1]$. Then,

$$r(t) = \frac{2t^{(5/2)}(1-t)}{\Gamma(7/2)} + \frac{t^{(5/2)}}{2\Gamma(9/2)} \geq 0, \quad (40)$$

$$r(t) = \left[\frac{2}{\Gamma(7/2)} + \frac{1}{7\Gamma(7/2)} \right] t^{(5/2)} - \frac{2}{\Gamma(7/2)} t^{(7/2)} \leq \left(2 + \frac{1}{7} \right) \frac{1}{\Gamma(7/2)} t^{(5/2)} < \frac{5}{4\Gamma(5/2)} t^{(3/2)} \leq Ht^{(3/2)} = h(t).$$

Furthermore, $\hat{r} = (5/(4\Gamma(5/2)))$. We can see that $f: [0, 1] \times [-(5/(4\Gamma(5/2))), +\infty) \longrightarrow (-\infty, +\infty)$ is continuous and increasing in the second variable, and

$$f(t, 0) = \left[\frac{15}{7\Gamma(7/2)} t^{(5/2)} - \frac{2}{\Gamma(7/2)} t^{(7/2)} \right]^{(1/3)} = (r(t))^{(1/3)} \geq 0, t \in [0, 1], \quad (41)$$

with $f(t, 0) \equiv 0, t \in [0, 1]$. Thus, conditions (H1') and (H3') are satisfied. Obviously, it can be expressed as $f(t, u(t)) = [(r(t)/\hat{r})u(t) + r(t)]^{(1/3)}$ and

$$\frac{r(t)}{\hat{r}} = \frac{8}{35} t^{(5/2)} - \frac{16}{25} t^{(7/2)}, t \in [0, 1]. \quad (42)$$

From the Remark 4 in [43], we obtain

$$f(t, \lambda x + (\lambda - 1)y) \geq \Psi(\lambda) f(t, x), \quad \forall t \in [0, 1], x \in (-\infty, +\infty), y \in [0, \hat{r}], \quad (43)$$

where $\Psi(\lambda) = \lambda^{(1/3)} > \lambda, \lambda \in (0, 1)$; then, condition (H1') is satisfied. In view of Corollary 1, problem (37) has a unique solution u^* in $P_{h,r}$. For $v_0 \in P_{h,r}$, let

$$v_n(t) = \int_0^1 G(t, s) f(s, v_{n-1}(s)) ds + (-2) \int_0^1 G(t, s) t ds$$

$$= \int_0^1 G(t, s) \left\{ \left(\frac{8}{35} v_{n-1}(s) + \frac{15}{7\Gamma(7/2)} \right) s^{(5/2)} - \left(\frac{16}{25} v_{n-1}(s) + \frac{2}{\Gamma(7/2)} \right) s^{(7/2)} \right\}^{(1/3)} ds$$

$$- \left[\frac{2}{\Gamma(7/2)} + \frac{1}{7\Gamma(7/2)} \right] t^{(5/2)} + \frac{2}{\Gamma(7/2)} t^{(7/2)}, \quad n = 1, 2, \dots \quad (44)$$

We have $v_n(t) \longrightarrow u^*(t)$ as $n \longrightarrow \infty, t \in [0, 1]$.

Example 2. Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{(5/2)} u(t) = \left\{ \left(\frac{5}{3} t^2 + t \right) (u + 2) \right\}^{(5/2)} + \gamma g(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & \end{cases} \quad (45)$$

$$u(1) = \frac{1}{2} \int_0^1 u(t) dt,$$

where $f(t, u) = \{((5/3)t^2 + t)(u + 2)\}^{(2/5)}, t \in [0, 1]$, and $\beta = (5/2), \rho = (1/2), \gamma = 0$, and $g(t) \geq 0, t \in [0, 1]$ are continuous. Take $h(t) = t^{\beta-1}, t \in [0, 1]$. It is clear that $f: [0, 1] \times [0, +\infty) \longrightarrow [0, +\infty)$ is continuous and

increasing in the second variable, $f(t, 0) = ((10/3)t^2 + 2t)^{(2/5)} \geq 0$ with $f(t, 0) \equiv 0, t \in [0, 1]$, and thus, conditions (H4') and (H5') are satisfied. Next, for $\lambda \in (0, 1)$,

$$\begin{aligned}
 f(t, \lambda u) &= \left\{ \left(\frac{5}{3}t^2 + t \right) (\lambda u + 2) \right\}^{(2/5)} \geq \left\{ \left(\frac{5}{3}t^2 + t \right) (\lambda u + 2\lambda) \right\}^{(2/5)}, \\
 &= \lambda^{(2/5)} \left\{ \left(\frac{5}{3}t^2 + t \right) (u + 2) \right\}^{(2/5)}, \quad t \in [0, 1],
 \end{aligned}
 \tag{46}$$

where $\Psi(\lambda) = \lambda^{(2/5)} > \lambda, \lambda \in (0, 1)$; then, condition (H6') is satisfied. In view of Corollary 2, problem (45) has a unique solution u^* in P_h . For $v_0 \in P_h$, set

$$v_n(t) = \int_0^1 G(t, s) \left\{ \left(\frac{5}{3}t^2 + t \right) (v_{n-1}(t) + 2) \right\}^{(2/5)}, \quad n = 1, 2, \dots, \tag{47}$$

And we have $v_n(t) \longrightarrow u^*(t)$ as $n \longrightarrow \infty, t \in [0, 1]$.

5. Conclusion

In this paper, we study an integral boundary value problem (4) with sign-changed parameter in Banach spaces. By taking two functions r and h and using fixed point theorems of increasing $\Psi - (h, r)$ -concave operators defined on ordered set $P_{h,r}$, we establish some new existence and uniqueness criteria for problem (4) dependent on different parameters, which also give some different answers to the same type of fractional differential equations' boundary value problem in the literature [3]. Our results can guarantee the existence of a unique solution with not supposing the existence of upper-lower solutions. As applications, two good examples are presented to illustrate the main results.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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