Research Article

Hopf Bifurcation for a FitzHugh–Nagumo Model with Time Delay in a Network

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A reaction diffusion system is used to study the interaction between species in a population dynamic system. It is not only used in a population dynamic system with the diffusion phenomenon but also used in physical chemistry, medicine, and animal and plant protection. It has been studied by more and more scholars in recent years. The FitzHugh–Nagumo model is one of the most famous reaction-diffusion models. This article takes a deeper look at a FitzHugh–Nagumo model in a network with time delay. Firstly, we studied the linear stability of the equilibrium, then the existence of Hopf bifurcation is given, and finally, the stability of the Hopf bifurcation is introduced.

1. Introduction

Reaction-diffusion equations are widely used in ecology and biology fields [1–3]. The Hopf bifurcation is very useful in practice; in [4–7], the Turing instability is analyzed. Turing [8] first introduced the coupled reaction-diffusion equations; he pointed out that the stable process could evolve into an instability with diffusive effects. As a time- and space-breaking symmetry, Turing–Hopf bifurcation is a well-known mechanism for generating spatiotemporal patterns [9, 10].

The FitzHugh–Nagumo model is one of the most famous reaction-diffusion models; Hodgkin and Huxley first introduced the FitzHugh–Nagumo model. They used the model to study electrical impulses on nerve fibers [11–13]. In the FitzHugh–Nagumo model, the variables of equations for the voltage r and s have many forms, and the following form is commonly used:

$$\begin{cases} \frac{\partial r}{\partial t} = \frac{\partial^2 r}{\partial x^2} - r + I + f(r), \\ \frac{\partial s}{\partial t} = ar - bs. \end{cases} \quad (1)$$

The function $f(r)$ is positive feedback, and it is a third-order polynomial; however, the function r is negative feedback. In [14–18], the pattern formation in the FitzHugh–Nagumo system is studied frequently. In 2001, Medvedev and Kopell [19] introduced the FitzHugh–Nagumo oscillators with coupled electrically, and the synchronization and transience are obtained; in 2004, Kostova et al. [20] investigated the revised FitzHugh–Nagumo equation. The FitzHugh–Nagumo model with time delays is considered by C. J. in 2013 [21]. Based on the abovementioned research, we discussed the FitzHugh–Nagumo model with delay as follows:

$$\begin{cases} \frac{\partial r}{\partial t} = -r^3(t, x) + (\alpha + 1)r^2(t, x) - \alpha r(t, x) - s(t - \tau, x) + D_r \Delta r, \\ \frac{\partial s}{\partial t} = a\beta r(t, x) - as(t, x) + bD_s \Delta s. \end{cases} \quad (2)$$

In the abovementioned system, $r(t, x)$ is the membrane potential, $s(t, x)$ is the recovery variable, D_r and D_s are the diffusive coefficients of $r(t, x)$ and $s(t, x)$, respectively, α is the excitatory threshold, b is the diffusion coefficient, a is the excitability and β, a are parameters, they can change the

rest-state and dynamics, Δ denotes the Laplacian operator which describes the diffusion, and τ represents the time delay. For detail, one can see [22]. This model is based on a

spatially continuous domain, that is, the entire landscape is spatially continuous; in this paper, we introduce a network that owns system (2).

$$\begin{cases} \dot{r}_i(t) = -r_i^3(t) + (\alpha + 1)r_i^2(t) - \alpha r_i(t) - s_i(t - \tau) + D_r \sum_{j=1}^n L_{ij}r_j(t), \\ \dot{s}_i(t) = a\beta r_i(t) - as_i(t) + D_s \sum_{j=1}^n L_{ij}s_j(t), \end{cases} \quad (3)$$

where $G = (g_{ij})_{n \times n}$ is an adjacency matrix, whose entry g_{ij} equals 1 if there exists an edge from node j to node i and 0 otherwise. It is obvious that $g_{ii} = 0$ for $i = 1, 2, \dots, n$. We assume that G is strongly connected, i.e., for any pair of distinct nodes, there exists a path from one node to the other. It is known that the adjacency matrix G is irreducible [23]. The definition of the Laplacian matrix L is as follows:

$$L = \begin{pmatrix} -\sum_{k=1}^n g_{1k} & g_{12} & \cdots & g_{1n} \\ g_{21} & -\sum_{k=1}^n g_{2k} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & -\sum_{k=1}^n g_{nk} \end{pmatrix}. \quad (4)$$

The rest of this paper is designed as follows. In Section 2, we give the linear stability of the equilibrium and the existence of Hopf bifurcation. In Section 3, the stability of the Hopf bifurcation is introduced.

2. Stability of the Equilibrium and Hopf Bifurcation

By simple calculation, we know that equation (3) has three equilibria $(0, 0)$; $(\hat{r}_1, \hat{s}_1)$; and $(\hat{r}_2, \hat{s}_2)$, where

$$\begin{aligned} \hat{r}_1 &= \frac{\alpha + 1 + \sqrt{(\alpha + 1)^2 - 4\beta}}{2}, & \hat{s}_1 &= \beta\hat{r}_1, \\ \hat{r}_2 &= \frac{\alpha + 1 - \sqrt{(\alpha + 1)^2 - 4\beta}}{2}, & \hat{s}_2 &= \beta\hat{r}_2. \end{aligned} \quad (5)$$

The interior equilibrium exists if and only if

$$(\alpha + 1)^2 > 4\beta. \quad (6)$$

Now, we focus on the stability properties of the coexisting equilibrium $(\hat{r}_1, \hat{s}_1)$, $(\hat{r}_2, \hat{s}_2)$, we define the small perturbations $\delta r_i = r_i - \hat{r}_i$, $\delta s_i = s_i - \hat{s}_i$, and system (3) can be expressed as follows:

$$\begin{pmatrix} \delta \dot{r}_i \\ \delta \dot{s}_i \end{pmatrix} = J_1 \begin{pmatrix} \delta r_i \\ \delta s_i \end{pmatrix} + J_2 \begin{pmatrix} \delta r_{i,\tau} \\ \delta s_{i,\tau} \end{pmatrix} + D \begin{pmatrix} \sum_{j=1}^n L_{ij} \delta r_j \\ \sum_{j=1}^n L_{ij} \delta s_j \end{pmatrix}, \quad (7)$$

where $\delta r_{i,\tau} = \delta r_i(t - \tau)$ and

$$\begin{aligned} J_1 &= \begin{pmatrix} a_{11} & -1 \\ b\beta & -b \end{pmatrix}, \\ J_2 &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\ D &= \begin{pmatrix} D_r & 0 \\ 0 & D_s \end{pmatrix}, \end{aligned} \quad (8)$$

where $a_{11} = -3\hat{r}_1^2 + 2(\alpha + 1)\hat{r}_1 - \alpha$.

As in [24], we use $0 = \Lambda^1 > \Lambda^2 > \cdots > \Lambda^n$ to express the eigenvalues of the Laplacian matrix L and $\{\Phi^\gamma: \gamma = 1, 2, \dots, n\}$ to express each eigenvector relevant to a topological eigenvalue Λ^γ , so we can get $\sum_{j=1}^n L_{ij} \Phi_j^\gamma = \Lambda^\gamma \Phi_i^\gamma$. Let T be the transpose and $(\delta r_i(t), \delta s_i(t))^T$ express any small disturbance in the equilibrium state; we can have the following basis decomposition:

$$\begin{pmatrix} \delta r_i(t) \\ \delta s_i(t) \end{pmatrix} = \sum_{\alpha=1}^n \begin{pmatrix} c_1^\gamma \\ c_2^\gamma \end{pmatrix} e^{\lambda_\gamma t} \Phi_\gamma^\alpha, \quad (i = 1, 2, \dots, n). \quad (9)$$

Inserting (9) into (7), using the orthogonality of the eigenvectors, we get

$$\lambda_\gamma \begin{pmatrix} c_1^\gamma \\ c_2^\gamma \end{pmatrix} e^{\lambda_\gamma t} = J_1 \begin{pmatrix} c_1^\gamma \\ c_2^\gamma \end{pmatrix} e^{\lambda_\gamma t} + J_2 \begin{pmatrix} c_1^\gamma \\ c_2^\gamma \end{pmatrix} e^{\lambda_\gamma (t-\tau)} + \Lambda^\gamma D \begin{pmatrix} c_1^\gamma \\ c_2^\gamma \end{pmatrix} e^{\lambda_\gamma t}. \quad (10)$$

So, the characteristic equation can be written as

$$\Delta(\lambda_{\gamma,\tau}) = \det(\lambda_\gamma I - (J_1 + e^{-\lambda_\gamma \tau} J_2 + \Lambda^\gamma D)). \quad (11)$$

Substituting (8) into (11), the characteristic equation of (3) can be written as

$$\lambda_\gamma^2 - (b + \Lambda^\gamma D_s + a_{11} + \Lambda^\gamma D_r) \lambda_\gamma + b\beta + b\beta b^{-\lambda_\gamma \tau} = 0. \quad (12)$$

Lemma 1. *If*

$$\begin{aligned} 3\hat{r}^2 + \alpha &> b, \\ 2b\beta &> A_1^2, \end{aligned} \quad (13)$$

and (6) holds, then we can get the following:

- (1) When $\tau = 0$, all characteristic roots of (12) have negative real parts
- (2) When $\tau > 0$, the characteristic equation (12) has a pair of purely imaginary roots $\pm i\omega^*$ at $\tau = \tau_j$, where

$$\omega^{*2} = 2b\beta - A_1^2,$$

$$A_1 = b + \Lambda^y D_s + a_{11} + \Lambda^y D_r,$$

$$\tau_j = \frac{1}{\omega^*} \left(2j\pi + \arccos \left(\frac{\omega^{*2} - b\beta}{b\beta} \right) \right), \quad j = 0, 1, 2, \dots \quad (14)$$

Proof. (1) When $\tau = 0$, the characteristic equation (12) can be reduced to

$$\lambda_\gamma^2 - \lambda_\gamma [e + \Lambda^y D_s - 3\hat{r}^2 + 2(\alpha + 1)\hat{r} - a + \Lambda^y D_r] + 2b\beta = 0. \quad (15)$$

Considering (13), we can obtain that $b + \Lambda^y D_s - 3\hat{r}^2 + 2(\alpha + 1)\hat{r} - a + \Lambda^y D_r < 0$ and $2b\beta > 0$. Therefore, the characteristic equation (12) has a root with a negative real part (3). Suppose $\pm i\omega$ are a pair of pure imaginary roots of (12). Substituting $\pm i\omega$ into (12), we can get

$$-\omega^2 + iA_1\omega + b\beta + b\beta(\cos \omega\tau i \sin \omega\tau) = 0, \quad (16)$$

and through separating real and imaginary parts of the root, we can get

$$\begin{aligned} b\beta \cos \omega\tau + b\beta - \omega^2 &= 0, \\ A_1\omega \pm b\beta \sin \omega\tau &= 0. \end{aligned} \quad (17)$$

It is noticed that $\cos^2 \omega\tau + \sin^2 \omega\tau = 1$, and we can get

$$\omega^2(\omega^2 + (A_1^2 - 2b\beta)) = 0. \quad (18)$$

Equation (13) has a unique positive real root ω^{*2} if and only if (13) holds.

Theorem 1.

- (1) When $\tau \in [0, \tau_0)$, system (3) is locally stable at the equilibrium $(\hat{u}, \hat{v})$ if (6) and (13) hold
- (2) When $\tau \in [\tau_0, \infty)$, system (3) is unstable at the equilibrium $(\hat{u}, \hat{v})$ if (6) and (13) hold
- (3) We can find the constant $\sigma > 0$ and a smooth curve $\lambda(\tau): (\tau_0 - \sigma, \tau_0 + \sigma) \rightarrow C$ when $\lambda(\tau_0) = i\omega^*$ and $\Delta(i\omega^*, \tau_0) = 0$ for all $\tau \in (\tau_0 - \sigma, \tau_0 + \sigma)$

Proof. We refer to the Theorem 3.6 in the work of Ruan [25]. In consideration of that τ_0 is the smallest value when the root of equation (12) has zero real part, when $t \in [0, \tau_0)$, the real parts of the characteristic roots are all negative. Through using G. J. Butcher's Lemma [26], we know that $(\hat{u}, \hat{v})$ is locally stable when $\tau \in [0, \tau_0)$.

Now, we check the transversality condition

$$\frac{d}{d\tau} \text{Re} \lambda(\tau)|_{\tau=\tau_0} > 0. \quad (19)$$

Suppose $\lambda_\gamma = \mu + i\omega$ is the eigenvalue of (12), and by substituting λ_γ into (12), we can obtain

$$\begin{aligned} \mu^2 - \omega^2 + A_1\mu + b\beta + b\beta b^{-\mu\tau} \cos \omega\tau &= 0, \\ 2\mu\omega + A_1\omega - b\beta b^{-\mu\tau} \sin \omega\tau &= 0. \end{aligned} \quad (20)$$

Differentiating (20) with respect to τ , we have

$$\begin{aligned} (2\mu + A_1 - b\beta\tau b^{-\mu\tau} \cos \omega\tau) \frac{d\mu}{d\tau} - (2\omega + b\beta\tau b^{-\mu\tau} \sin \omega\tau) \frac{d\omega}{d\tau} &= b\beta b^{-\mu\tau} (\omega \sin \omega\tau + \mu \cos \omega\tau), \\ (2\omega + b\beta\tau b^{-\mu\tau} \sin \omega\tau) \frac{d\mu}{d\tau} + (2\mu + A_1 - A_2\tau b^{-\mu\tau} \cos \omega\tau) \frac{d\omega}{d\tau} &= b\beta b^{-\mu\tau} (\omega \cos \omega\tau + \mu \sin \omega\tau). \end{aligned} \quad (21)$$

Eliminating the terms $(d\omega/d\tau)$ from the above-mentioned equation, we get

$$\begin{aligned} [(2\mu + A_1 - b\beta\tau e^{-\mu\tau} \cos \omega\tau)^2 + (2\omega + b\beta\tau e^{-\mu\tau} \sin \omega\tau)^2] \frac{d\mu}{d\tau} &= b\beta b^{-\mu\tau} [(\omega \sin \omega\tau + \mu \cos \omega\tau)(2\mu + A_1 - b\beta\tau b^{-\mu\tau} \cos \omega\tau)] \\ &\quad + b\beta b^{-\mu\tau} [(\mu \sin \omega\tau + \omega \cos \omega\tau)(2\omega + b\beta\tau b^{-\mu\tau} \sin \omega\tau)]. \end{aligned} \quad (22)$$

To verify $(d\mu/d\tau) > 0$, we just need to verify that

$$(\omega \sin \omega \tau + \mu \cos \omega \tau)(2\mu + A_1 - b\beta \tau b^{-\mu\tau} \cos \omega \tau) + (\mu \sin \omega \tau + \omega \cos \omega \tau)(2\omega + b\beta \tau b^{-\mu\tau} \sin \omega \tau) > 0. \quad (23)$$

When $\tau = \tau_0, \mu = 0, \omega = \omega^*$, (20) becomes

$$\begin{aligned} b\beta \cos \omega^* \tau_0 &= \omega^{*2} - b\beta, \\ b\beta \sin \omega^* \tau_0 &= A_1 \omega^*. \end{aligned} \quad (24)$$

By some simple calculations, we can get

$$\omega^{*4} + (A_1^2 - 2b\beta)\omega^{*2} = 0. \quad (25)$$

So, (23) becomes

$$\omega^* \sin \omega^* \tau_0 (A_1 \mp b\beta \tau_0 \cos \omega^* \tau_0) + \omega^* \cos \omega^* \tau_0 (2\omega^* \pm b\beta \tau_0 \sin \omega^* \tau_0) > 0. \quad (26)$$

We must verify

$$A_1 \omega^* \sin \omega^* \tau_0 + 2\omega^* \cos \omega^* \tau_0 > 0. \quad (27)$$

Substituting (25) into (27) yields

$$\frac{1}{eb} A_1^2 \omega^{*2} - \frac{1}{eb} [2\omega^{*2} (A_2 - \omega^{*2})] > 0. \quad (28)$$

Since $b\beta > 0$, we need only to verify that

$$2\omega^{*4} + (A_1^2 - 2b\beta)\omega^{*2} = 2\omega^{*4} - \omega^{*4} = \omega^{*4} > 0. \quad (29)$$

The abovementioned inequality holds, so system (3) undergoes a Hopf bifurcation at $\tau = \tau_0$.

By applying the implicit function to $\Delta(\lambda, \tau)$, we can find the constant $\sigma > 0$ and a smooth curve $\lambda(\tau): (\tau_0 - \sigma, \tau_0 + \sigma) \rightarrow \mathbb{C}$ when $\lambda(\tau_0) = i\omega^*$ and $\Delta(i\omega^*, \tau_0) = 0$ for all $\tau \in (\tau_0 - \sigma, \tau_0 + \sigma)$.

Now, by using proposition 6.5 in [28], we obtain that $(\hat{u}, \hat{v})$ is locally unstable for $\tau \in [\tau_0, \infty)$.

3. The Stability of the Hopf Bifurcation

As a matter of convenience, let $\tau = \tau_0 + \nu$. Then, $\nu = 0$ is the Hopf bifurcation value for system (3). Let $r_i - \hat{r} = \bar{r}_i, s_i - \hat{s} = \bar{s}_i$. For convenience, we leave off the bars of $\bar{r}_i$ and $\bar{s}_i$. System (3) can be rewritten as

$$\begin{cases} \dot{r}_i(t) = a_{11}r_i(t) - s_i - s_i(t - \tau) + (2(\alpha + 1) - 6\hat{r})r_i^2(t) + D_r \sum_{j=1}^n L_{ij}r_j(t), \\ \dot{s}_i(t) = b\beta r_i(t) - bs_i(t) + D_s \sum_{j=1}^n L_{ij}s_j(t). \end{cases} \quad (30)$$

We rewrite the abovementioned equation as follows through using the orthogonality of the eigenvectors.

$$\begin{cases} \dot{r}_i(t) = a_{11}r_i(t) - s_i - s_i(t - \tau) + (2(\alpha + 1) - 6\hat{r})r_i^2(t) + D_r \Lambda^\alpha r_i(t), \\ \dot{s}_i(t) = b\beta r_i(t) - bs_i(t) + D_s \Lambda^\alpha s_i(t). \end{cases} \quad (31)$$

We use $X(t) = (r_i(t), s_i(t))^T$ to denote the solution of (31) and let $X_t(\theta) = X(t + \theta), \theta \in [-\tau, 0]$, where T denotes the transpose, $X_t \in C([-\tau, 0], \mathbb{R}^2) \triangleq C$. System (31) can become the functional differential equation as follows:

$$\dot{X}_t = L_\nu(X_t) + R_\nu(X_t), \quad (32)$$

where $L_\nu: C \rightarrow \mathbb{R}^2, R_\nu: C \rightarrow \mathbb{R}^2$; they have the following forms:

$$\begin{aligned} L_\nu(\phi) &= M_0 \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + M_1 \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix}, \\ M_0 &= \begin{pmatrix} a_{11} + D_r \Lambda^\nu & -1 \\ b\beta & -b + D_s \Lambda^\nu \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\ R_\nu(\phi) &= \begin{pmatrix} (-6\hat{r} + 2(\alpha + 1))\phi_1^2(0) \\ 0 \end{pmatrix}. \end{aligned} \quad (33)$$

We can let

$$\eta(\theta, \nu) = M_0 \delta(\theta) + M_1 \delta(\theta + \tau), \quad (34)$$

where $\delta(\cdot)$ is the Dirac function, and we can get

$$L_\nu(\phi) = \int_{-\tau}^0 \phi(\theta) d\eta(\theta, \nu), \quad \text{for } \phi \in C, \quad (35)$$

where

$$d\eta(\theta, \nu) = M_0 \delta(\theta) d\theta + M_1 \delta(\theta + \tau) d\theta. \quad (36)$$

Then, two operators A_ν and F_ν on $C^1([-\tau, 0], R^2) \triangleq C^1$ can be defined by

$$(A_\nu \phi)(\theta) = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-\tau, 0), \\ \int_{-\tau}^0 \phi(\xi) d\eta(\xi, \nu), & \theta = 0, \end{cases} \quad (37)$$

$$(F_\nu \phi)(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ R_\nu(\phi), & \theta = 0. \end{cases}$$

Then, (32) is transformed into

$$\dot{X}_t = A_\nu(X_t) + F_\nu(X_t), \quad (38)$$

$$(A_\nu^* \psi)(\theta^*) = \begin{cases} \frac{d\psi}{d\theta^*}, & \theta^* \in (0, \tau], \\ \int_{-\tau}^0 \psi(-\xi) d\eta^T(\xi, \nu), & \theta^* = 0, \end{cases} \quad (39)$$

where $\psi \in C^1([0, \tau], R^2)$. We can define the bilinear form by taking the solution space of (38) as the complex space C^2 .

$$\langle \psi(\theta^*), \phi(\theta) \rangle = \bar{\psi}^T(0) \phi(0) - \int_{\theta=-\tau}^0 \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \quad (40)$$

for $\phi \in C([-\tau, 0], C^2)$, $\psi \in C^1([0, \tau], C^2)$, where $\eta(\theta) = \eta(\theta, 0)$.

From the proof in Lemma 1, we can get that $\pm i\omega^*$ are eigenvalues of (12); thus, $\pm i\omega^*$ are the eigenvalues of A_0 and A_0^* , respectively. We set

$$A_0 q(\theta) = i\omega^* q(\theta), \quad \text{and } A_0^* q^*(\theta^*) = -i\omega^* q^*(\theta^*), \quad (41)$$

and we know that $\langle q^*, q \rangle = 1$ and $\langle q^*, \bar{q} \rangle = 0$. Let

$$q(\theta) = q(0) b^{i\omega^* \theta} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} b^{i\omega^* \theta}, \quad \text{and } q^*(\theta^*) = q^*(0) b^{i\omega^* \theta^*} = \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} b^{i\omega^* \theta^*}, \quad (42)$$

for $\theta \in [-\tau, 0]$, $\theta^* \in (0, \tau)$.

So, we have

$$q(\theta) = \begin{pmatrix} 1 \\ b\beta \\ i\omega^* + b \end{pmatrix} b^{i\omega^* \theta}, \quad \text{and } q^*(\theta^*) = \rho \begin{pmatrix} \frac{b\beta}{-i\omega^* - a_{11}} \\ 0 \end{pmatrix} b^{i\omega^* \theta^*}. \quad (43)$$

In order to ensure $\langle q^*(s), q(\theta) \rangle = 1$, we should determine the value of ρ , and from (40), we get

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{q}^*(0) q(0) - \int_{\theta=-\tau}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta) d\eta(\theta) q(\xi) d\xi, \\ &= \bar{\rho}(\bar{q}_1^*)(q_1, q_2)^T - \int_{-\tau}^0 \int_0^{\theta} \bar{\rho}(\bar{q}_1^*, \bar{q}_2^*) b^{-i\omega^*(\xi-\theta)} d\eta(\theta) (q_1, q_2)^T b^{i\omega^* \xi} d\xi, \\ &= \bar{\rho}(\bar{q}_1^* + q_2 + \tau_0 b^{-i\omega^*} \tau_0 a_{13} q_2), \\ &= \bar{\rho}(\bar{q}_1^* + q_2 - \tau_0 b^{-i\omega^*} \tau_0 q_2). \end{aligned} \quad (44)$$

Therefore, we can let

$$\bar{\rho} = \frac{1}{(\bar{q}_1^* + q_2 - \tau_0 b^{-i\omega^*} \tau_0 q_2)}. \quad (45)$$

Then, we construct a coordinate for the center manifold C_0 at $\nu = 0$ by using q and q^* ; the method is similar to

that in [27]. Let $x_t = x_t(\theta)$ be a solution of (38), and we define

$$z(t) \triangleq \langle q^*, x_t \rangle, \quad (46)$$

$$U(t, \theta) \triangleq x_t(\theta) - 2\text{Re}\{z(t)q(\theta)\}. \quad (47)$$

We further get

$$U(t, \theta) = U(z(t), \bar{z}(t), \theta)$$

$$\triangleq U_{20}(\theta) \frac{z^2}{2} + U_{11}(\theta) z \bar{z} + U_{02}(\theta) \frac{\bar{z}^2}{2} + U_{30}(\theta) \frac{z^3}{6} + \dots, \quad (48)$$

where z and $\bar{z}$ are the local coordinates of the center manifold C_0 in directions q and q^* , respectively.

We only take real solutions into account. For a solution $X(t) \in C_0$ of (26) with $\nu = 0$, from (47), we have

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{x}_t \rangle, \\ &= i\omega^* z + \bar{q}^*(0) F_0(U(z, \bar{z}, \theta) + 2\text{Re}\{z(t)q(\theta)\}), \\ &\triangleq i\omega^* z + \bar{q}^*(0) F_0(z, \bar{z}). \end{aligned} \quad (49)$$

Thus, (49) can become

$$\dot{z}(t) = i\omega^* z + g(z, \bar{z}), \quad (50)$$

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) F_0(z, \bar{z}), \\ &\triangleq \frac{g_{20}}{2} z^2 + g_{11} z \bar{z} + \frac{g_{02}}{2} \bar{z}^2 + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} \\ &\quad + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + \dots. \end{aligned} \quad (51)$$

Lemma 2. If we set

$$k_1 = -6\hat{u} + 2(\alpha + 1), \quad (52)$$

then g_{ij} can be explicitly expressed by (55).

Proof. Noticing (47) and (48), we obtain

$$\begin{aligned} x_t(\theta) &= U(t, \theta) + 2\text{Re}\{z(t)q(\theta)\}, \\ &= U_{20}(\theta) \frac{z^2}{2} + U_{11}(\theta) z \bar{z} + U_{02}(\theta) \frac{\bar{z}^2}{2} + zq + \bar{z}\bar{q} + \dots, \\ &= U_{20}(\theta) \frac{z^2}{2} + U_{11}(\theta) z \bar{z} + U_{02}(\theta) \frac{\bar{z}^2}{2} + (1, q_2)^T b^{i\omega^* \theta} \\ &\quad + (1, \bar{q}_2)^T e^{-i\omega^* \theta} \bar{z} + \dots. \end{aligned} \quad (53)$$

In view of (52), substituting (53) into (51),

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^* F_0(z, \bar{z}) = \bar{q}^*(0) F_0(x_t(\theta)), \\ &= \bar{p}(\bar{q}_1^*, 1) \begin{pmatrix} K_1 \varphi_1^{(0)}(0) \\ 0 \end{pmatrix}, \end{aligned} \quad (54)$$

$$\begin{aligned} g_{20} &= 2\bar{p}\bar{q}^* k_1, \\ g_{11} &= 2\bar{p}\bar{q}^* k_1, \\ g_{02} &= 2\bar{p}\bar{q}^* k_1, \\ g_{30} &= 6\bar{p}\bar{q}^* k_1 U_{20}^{(1)}(0), \\ g_{21} &= 2\bar{p}\bar{q}^* k_1 [U_{20}^{(1)}(0) + 2U_{11}^{(0)}(0)], \\ g_{12} &= 2\bar{p}\bar{q}^* k_1 [U_{02}^{(1)}(0) + 2U_{11}^{(1)}(0)], \\ g_{03} &= 6\bar{p}\bar{q}^* k_1 [U_{02}^{(1)}(0)], \end{aligned} \quad (55)$$

where $U_{20}^{(k)}, U_{11}^{(k)}$, and $U_{02}^{(k)}$ are the k -th components of U_{20}, U_{11} , and U_{02} , respectively, for $-\tau \leq \theta \leq 0$. The results in the theorem are valid.

Next, we will get the values of $U_{20}(\theta)$ and $U_{11}(\theta)$. From (38) and (47), we obtain

$$\dot{U} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} A_0 U - 2\text{Re}\{\bar{q}^*(0)F_0 q(\theta)\}, & -\tau \leq \theta < 0, \\ A_0 U - 2\text{Re}\{\bar{q}^*(0)F_0 q(\theta)\} + F_0, & \theta = 0, \end{cases} \quad (56)$$

where

$$K(z, \bar{z}, \theta) = K_{20} \frac{z^2}{2} + K_{11} z \bar{z} + K_{02} \frac{\bar{z}^2}{2} + \dots. \quad (57)$$

From (48), we have

$$\begin{aligned} \dot{U} &= \dot{U}_z \dot{z}(t) + \dot{U}_{\bar{z}} \dot{\bar{z}}(t) = (U_{20}(\theta)z + U_{11}(\theta)\bar{z} + \dots)(i\omega^* z(t) + g(z, \bar{z})) \\ &\quad + (U_{11}(\theta)z + U_{02}(\theta)\bar{z} + \dots)(-i\omega^* \bar{z}(t) + g(z, \bar{z})). \end{aligned} \quad (58)$$

Substituting (48) into (56), we can get that

$$\begin{aligned}\dot{U} &= A_0 \left(U_{20}(\theta) \frac{z^2}{2} + U_{11}(\theta) z\bar{z} + U_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \right) + K_{20}(\theta) \frac{z^2}{2} + K_{11}(\theta) z\bar{z} + K_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \\ &= (A_0 U_{20}(\theta) + K_{20}(\theta)) \frac{z^2}{2} + (A_0 U_{11}(\theta) + K_{11}(\theta) z\bar{z}) + (A_0 U_{02}(\theta) + K_{02}(\theta) \frac{\bar{z}^2}{2}) + \dots.\end{aligned}\quad (59)$$

By comparing the coefficients of z^2 and $z\bar{z}$ from (58) and (59), we get

$$\begin{aligned}(A_0 - 2i\omega^* I)U_{20}(\theta) &= -K_{20}(\theta), \\ A_0 U_{11}(\theta) &= -K_{11}(\theta).\end{aligned}\quad (60)$$

For $\theta \in (0, \tau]$, it follows from (47), (48), (57), and (58) that

$$\begin{aligned}K(z, \bar{z}, \theta) &= -2\text{Re}\{\bar{q}^*(0)F_0 q(\theta)\}, \\ &= -\bar{q}^* F_0 q(\theta) - q^*(0)\bar{F}_0 \bar{q}(\theta), \\ &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta), \\ &= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \dots\right)q(\theta) - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \dots\right)\bar{q}(\theta).\end{aligned}\quad (61)$$

By comparing the coefficients of z^2 and $z\bar{z}$ from (57) and (61), we get

$$\begin{aligned}K_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ K_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).\end{aligned}\quad (62)$$

Thus, we can get

$$\dot{U}_{20}(\theta) = 2i\omega^* U_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \quad (63)$$

Since $q(\theta) = (q_1, q_2)^T e^{i\omega^* \theta}$, we obtain

$$U_{20}(\theta) = \frac{ig_{20}}{\omega^*} q(0)e^{i\omega^* \theta} + \frac{i\bar{g}_{02}}{\omega^*} \bar{q}(0)e^{-i\omega^* \theta} + G_1 e^{2i\omega^* \theta}, \quad (64)$$

where $G_1 = (G_1^{(1)}, G_1^{(2)})^T$ is a constant vector. We also can have

$$U_{11}(\theta) = \frac{-ig_{11}}{\omega^*} q(0)e^{i\omega^* \theta} + \frac{i\bar{g}_{11}}{\omega^*} \bar{q}(0)e^{-i\omega^* \theta} + G_2, \quad (65)$$

where $G_2 = (G_2^{(1)}, G_2^{(2)})^T$ is a constant vector.

From the definition of A_0 and (60), we have

$$\int_{-\tau}^0 d\eta(\theta) U_{20}(\theta) = 2i\omega^* U_{20}(0) - K_{20}(0), \quad (66)$$

and

$$\int_{-\tau}^0 d\eta(\theta) U_{11}(\theta) = -K_{11}(0), \quad (67)$$

where $\eta(\theta) = \eta(\theta, 0)$. From (56), when $\theta = 0$, we get

$$\begin{aligned}K(z, \bar{z}, 0) &= -2\text{Re}\{\bar{q}^*(0)F_0 q(0)\} + F_0, \\ &= -g(z, \bar{z})q(0) - \bar{g}(z, \bar{z})\bar{q}(0) + F_0.\end{aligned}\quad (68)$$

Then, we obtain

$$\begin{aligned}K(z, \bar{z}, 0) &= -2\text{Re}\{\bar{q}^*(0)F_0 q(0)\} + F_0, \\ &= -g(z, \bar{z})q(0) - \bar{g}(z, \bar{z})\bar{q}(0) + F_0.\end{aligned}\quad (69)$$

Therefore, we obtain

$$K_{20}\frac{z^2}{2} + K_{11}z\bar{z} + K_{02}\frac{\bar{z}^2}{2} + \dots = -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \dots\right)q(0) - \left(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \dots\right)\bar{q}(0) + F_0. \quad (70)$$

Comparing both sides of (70), we have

$$K_{20} = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\begin{pmatrix} k_1 \\ 0 \end{pmatrix}, \quad (71)$$

$$K_{11} = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\begin{pmatrix} 2k_1 \\ 0 \end{pmatrix}. \quad (72)$$

Since $i\omega^*$ is the eigenvalue of A_0 and $q(0)$ is the corresponding eigenvector, we get

$$\begin{aligned} \left(i\omega^* I - \int_{-\tau}^0 e^{i\omega^*\theta} d\eta(\theta) \right) q(0) &= 0, \\ \left(-i\omega^* I - \int_{-\tau}^0 e^{-i\omega^*\theta} d\eta(\theta) \right) \bar{q}(0) &= 0. \end{aligned} \quad (73)$$

Then, substituting (65) and (67) into (72), we obtain

$$\left(2i\omega^* I - \int_{-\tau}^0 e^{2i\omega^*\theta} d\eta(\theta) \right) G_1 = 2\begin{pmatrix} k_1 \\ 0 \end{pmatrix}, \quad (74)$$

that is,

$$K^* G_1 = 2\begin{pmatrix} k_1 \\ 0 \end{pmatrix}, \quad (75)$$

where

$$K^* = \begin{pmatrix} 2i\omega^* - a_{11} - D_r\Lambda^\alpha & e^{-2i\omega^*\tau} \\ 0 & 2i\omega^* - D_s\Lambda^\alpha \end{pmatrix}. \quad (76)$$

Thus,

$$\begin{aligned} G_1^{(1)} &= \frac{\begin{pmatrix} 2k_1 & e^{-2i\omega^*\tau} \\ 0 & 2i\omega^* - D_s\Lambda^\alpha \end{pmatrix}}{\text{Det}(K^*)}, \\ G_1^{(2)} &= \frac{\begin{pmatrix} 2i\omega^* - a_{11} - D_s\Lambda^\alpha & 2k_1 \\ 0 & 0 \end{pmatrix}}{\text{Det}(K^*)}. \end{aligned} \quad (77)$$

Therefore, we can get the values of $U_{20}(\theta)$ and $U_{11}(\theta)$ from (64) and (65).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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