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Research Article

Partially Accelerated Model for Analyzing Competing Risks Data from Gompertz Population under Type-I Generalized Hybrid Censoring Scheme

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In reliability engineering and lifetime analysis, many units of the product fail with different causes of failure, and some tests require stress higher than normal stress. Also, we need to design the life experiments which present methodology for formulating scientific and engineering problems using statistical models. So, in this paper, we adopted a partially constant stress accelerated life test model to present times to failure in a small period of time for Gompertz life products. Also, considering that, units are failing with the only two independent causes of failure and tested under type-I generalized hybrid censoring scheme the data built. Obtained data are analyzed with two methods of estimations, maximum likelihood and Bayes methods. These two methods are used to construct the point and interval estimators with the help of the MCMC method. The developed results are measured and compared under Monte Carlo studying. Also, a data set is analyzed for illustration purposes. Finally, some comments are presented to describe the numerical results.

1. Introduction

The data under life-testing experiments may be complete or censoring data; the words of complete data set are used when the time to failure for all units under the test is obtained. But, the word censoring data is used when some but not all data about tested units is obtained. Type-I and type-II censoring schemes are from the oldest censoring schemes in life testing experiments. In the type-I censoring scheme, tested time is prefixed and the number of failure units is random. But, in type-II censoring scheme, the tested time is random and the number of failure units is prefixed. The two types of censoring have the lack of memory where the number of failure units may be very small or zero in type-I censoring scheme, but the total time of the test may be very large in type-II censoring scheme. And, the joint censoring scheme of type-I and type-II is called hybrid censoring scheme.

In the plan of type-I hybrid censoring scheme (type-I HCS), n units are randomly selected from the product. The ideal test time and a suitable number of failure units that need statistical inference are proposed to be η^* and m, respectively. The experimenter terminates the test at the min (η^*, X_m). Type-I HCS is exposed and studied by different authors, [1, 2] and recently by [3]. But, in the plan of type-II hybrid censoring scheme (type-II HCS), also, n identical units are randomly selected from the product. And, the ideal test time and a suitable number of failure units that need statistical inference are proposed to be η^* and m, respectively. The experimenter terminates the test at the max (η^*, X_m); for more details, see [4]. The types of censoring, type-I HCS and type-II HCS also, satisfy the property that a smaller number of failures may be zero and there is a large test time, respectively; see [5]. Then, this problem has been treated with a generalized form of two types of censoring schemes known as a generalized hybrid censoring scheme (GHCS) [6].

For type-I GHCS, n identical units selected from product to put to the test and two prefixed integers s and m satisfying that $1 \leq s < m \leq n$ and prefixed time $\eta^* \in (0, \infty)$ are proposed. The plan of type-I GHCS can be described as follows. When the experiment is running, the failure time is recorded until s number of failures is observed. If $X_s < \eta^*$, then the test is terminated at a minimum time of (η^*, X_m). In another case, if $\eta^* < X_s < X_m$, the test is terminated at X_s. Then, in type-I GHCS, the minimum number s of failure must be satisfied and the data is summarized as $\mathbf{X} = (X_{1;n}, X_{1;n}, \ldots, X_{\nu;n})$ where ν is defined by

$$\nu = \begin{cases} \mathbf{s}, & \text{at } \eta^* < X_s < X_m, \\ m, & \text{at } X_s < X_m < \eta^*, \\ \nu, & s < \nu < m \text{ at } X_s < \eta^* < X_m. \end{cases} \tag{1}$$

For type-II GHCS, n identical units selected from the product to put to the test and prefixed integers m as well as two prefixed times $\eta_1^*, \eta_2^* \in (0, \infty)$ satisfying $\eta_1^* < \eta_2^*$ are proposed. The plan of type-II GHCS is described as follows. After the experiment is running, the failure time is recorded until the time η_1^* is reached. If $X_m < \eta_1^*$, then the test is terminated at η_1^*. In another case, if $\eta_1^* < X_m < \eta_2^*$, the test is terminated at X_m. But, if $\eta_1^* < \eta_2^* < X_m$, then, the test is terminated at η_2^*. Then, we say that the minimum time η_1^* must be observed and the maximum time η_2^* cannot be beyond it, and the random time data $\mathbf{X} = (X_{1;n}, X_{1;n}, \ldots, X_{\nu;n})$ where ν is defined by

$$\nu = \begin{cases} m, & \text{at } \eta_1^* < X_m < \eta_2^*, \\ \nu, & \nu < m \text{ at } \eta_1^* < \eta_2^* < X_m, \\ \nu, & \nu > m \text{ at } X_m < \eta_1^*. \end{cases} \tag{2}$$

In life testing experiments, the problem of obtaining sufficient information about the life of a product under recent technology is more difficult for a long-life product. Then, the related statistical inferences became more difficult. This problem can be solved with a good choice of the type of censoring scheme. Another solution to this problem is exposing the test unit to stress higher than normal stress conditions which is known as accelerated life tests (ALTs). Studies [7, 8] presented the key reference of ALTs. Recently, this problem is handled in [9, 10]. Different forms of ALTs are available. The first is known as constant stress ALTs, in which the experiment is loaded under constant stress until the final point of the experiments. The second type is called step stress ALTs, in which the experiment is running at different stress levels and changing at a prefixed time or number [11]. The last type is progressive stress ALTs, in which the stress is kept with a continuous increase at all experiment steps [12]. In some tests, units are running under normal stress and accelerated stress; then, this type of acceleration is called partially accelerated life test. The model of partially constant ALTs can be built as follows. For n identical tested units randomly chosen from population, n_1 and $(n - n_1)$ units are selected randomly to test under used and accelerated conditions, respectively. Then, the failure times at each stage are recorded under a determined censoring scheme. Units under test can fail with different fetal risks; one of these risks is caused by the failure and the problem of measuring the risk of one cause of the failure known as competing risks. This problem is discussed by different authors [13–16]. Recently, this problem is handled for the accelerated model in [17].

The main objective in this paper is adopting the type-I GHCS with a partially constant ALT model when test units fail with only two independent causes of failure and the failure time is distributed with Gompertz distribution (GD). The Gompertz lifetime population with random variable X has probability density function (PDF) given by

$$f(x) = \theta \exp\left\{\exp(\alpha x) - \frac{\theta}{\alpha}(\exp(\alpha x) - 1)\right\}, \quad x > 0, \alpha, \theta > 0, \tag{3}$$

and cumulative distribution function (CDF) is presented by

$$F(x) = 1 - \exp\left\{-\frac{\theta}{\alpha}(\exp(\alpha x) - 1)\right\}, \tag{4}$$

where θ and α are shape parameters. Then, we describe the mechanism of the model and formulate the likelihood function. Also, we present under observed data the point and interval estimators of model parameters with maximum likelihood and Bayes estimations. The theoretical results are measured and compared with Monte Carlo simulation and data analysis.

The paper is organized as follows. Section 2 presents some abbreviations and the model description. Section 3 gives the classical estimation with the MLE method. Section 4 presents the Bayes estimation with the MCMC method. Section 5 reported the results of the Monte Carlo studying. Section 6 presents lifetime data analysis for illustrating purpose. In Section 7, we give a report about the numerical results obtained from the simulation study and data analysis.

2. The Model

In this section, we present the list of abbreviations that are used in the paper as well as a complete description of the model mechanism and the corresponding likelihood function.

2.1. Abbreviations

GD : Gompertz distribution
MLE : maximum likelihood estimation
ME : mean
PC : probability coverage
CDF : cumulative distribution function
HRF : hazard failure rate function
x_{ij}: failure time under stress i and cause j
MH : Metropolis–Hastings algorithm
CI : credible intervals
MCMC : Markov chain Monte Carlo
MSE : mean squared error
ML : mean interval length
PDF : probability density function
SF : survival function
SEL : squared error loss

δ_j: the indicator value expressed to cause

ACI : approximate confidence interval

CDF : cumulative distribution function

2.2. The Model under Type-I GHCS. Suppose that n identical units are randomly chosen from the population to put to the test, and let n_1 be randomly selected to test under used condition and $n_2 = n - n_1$ units to test under stress condition. Previously, two integers s and m such that $s < m$ and $m \le \min(n_1, n_2)$ with ideal test time η^* are determined. Firstly, when the test is running, the times to failure $X_{ij;n_j}$, $i = 1, 2, \ldots, \nu_j$ and $j = 1, 2$, denoting used and accelerated conditions, respectively, are recorded. The values of ν_j are defined in (1), and the mechanism is described as follows. If $X_{sj} < \eta^*$, then, the test is terminated at the minimum time of (η^*, X_{mj}). In another case, if $\eta^* < X_{sj} < X_{mj}$, the test is terminated at X_{sj}. Considering only the two causes of failure, the time to failure and the corresponding cause of failure are recorded. Then, type-I GHCS under the competing risks model is described by

$$\left(X_{1j;n_j}, \delta_{1j}\right) < \left(X_{2j;n_j}, \delta_{2j}\right) < \cdots < \left(X_{\nu_j j;n_j}, \delta_{\nu_j j}\right), \tag{5}$$

where

$$\nu_j = \begin{cases} \mathbf{s}, & \text{at } \eta^* < X_{sj} < X_{mj}, \\ m, & \text{at } X_{sj} < X_{mj} < \eta^*, \\ \nu, & s < \nu < m \text{ at } X_{sj} < \eta^* < X_{mj}, \end{cases} \tag{6}$$

and $\delta_{ij} = \{1, 0\}$ means cause one or cause two, $i = 1, 2, \ldots, \nu_j$. The joint likelihood function of observed data $\{(x_{1j;n_j}, \delta_{1j}) < (x_{2j;n_j}, \delta_{2j}) < \cdots < (x_{\nu_j j;n_j}, \delta_{\nu_j j})\}$ with the CDF and PDF of random variables given by$F_{lj}(x)$ and $f_{lj}(x)$, $l = 1, 2$, denotes cause 1 and cause 2, respectively; then, it is given by

$$\begin{aligned} L(\underline{\varphi} \mid \underline{x}) = \prod_{j=1}^{2} Q_j \left\{ \prod_{i=1}^{\nu_j} \left[f_{1j}\left(x_{ij;n_j}\right) S_{2j}\left(x_{ij;n_j}\right) \right]^{\delta_i j} \left[f_{2j}\left(x_{ij;n_j}\right) S_{1j}\left(x_{ij;n_j}\right) \right]^{1-\delta_i j} \right\} \\ \times \left(S_{1j}\left(x_{\nu_j;,n_j}\right) S_{2j}\left(x_{\nu_j;n_j}\right) \right)^{n_j - \nu_j}, \end{aligned} \tag{7}$$

where $Q_j = (n_j!/(n_j - \nu_j)!)$ and

$$\delta_{il} = \begin{cases} 1, & l = 1, \\ 0, & l = 2. \end{cases} \tag{8}$$

Considering that, tested units with CDF given by (4) for used condition and for independent two causes of failure that are reduced to the distribution have PDFs given by

$$f_{1l}(x) = \theta_l \exp\left\{ \exp(\alpha x) - \frac{\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}, \quad l = 1, 2. \tag{9}$$

The Gompertz lifetime distribution with common shape parameters α and different shape parameters θ_l, $l = 1, 2$ and also the CDFs and SFs are given by

$$F_{1l}(x) = 1 - \exp\left\{ -\frac{\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}, \tag{10}$$

$$S_{1L}(x) = \exp\left\{ -\frac{\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}. \tag{11}$$

Consider that the proportional hazard model (also named Cox model) is relevant to handle the effects of the environment or stress on the lifetime distribution. Thus, the survival function $S_{1l}(\cdot)$ under used stress is GD but the survival function under the higher stress takes the form

$$S_{2l}(x) = [S_{1l}(x)]^{\beta}. \tag{12}$$

Therefore, $S_{2l}(x)$, CDFs, and PDFs under accelerated condition are given, respectively, by

$$S_{2l}(x) = \exp\left\{ -\frac{\beta\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}, \tag{13}$$

$$F_{2l}(x) = 1 - \exp\left\{ -\frac{\beta\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}, \tag{14}$$

$$f_{2l}(x) = \theta_l \beta \exp\left\{ \exp(\alpha x) - \frac{\beta\theta_l}{\alpha} (\exp(\alpha x) - 1) \right\}. \tag{15}$$

3. Estimation under ML Method

The results of the point and asymptotic confidence intervals of model parameters are discussed in this section with MLE for two independent causes of failure.

Let $x = (X_{1j;n_j}, \delta_{1j}), (X_{2j;n_j}, \delta_{2j}), \ldots, (X_{\nu_j j;n_j}, \delta_{\nu_j j})$ be the sample of type-I GHC competing risks data from GD; for distribution (10) and (14), the joint function (7) is reduced to

$$L(\alpha,\theta_1,\theta_2,\beta|\underline{x}) = \theta_1^{m_1}\theta_2^{m_2}\beta^{\nu_2}\exp\left\{\alpha\left(\sum_{i=1}^{\nu_1}x_{i1}+\sum_{i=1}^{\nu_2}x_{i2}\right)-\frac{(\theta_1+\theta_2)}{\alpha}\right.$$
$$\left.\times\sum_{i=1}^{\nu_1}\delta_{i1}\left(\exp\left(\alpha x_{i1}\right)-1\right)-\frac{\beta(\theta_1+\theta_2)}{\alpha}\sum_{i=1}^{\nu_2}\delta_{i2}\left(\exp\left(\alpha x_{i2}\right)-1\right)-\frac{(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)-\frac{\beta(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right\}, \tag{16}$$

where ν_1 and ν_2 denote the number of unit failures under used and accelerated conditions, respectively. Also, $m_1 = \sum_{j=1}^{2}\sum_{i=1}^{\nu_j}\delta_{ij}$ and $m_2 = \sum_{j=1}^{2}\sum_{i=1}^{\nu_j}(1-\delta_{ij})$ are the number of unit failures under the first and second causes, respectively. Then, the natural logarithm of the likelihood function (16) is reduced to

$$\begin{aligned}\ell(\alpha,\theta_1,\theta_2,\beta|\underline{x}) = {} & m_1\log\theta_1 + m_2\log\theta_2 + \nu_2\log\beta + \alpha\left(\sum_{i=1}^{\nu_1}x_{i1}+\sum_{i=1}^{\nu_2}x_{i2}\right)\\ & -\frac{(\theta_1+\theta_2)}{\alpha}\sum_{i=1}^{\nu_1}\delta_{i1}\left(\exp\left(\alpha x_{i1}\right)-1\right)-\frac{\beta(\theta_1+\theta_2)}{\alpha}\\ & \times\sum_{i=1}^{\nu_2}\delta_{i2}\left(\exp\left(\alpha x_{i2}\right)-1\right)-\frac{(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)\\ & -\frac{\beta(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_2}\right)-1\right).\end{aligned} \tag{17}$$

3.1. Point Estimators. The point MLE of model parameters can be obtained after taking the partial derivatives of (17) with respect to vector $\varphi = \{\alpha,\theta_1,\theta_2,\beta\}$. Then, the derivative with respect to θ_1 and θ_2 is reduced to

$$\theta_1(\alpha,\beta) = \frac{\alpha m_1}{D}, \tag{18}$$

$$\theta_2(\alpha,\beta) = \frac{\alpha m_2}{D}, \tag{19}$$

$$\beta(\alpha) = \frac{\nu_2\left(\sum_{i=1}^{\nu_1}\delta_{i1}\left(\exp\left(\alpha x_{i1}\right)-1\right)+\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)\right)}{(m_1+m_2-\nu_2)\left(\sum_{i=1}^{\nu_2}\delta_{i2}\left(\exp\left(\alpha x_{i2}\right)-1\right)+\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right)}, \tag{20}$$

$$\begin{aligned}&\sum_{i=1}^{\nu_1}x_{i1}+\sum_{i=1}^{\nu_2}x_{i2}+\frac{(\theta_1+\theta_2)}{\alpha^2}\left(\sum_{i=1}^{\nu_1}\delta_{i1}\left[(1-\alpha x_{i1})\exp\left(\alpha x_{i1}\right)-1\right]\right)+\frac{\beta(\theta_1+\theta_2)}{\alpha^2}\\ &\times\left(\sum_{i=1}^{\nu_2}\delta_{i2}\left[(1-\alpha x_{i2})\exp\left(\alpha x_{i2}\right)-1\right]\right)+\frac{(\theta_1+\theta_2)}{\alpha^2}\left[\left(1-\alpha x_{\nu_1}\right)\exp\left(\alpha x_{\nu_1}\right)-1\right]\\ &\quad+\frac{\beta(\theta_1+\theta_2)}{\alpha^2}\left[(1-\alpha x_{\nu 2})\exp\left(\alpha x_{\nu_2}\right)-1\right]=0,\end{aligned} \tag{21}$$

where

$$D = \sum_{i=1}^{\nu_1}\delta_{i1}\left(\exp\left(\alpha x_{i1}\right)-1\right)+\beta\sum_{i=1}^{\nu_2}\delta_{i2}\left(\exp\left(\alpha x_{i2}\right)-1\right)+\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)+\beta\left(\exp\left(\alpha x_{\nu_2}\right)-1\right). \tag{22}$$

Then, the likelihood equations are reduced to one linear equation of α obtained after replacing θ_1, θ_2, and β by (18)–(20) in (21). The initial value of any iteration can be obtained from the profile likelihood function obtained from (16) after replacing θ_1, θ_2, and β. Then, the estimates are obtained $\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2$, and $\widehat{\beta}$.

3.2. Interval Estimation. The Fisher information matrix is defined as the minus expectation of second derivatives from the log-likelihood function with respect to model parameters. In practice, the expectation problem of the second derivative is more difficult to practice; then, the approximate Fisher information presents a suitable approximation that is used to build interval estimation as follows. Let Φ present the second derivative of parameters vector $\varphi = \{\alpha, \theta_1, \theta_2, \beta\}$ given by

$$\Phi = \left[-\frac{\partial^2 \ell(\alpha, \theta_1, \theta_2, \beta | \underline{x})}{\partial \varphi_i \partial \varphi_j} \right]. \tag{23}$$

Then, the approximate information matrix Φ at the values of the MLE of the parameters vector φ is denoted by $\Phi_0(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta})$ where $\varphi = \{\alpha, \theta_1, \theta_2, \beta\}$. Then, the estimate $\widehat{\varphi} = \{\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta}\}$ is distributed as a normal distribution with mean $\varphi = \{\alpha, \theta_1, \theta_2, \beta\}$ and variance-covariance matrix $\Phi_0(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta})$ described by

$$\widehat{\varphi} \propto \mathbf{N}\left((\alpha, \theta_1, \theta_2, \beta), \Phi_0^{-1}\left(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta} \right) \right). \tag{24}$$

Then, $100(1-2\gamma)\%$ approximate interval estimate of $\varphi = \{\alpha, \theta_1, \theta_2, \beta\}$ is given by

$$\begin{cases} \widehat{\alpha} \mp z_\gamma \sqrt{\sigma_{11}}, \\ \widehat{\theta}_1 \mp z_\gamma \sqrt{\sigma_{22}}, \widehat{\theta}_2 \mp z_\gamma \sqrt{\sigma_{33}}, \widehat{\beta} \mp z_\gamma \sqrt{\sigma 44}, \end{cases} \tag{25}$$

where z_γ presents the standard normal values with probability tailed γ and the values σ_{11}, σ_{22}, σ_{33}, and σ_{44} are the diagonal of the matrix $\Phi_0^{-1}(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta})$.

4. Bayes Estimation

In this section, we present the Bayes point and interval estimation for the unknown model parameters to formulate the available information in the form of statistical distribution. The available information is exposed in the prior information and information exposed in the data. So, we consider the independent gamma priors for Gompertz parameters $\{\alpha, \theta_1, \theta_2\}$ and noninformative prior for accelerated factor β as follows:

$$\begin{cases} \alpha \propto \alpha^{a_1 - 1} \exp(-b_1 \alpha), \\ \theta_1 \propto \theta_1^{a_2 - 1} \exp(-b_2 \theta_1), \\ \theta_2 \propto \theta_2^{a_3 - 1} \exp(-b_3 \theta_2), \\ \beta \propto \beta^{-1}. \end{cases} \tag{26}$$

Then, the joint prior density is given by

$$h^*(\alpha, \theta_1, \theta_2, \beta) \propto \alpha^{a_1 - 1} \theta_1^{a_2 - 1} \theta_2^{a_3 - 1} \beta^{-1} \exp(-b_1 \alpha - b_2 \theta_1 - b_3 \theta_2). \tag{27}$$

The joint posterior distribution can be formulated by

$$h(\alpha, \theta_1, \theta_2, \beta | \underline{x}) = \frac{h^*(\alpha, \theta_1, \theta_2, \beta) L(\alpha, \theta_1, \theta_2, \beta | \underline{x})}{\int \cdots \int h^*(\alpha, \theta_1, \theta_2) L(\alpha, \theta_1, \theta_2, \beta | \underline{x}) \mathrm{d}\alpha \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\beta}. \tag{28}$$

The Bayes estimate of model parameters depends on the posterior distribution and the choice of the loss function. Without loss of generality, considering squared error loss function (SEL), the Bayes estimators for any function $g(\alpha, \theta_1, \theta_2, \beta)$ are formulated by

$$\widehat{g}_B = \int \cdots \int g(\alpha, \theta_1, \theta_2, \beta) h(\alpha, \theta_1, \theta_2, \beta) \mathrm{d}\alpha \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\beta. \tag{29}$$

Integration in (28) and (29) is generally more difficult; hence, we need some approximation to compute these integrations. Different methods are available to approximate the integral such as numerical integration from the important ones applied in the Bayes context called MCMC method described as follows.

4.1. MCMC Approach. The problem estimation with the Bayesian approach with the help of the MCMC method is needed to build the posterior conditional distributions of model parameters as follows:

$$h(\alpha,\theta_1,\theta_2,\beta|\underline{x}) \propto \alpha^{a_1-1}\theta_1^{m_1+a_2-1}\theta_2^{m_2+a_3-1}\beta^{\nu_2-1}\exp\left\{-\alpha\left(b_1-\sum_{i=1}^{\nu_1}x_{i1}-\sum_{i=1}^{\nu_2}x_{i2}\right)\right.$$
$$-\frac{(\theta_1+\theta_2)}{\alpha}\sum_{i=1}^{\nu_1}\delta_{i1}(\exp(\alpha x_{i1})-1)-\frac{\beta(\theta_1+\theta_2)}{\alpha}\sum_{i=1}^{\nu_2}\delta_{i2}$$
$$\times(\exp(\alpha x_{i2})-1)-b_2\theta_1-b_3\theta_2-\frac{(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)$$
$$\left.-\frac{\beta(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right\}. \tag{30}$$

Therefore, the problem of building conditional distributions of (28) given data is presented by

$$h_1(\theta_1|\alpha,\theta_2,\beta\,\underline{x}) \propto \exp\left\{-\frac{\theta_1}{\alpha}\left[b_2\alpha+\sum_{i=1}^{\nu_1}\delta_{i1}(\exp(\alpha x_{i1})-1)+\beta\sum_{i=1}^{\nu_2}\delta_{i2}(\exp(\alpha x_{i2})-1)\right.\right.$$
$$\left.\left.+\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)+\beta\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right]\right\}\theta_1^{m_1+a_2-1}, \tag{31}$$

$$h_2(\theta_2|\alpha,\theta_2,\beta\,\underline{x}) \propto \exp\left\{-\frac{\theta_2}{\alpha}\left[b_3\alpha+\sum_{i=1}^{\nu_1}\delta_{i1}(\exp(\alpha x_{i1})-1)+\beta\sum_{i=1}^{\nu_2}\delta_{i2}(\exp(\alpha x_{i2})-1)\right.\right.$$
$$\left.\left.+\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)+\beta\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right]\right\}\theta_2^{m_2+a_3-1}, \tag{32}$$

$$h_3(\beta|\alpha,\theta_1,\theta_2,\underline{x}) \propto \beta^{\nu_2-1}\exp\left\{-\frac{\beta(\theta_1+\theta_2)}{\alpha}\left[\sum_{i=1}^{\nu_2}\delta_{i2}(\exp(\alpha x_{i2})-1)+\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right]\right\}, \tag{33}$$

$$h_4(\alpha|\theta_1,\theta_2,\beta,\underline{x}) \propto \alpha^{a_1-1}\exp\left\{-\alpha\left(b_1-\sum_{i=1}^{\nu_1}x_{i1}-\sum_{i=1}^{\nu_2}x_{i2}\right)-\frac{(\theta_1+\theta_2)}{\alpha}\right.$$
$$\times\sum_{i=1}^{\nu_1}\delta_{i1}(\exp(\alpha x_{i1})-1)-\frac{\beta(\theta_1+\theta_2)}{\alpha}\sum_{i=1}^{\nu_2}\delta_{i2}$$
$$\times(\exp(\alpha x_{i2})-1)-\frac{(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_1}\right)-1\right)$$
$$\left.-\frac{\beta(\theta_1+\theta_2)}{\alpha}\left(\exp\left(\alpha x_{\nu_2}\right)-1\right)\right\}. \tag{34}$$

The conditional distribution (31) to (34) shows that the conditional posterior distribution of θ_1, θ_2, and β takes gamma distribution. But, the conditional distribution of α is more similar to the normal distribution. Then, the suitable scheme of the MCMC method is Metropolis–Hastings (MH) under Gibbs algorithms [18] described as follows.

4.2. MCMC Algorithm

Step 1: begin with initial parameter values $(\alpha^{(0)},\theta_1^{(0)},\theta_2^{(0)},\beta^{(0)}) = (\widehat{\alpha},\widehat{\theta}_1,\widehat{\theta}_2,\widehat{\beta})$ put $\kappa = 1$.

Step 2: gamma distribution is used to generate $\theta_1^{(\kappa)}$ from (29), $\theta_2^{(\kappa)}$ from (30), and $\beta^{(\kappa)}$ from (31) with the Gibbs technique.

Step 3: normal proposal distribution is used to generate $\alpha^{(\kappa)}$ from the conditional distribution (32).

Step 4: report the value of the parameters vector $\varphi^{(\kappa)} = (\alpha^{(\kappa)},\theta_1^{(\kappa)},\theta_2^{(\kappa)},\beta^{(\kappa)})$.

Step 5: put κ to be $\kappa+1$.

Step 6: repeat steps (2) to (5) N times.

Step 7: determine the iteration number that is needed for the stationary state N^* (burn-in); then, for any function $g(\alpha,\theta_1,\theta_2,\beta)$, Bayes estimators are presented by

$$\widehat{g}_B = \frac{1}{N-N^*}\sum_{i=N^*+1}^{N} g^{(i)}, \tag{35}$$

Table 1: MEs and MSEs for $(\alpha, \theta_1, \theta_2, \beta) = (0.1, 0.05, 0.08, 2.0)$.

η^*	(n, n_1, n_2, s, m)	Par.	MLE MEs	MLE MSEs	MCMC$_{\text{prior0}}$ MEs	MCMC$_{\text{prior0}}$ MSEs	MCMC$_{\text{prior1}}$ MEs	MCMC$_{\text{prior1}}$ MSEs
5.0	(70, 35, 35, 15, 30)	α	0.1231	0.0825	0.1202	0.0811	0.1187	0.0642
		θ_1	0.0624	0.0342	0.0611	0.0325	0.0600	0.0227
		θ_2	0.0872	0.0536	0.0851	0.0514	0.0817	0.0324
		β	3.3124	1.2251	3.3100	1.2227	2.4250	1.0011
	(70, 35, 35, 25, 30)	α	0.1175	0.0724	0.1145	0.0692	0.1101	0.0592
		θ_1	0.0584	0.0300	0.0571	0.0297	0.0542	0.0201
		θ_2	0.0832	0.0482	0.0817	0.0491	0.0813	0.0287
		β	2.8142	1.1243	2.7842	1.1145	2.254	0.9852
	(70, 30, 40, 15, 30)	α	0.1250	0.0831	0.1215	0.0831	0.1199	0.0651
		θ_1	0.0641	0.0357	0.0624	0.0342	0.0621	0.0240
		θ_2	0.0893	0.0548	0.0867	0.0528	0.0825	0.0329
		β	3.3139	1.2263	3.3115	1.2233	2.4262	1.0026
	(70, 30, 40, 25, 30)	α	0.1182	0.0731	0.1132	0.0699	0.1122	0.0601
		θ_1	0.0591	0.0314	0.0582	0.0299	0.0556	0.0214
		θ_2	0.0850	0.0480	0.0840	0.0481	0.0825	0.0298
		β	2.8155	1.1251	2.7856	1.1152	2.2539	0.9857
10	(70, 35, 35, 15, 30)	α	0.1241	0.0782	0.1199	0.0772	0.1101	0.0574
		θ_1	0.0615	0.0302	0.0574	0.0289	0.0564	0.0200
		θ_2	0.0860	0.0489	0.0841	0.0470	0.0821	0.0241
		β	3.3002	0.9945	3.0001	0.9880	2.2135	0.8852
	(70, 35, 35, 25, 30)	α	0.1177	0.0623	0.1199	0.0711	0.1082	0.0452
		θ_1	0.0562	0.0278	0.0541	0.0255	0.0522	0.0185
		θ_2	0.0811	0.0350	0.0810	0.0442	0.0817	0.0211
		α	2.7142	0.9799	2.6452	0.9777	2.2101	0.8800
	(70, 30, 40, 15, 30)	α	0.1271	0.0782	0.1199	0.0792	0.1117	0.0574
		θ_1	0.0632	0.0321	0.0588	0.0301	0.0575	0.0215
		θ_2	0.0871	0.0499	0.0857	0.0490	0.0829	0.0254
		β	3.3022	0.9962	3.0025	0.9898	2.2128	0.8870
	(70, 30, 40, 25, 30)	α	0.1181	0.0636	0.1214	0.0718	0.1099	0.0460
		θ_1	0.0577	0.0291	0.0562	0.0263	0.0538	0.0197
		θ_2	0.0815	0.0370	0.0819	0.0460	0.0822	0.0215
		β	2.7155	0.9812	2.6471	0.9772	2.2122	0.8817

and the corresponding variance is defined by

$$V(g) = \frac{1}{N - N^*} \sum_{i=N^*+1}^{N} \left(g^{(i)} - \widehat{g}_B\right)^2, \qquad (36)$$

where $g^{(i)} = g(\alpha^{(i)}, \theta_1^{(i)}, \theta_2^{(i)}, \beta^{(i)})$ and $i = N^* + 1, 2, \ldots, N$.

Step 8: the ordered value of the vector $g^{(i)} = g(\alpha^{(i)}, \theta_1^{(i)}, \theta_2^{(i)}, \beta^{(i)})$ is denoted by $g_{(i)}$, and then, the corresponding $100(1 - 2\gamma)\%$ credible interval of g is given by

$$\left(g_{\gamma(N-N^*)}, g_{(1-\gamma)(N-N^*)}\right). \qquad (37)$$

5. Monte Carlo Simulation Study

In this section, we assess the developed results in classical MLE or Bayesian approach for different combinations of sample size n and different combinations of n_1 and n_2. Also, the study reported different effect sizes s, m and different test times η^*. We adopt one parameter set to be $(\alpha, \theta_1, \theta_2, \beta) = (0.1, 0.05, 0.08, 2.0)$. For prior information, we adopt noninformative prior0 (posterior is proportional with likelihood function) and informative prior1 $(a_i, b_i) = \{(1, 5), (0.1, 2), (0.1, 1)\}$. Mathematics Vr 10 is used, and iteration is reported for 1000 samples of type-I GHC data generated from the Gompertz distribution. For the Bayesian approach with MCMC methods, we generate $N = 11000$ and discard the first $N^* = 1000$. For the point estimate, we compute the mean value of parameter estimates (MEs) and mean squared error (MSE). But, interval estimation is measured with probability coverage and the mean interval length. The results of the simulation study are reported in Tables 1 and 2 as follows.

6. Data Analysis Simulation

In this section, we choose the set of data generated from GD with respect to type-I GHCS and accelerated under partially constant stress ALTs. The random sample is generated from two GDs over the following algorithms.

Table 2: MLs and PCs for 95% interval estimation $(\alpha, \theta_1, \theta_2, \beta) = (0.1, 0.05, 0.08, 2.0)$.

			MLE		$\mathrm{MCMC}_{\mathrm{prior0}}$		$\mathrm{MCMC}_{\mathrm{prior1}}$	
η^*	(n, n_1, n_2, s, m)	Par.	ML	PC	ML	PC	ML	PC
5.0	(70, 35, 35, 15, 30)	α	0.421	0.90	0.399	0.90	0.284	0.91
		θ_1	0.124	0.89	0.134	0.91	0.114	0.92
		θ_2	0.215	0.88	0.200	0.90	0.141	0.91
		β	4.125	0.90	4.101	0.91	3.521	0.96
	(70, 35, 35, 25, 30)	α	0.321	0.90	0.311	0.92	0.184	0.92
		θ_1	0.101	0.91	0.094	0.91	0.089	0.92
		θ_2	0.187	0.90	0.160	0.93	0.118	0.92
		β	3.421	0.91	3.405	0.91	3.011	0.93
	(70, 30, 40, 15, 30)	α	0.435	0.89	0.412	0.90	0.295	0.91
		θ_1	0.135	0.89	0.155	0.89	0.127	0.92
		θ_2	0.220	0.89	0.214	0.91	0.161	0.93
		β	4.131	0.89	4.117	0.91	3.528	0.90
	(70, 30, 40, 25, 30)	α	0.341	0.90	0.318	0.92	0.197	0.93
		θ_1	0.118	0.90	0.112	0.90	0.096	0.91
		θ_2	0.199	0.90	0.172	0.92	0.124	0.92
		β	3.434	0.90	3.414	0.91	3.018	0.91
10	(70, 35, 35, 15, 30)	α	0.314	0.90	0.305	0.91	0.200	0.93
		θ_1	0.095	0.91	0.088	0.91	0.078	0.92
		θ_2	0.147	0.90	0.130	0.92	0.095	0.91
		β	4.098	0.91	4.001	0.91	3.324	0.95
	(70, 35, 35, 25, 30)	α	0.275	0.93	0.266	0.92	0.170	0.94
		θ_1	0.082	0.91	0.076	0.93	0.051	0.92
		θ_2	0.095	0.93	0.081	0.92	0.068	0.93
		β	4.072	0.91	3.865	0.92	3.214	0.94
	(70, 30, 40, 15, 30)	α	0.331	0.91	0.322	0.91	0.211	0.91
		θ_1	0.112	0.90	0.093	0.92	0.085	0.94
		θ_2	0.162	0.90	0.148	0.92	0.099	0.94
		β	4.107	0.92	4.014	0.90	3.341	0.94
	(70, 30, 40, 25, 30)	α	0.282	0.90	0.271	0.90	0.173	0.94
		θ_1	0.087	0.91	0.085	0.93	0.064	0.93
		θ_2	0.107	0.92	0.092	0.91	0.072	0.95
		β	4.078	0.92	3.870	0.94	3.223	0.92

Table 3: The generated data under used conditions.

0.1844	0.1900	0.2027	0.5829	0.7032	0.9103	1.2755	1.4515	1.5733	1.6700
0	0	1	1	1	0	0	0	0	0
1.747	1.7529	1.7958	1.9991	2.0656	2.3661	2.3773	2.458	2.5426	2.9961
1	1	0	0	0	1	0	1	0	0
3.3552	3.3924	3.4199	3.4412	3.4652	3.8778	4.1908	4.2606	4.811	4.9416
1	1	0	1	1	0	1	0	0	0

Table 4: The generated data under accelerated conditions.

0.0746	0.1233	0.1298	0.1596	0.1950	0.3845	0.4421	0.5080	0.6562	0.8073
1	1	1	1	1	0	1	1	0	0
0.9045	0.9650	1.1242	1.3739	1.4121	1.5416	1.5892	1.6191	1.6475	1.6705
0	1	0	1	1	1	0	0	0	0
1.7388	1.7909	1.8932	2.1856	2.2012	2.2866	2.2870	2.6332	2.6470	2.7982
0	0	0	1	1	0	0	0	0	1

Table 5: Point and 95% confidence and credible intervals (ACIs and CIs) of MLE Bayes estimates.

Pa.s	$(\cdot)_{ML}$	$(\cdot)_{BMCMC}$	95% ACIs	Length	95% CIs	Length
$\alpha = 0.5$	0.7431	0.7271	(0.2894, 1.1967)	0.9073	(0.3131, 1.1825)	0.8694
$\theta_1 = 0.1$	0.2808	0.2689	(0.0454, 0.5162)	0.4708	(0.0644, 0.5542)	0.3898
$\theta_2 = 0.15$	0.3229	0.3076	(0.0571, 0.5888)	0.5317	(0.1327, 0.5749)	0.4422
$\beta = 2.0$	1.4976	1.7884	(0.5732, 2.4220)	1.8488	(0.8346, 3.4357)	2.6011

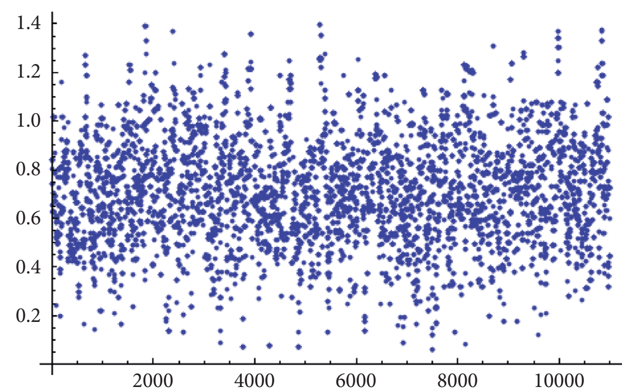

Figure 1: Simulation number of α generated by the MCMC method.

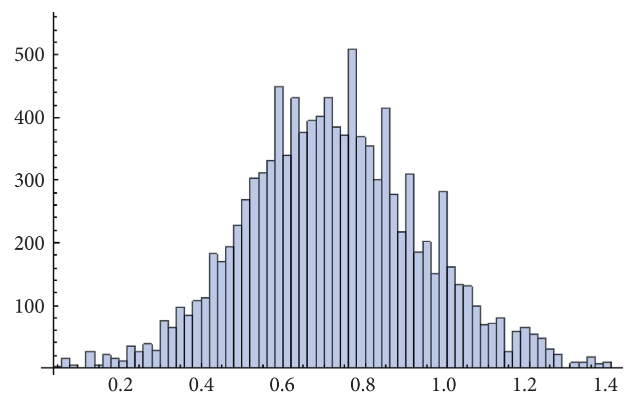

Figure 2: The histogram of α generated by the MCMC method.

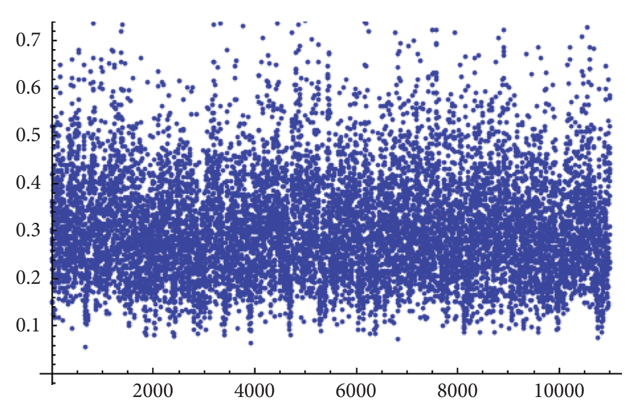

Figure 3: Simulation number of θ_1 generated by the MCMC method.

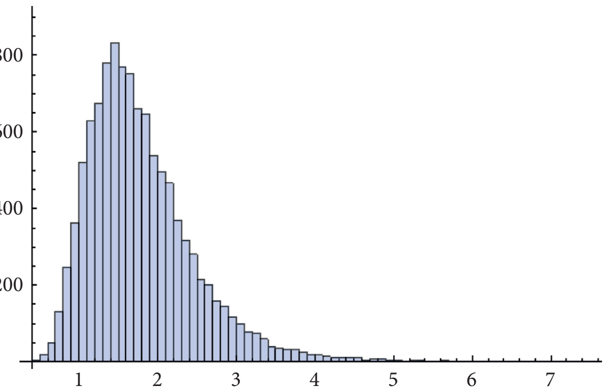

Figure 4: The histogram of θ_1 generated by the MCMC method.

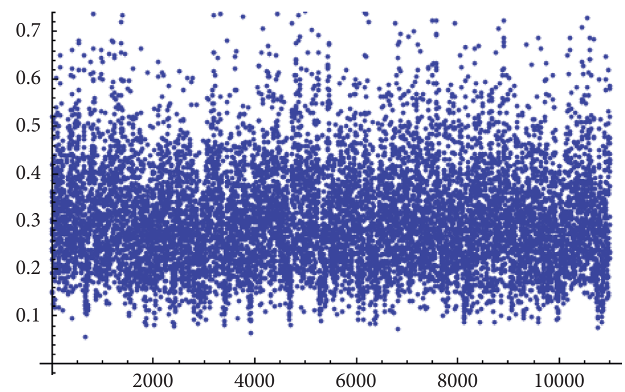

Figure 5: Simulation number of θ_2 generated by the MCMC method.

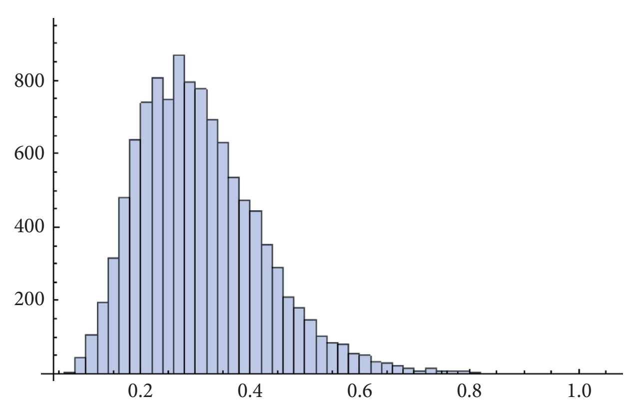

Figure 6: The histogram of θ_2 generated by the MCMC method.

Step 1: let the total sample take the size $n = 60$ and $n_1 = n_2 = 30$ and let $s = 15$ and $m = 25$.

Step 2: suppose that the parameter vectors are randomly chosen to be $\varphi = \{\alpha, \theta_1, \theta_2, \beta\} = \{0.5, 0.1, 0.15, 2.0\}$; then, a suitable η^* is chosen to be 2.5.

Step 3: the prior information is almost selected to satisfy $E(\varphi_i) \simeq (a_i/b_i)$; then $(a_i, b_i) = \{(1,2), (1,5), (1,5)\}, i = 1, 2, 3$.

Step 4: generate two samples of size $n_1 = 30$ from the two populations (10) with causes of failure. The two samples are put in ordered pairs to choose the

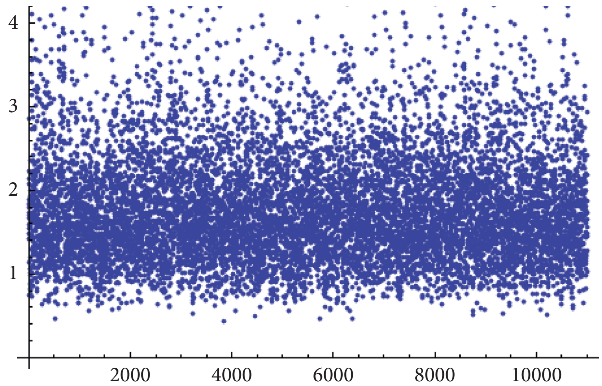

Figure 7: Simulation number of β generated by the MCMC method.

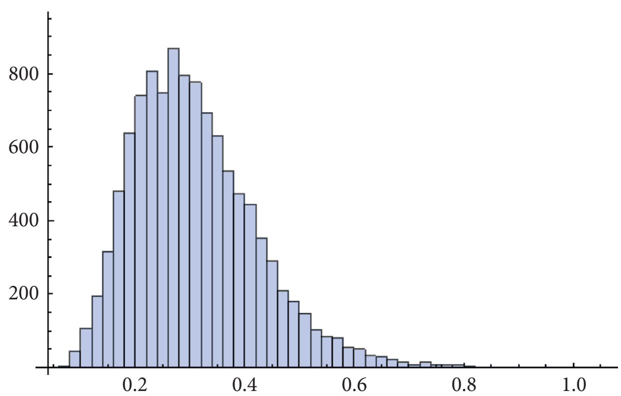

Figure 8: The histogram of β generated by the MCMC method.

minimum from each pair. Then, the minimum values are put in ascending order and determine the sample with used conditions, random values, and its cause of failure. If $X_s < \eta^*$, then we terminated time min (η^*, X_m). In another case, if $\eta^* < X_s < X_m$, the terminated time is X_s. Then, the value ν_1 is observed.

Step 5: step 4 is repeated for distribution (13) to determine ν_2.

Step 6: compute $m_1 = \sum_{j=1}^{2} \sum_{i=1}^{\nu_j} \delta_{ij}$ and $m_2 = \sum_{j=1}^{2} \sum_{i=1}^{\nu_j} (1 - \delta_{ij})$.

The data under used and accelerated conditions with its cause of failure is obtained from Tables 3 and 4 with $\nu_1 = 18$ and $\nu_2 = 25$. The point MLE and corresponding confidence interval are summarized in Table 5. Also, for the Bayesian approach, we run the chain 11000 with the first 1000 values as burn-in, and the point and interval estimates are summarized in Table 5. Figures 1–8 describe the generated MCMC sample which describes the convergence satisfied by the MCMC method. The results obtained from each figure have shown that the MCMC method serves very well.

7. Conclusions

Modern technology products have a long period of time, and information about the life product is more difficult. Then, to overcome this problem, the experimenter determined the censoring scheme that serves this problem. In this paper, we choose type-I GHCS which keeps the minimum number needed in statistical inference in a small period of time. Also, this type of censoring is applied with the concept of partially constant ALTs for units that fail under two independent causes of failure and units that have Gompertz lifetime distribution. Moreover, we have seen that the proposed model can be easily extended for different populations. Also. we can mention that we can use a more general class of prior information such as priors with log-concave density functions. The simulation study is conducted, and the results are reported in Tables 1 and 2 which show the following:

(1) The proposed model is more acceptable
(2) The results for the large value of η^* are more acceptable in terms of MSEs, PCs, and AL
(3) The results are getting better for increasing values of increasing (s, m)
(4) The results are getting better for closed values of (n_1, n_2)
(5) The results under MLEs and noninformative priors are closed
(6) Bayesian estimation under informative prior is better than MLEs
(7) The results for the selected set of parameters are more acceptable

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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