Research Article

The Second-Order Differential Equation System with the Controlled Process for Variational Inequality with Constraints

Li Wang , Xingxu Chen, and Juhe Sun

School of Science, Shenyang Aerospace University, Shenyang 110136, China

Correspondence should be addressed to Li Wang; liwang211@163.com

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In this paper, the variational inequality with constraints can be viewed as an optimization problem. Using Lagrange function and projection operator, the equivalent operator equations for the variational inequality with constraints under the certain conditions are obtained. Then, the second-order differential equation system with the controlled process is established for solving the variational inequality with constraints. We prove that any accumulation point of the trajectory of the second-order differential equation system is a solution to the variational inequality with constraints. In the end, one example with three kinds of different cases by using this differential equation system is solved. The numerical results are reported to verify the effectiveness of the second-order differential equation system with the controlled process for solving the variational inequality with constraints.

1. Introduction

We consider the variational inequality with constraints, denoted by VIP (K, F) : find $x^* \in K$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K, \quad (1)$$

where $K = \{x \in \Omega | g(x) \leq 0\}$, $F: \mathcal{R}^n \rightarrow \mathcal{R}^n$ is a monotone mapping, $g: \mathcal{R}^n \rightarrow \mathcal{R}^m$ is a convex and differentiable mapping, and $\Omega \subseteq \mathcal{R}^n$ is a nonempty closed convex set.

Variational inequality problems arise in physics, mechanics, economics, optimization, control, equilibrium model in transportation, and so forth. The finite dimensional variational inequality is an active field with abundant intension. In the past 30 years, great progress has been made in the study of numerical algorithms for solving the variational inequality problems and many numerical methods have emerged in large numbers, such as nonsmooth equation methods, smoothing methods, projection iteration methods, interior methods, multisplitting methods, homotopy approaches, Tikhonov regularization, and proximal point methods. The book by Facchinei and Pang [1] is the most famous reference to give a detailed numerical treatment of variational inequality problems and complementarity problems in the area of mathematical programming. But this

book did not include the differential equation approaches for solving variational inequality problems.

The differential equation method is used to solve constrained nonlinear optimization problems, which was firstly proposed by Arrow and Hurwicz [2]; and the equity-constrained optimization problem was considered. Fiacco and McCormick [3] studied the constraint qualification by using differential equation method. Evtushenko [4] studied the problem of equality constraint earlier. Yamashita [5], Evtushenko et al. [6–10], and Pan [11] developed and improved the differential equation methods. In particular, Evtushenko and Zhadan have carried out a great deal of research on differential equation methods for nonlinear programming problems and constraint problems on general closed sets by using the stability theory of differential equations since 1973, which enriches the differential equation method of nonlinear programming problem. There are other scholars who have done a lot of research work on differential equation methods, and they have established a variety of differential equation systems to solve the optimization problems. Brown et al. [12, 13] proposed the differential equation system based on the penalty function. Zhang and Constantinides [14] presented a differential equation system based on Lagrange function to solve the

equality constraint problem. Since the nonlinear Lagrange function can be used to construct a dual algorithm for solving nonlinear optimization problems and the dual algorithm has no limitation on the feasibility of the original variables, many differential equation systems involved dual variables, such as those in the works of Zhou et al. [15] and Jin et al. [16]. Recently, Jin et al. [17] formulated the differential systems based on the first derivatives and the second derivatives of the approximate augmented Lagrangian. Under suitable conditions, the asymptotic stability of the differential systems and local convergence properties of their Euler discrete schemes are analyzed. Zhang [18] studied the differential equation method for quadratic programming.

At the same time, there are a series of neural network methods for solving variational inequality and complementarity problems. In fact, the model of neural network is often described by the system of differential equations. Zhou et al. [19] transformed the nonlinear complementarity problem into an equivalent optimization problem, and a system of differential equations is constructed to solve the optimization problem. On the basis of projection operator, Gao et al. [20] presented a system of differential equations for solving variational inequality problems with linear and nonlinear constraints. He and Yang [21] studied a new neural network model for nonsymmetric linear variational inequality. The classical neural network model for solving linear variational inequality based on projection operator and compression method is extended. Liao et al. [22] studied the differential equation system of nonlinear complementarity problem. Nazemi and Sabeghi [23] applied a gradient neural network model to efficiently solve the convex second-order cone constrained variational inequality problem. Nazemi and Sabeghi [24] considered a new neural network model to simply solve the convex second-order cone constrained variational inequality problem.

Different from the above results, Antipin [25] considered the first-order and second-order differential equation systems for solving the minimization function on a simple set. The asymptotic and exponential stability of such processes was proved without using Lyapunov function. Furthermore, Antipin [26] studied the differential equation system and the controlled proximal differential equation system for solving the minimization function. The convergence theorems were obtained. Antipin and Antipin [27, 28] established the differential equation system and the controlled proximal differential equation system for solving the saddle problem and proved that the trajectory of the system converges monotonically in norm to one of the equilibrium points. Antipin [29] considered the differential equation system with the controlled process for solving the fixed point of the extremal mapping and proved that the trajectory of the system converges to the solution with exponential rate. Antipin [30] established the differential equation systems and the differential equation system with the controlled process for optimization problem, saddle problem, and the extremal mapping, respectively. The corresponding

convergence theorems were also proved. The variational inequality problem with coupled constraints and the fixed point of the extremal mapping with coupled constraints were studied by Antipin and Antipin [31, 32], where symmetric functions were introduced and the differential equation systems with the controlled process for global convergence were proposed. Wang and Wang [33] established the first-order differential equation system to solve the variational inequality with constraints problem (1).

We think that the research of differential equation method for solving variational inequality is not much. In the past, the problem is firstly transformed into a smooth unconstrained optimization problem; then the differential equation method is constructed by using negative gradient for the unconstrained optimization. Inspired by the work of Antipin, our approach is different, which directly starts from the nonsmooth system of equations to establish the differential equation method, which requires relatively weak conditions. At the same time, due to the fact that the differential equation system has mature numerical solution methods, such as Runge-Kutta method, using these pieces of numerical software to solve the problem of variational inequality may achieve good results in practical calculation. It is hoped that our research on the differential equation method for the variational inequality will contribute to the development of the neural network method for solving the variational inequality problem.

In this paper, we will establish the second-order differential equation system with the controlled process for solving the variational inequality with constraints problem (1). The remainder of this paper is organized as follows. In the next section, based on Lagrange function and projection operator, we prove a lemma that gives the equivalent operator equation under certain conditions for problem (1). In Section 3, we establish the second-order differential equation system with the controlled process for the variational inequality with constraints (1) and prove that any accumulation point of the trajectory of the second-order differential equation system is a solution to problem (1). One example with three kinds of different cases is solved by the second-order differential equation system with the controlled process in Section 4, and the transient behaviors of the trajectories of this kind of differential equation system for every case are illustrated.

2. Preliminaries

The projection operator to a convex set is quite useful in reformulating the variational inequality with constraints (1) as an equation.

Let C be a convex closed set; for every $x \in \mathfrak{R}^n$, there is a unique $\hat{x}$ in C such that

$$\|x - \hat{x}\| = \min\{\|x - y\| \mid y \in C\}. \quad (2)$$

The point $\hat{x}$ is the projection of x onto C , denoted by $\Pi_C(x)$. The projection operator $\Pi_C: \mathfrak{R}^n \rightarrow C$ is well defined over $\mathfrak{R}^n$ and it is a nonexpansive mapping.

Lemma 1 (see [34]). Let H be a real Hilbert space and let $C \subset H$ be a closed convex set. For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C, \quad (3)$$

if and only if $u - \Pi_C(z) = 0$.

In order to obtain the second-order differential equation system, we need to transform the variational inequality with constraints problem (1), which can be viewed as the following optimization problem:

$$\begin{aligned} \min & f(y) \\ \text{s.t.} & g(y) \leq 0, y \in \Omega, \end{aligned} \quad (4)$$

$$\begin{aligned} \langle F(x^*), x^* - x^* \rangle + \langle u, g(x^*) \rangle &\leq \langle F(x^*), x^* - x^* \rangle + \langle u^*, g(x^*) \rangle \\ &\leq \langle F(x^*), y - x^* \rangle + \langle u^*, g(y) \rangle, \end{aligned} \quad (6)$$

$\forall y \in \Omega$ and $u \in \mathfrak{R}_+^m$.

System (6) can be represented in an equivalent manner as

$$\begin{aligned} x^* &\in \arg \min \{ \langle F(x^*), y - x^* \rangle + \langle u^*, g(y) \rangle \mid y \in \Omega \}, \\ u^* &\in \arg \max \{ \langle u, g(x^*) \rangle \mid u \in \mathfrak{R}_+^m \}. \end{aligned} \quad (7)$$

Lemma 2. Suppose that $g: \mathfrak{R}^n \rightarrow \mathfrak{R}^m$ is a differentiable mapping and $\Omega \subset \mathfrak{R}^n$ is a nonempty closed convex set. Then the pair (x^*, u^*) is the saddle point of the Lagrange function $\mathcal{L}(x^*, y, u)$ if and only if (x^*, u^*) satisfies the following operator equation:

$$\begin{aligned} x^* &= \Pi_\Omega(x^* - \alpha(F(x^*) + \mathcal{J}g(x^*)^T u^*)), \\ u^* &= \Pi_+(u^* + \alpha g(x^*)), \end{aligned} \quad (8)$$

where $\Pi_+(\cdot)$ and $\Pi_\Omega(\cdot)$ are the operators that project a vector onto the positive octant R_+^m and the set Ω , respectively, and $\alpha > 0$ is a parameter.

Proof. System (4) or system (5) implies that (x^*, u^*) satisfies the following inequalities:

$$\begin{aligned} \langle F(x^*), y - x^* \rangle + \langle u^*, g(y) - g(x^*) \rangle &\geq 0, \quad \forall y \in \Omega, \\ \langle -g(x^*), u - u^* \rangle &\geq 0, \quad \forall u \in \mathfrak{R}_+^m. \end{aligned} \quad (9)$$

where $f(y) = \langle F(x^*), y - x^* \rangle$ and $f(y) \geq 0$.

Define the Lagrange function of the minimization problem (4):

$$\mathcal{L}(x^*, y, u) = \langle F(x^*), y - x^* \rangle + \langle u, g(y) \rangle, \quad \forall y \in \Omega, u \in \mathfrak{R}_+^m, \quad (5)$$

where y and u are the primal and dual variables, respectively.

Since x^* is the minimum of $f(y)$, the pair (x^*, u^*) (under certain regularity conditions) is a saddle point of $\mathcal{L}(x^*, y, u)$. That is,

If the function g is differentiable, we have

$$\begin{aligned} \langle F(x^*) + \mathcal{J}g(x^*)^T u^*, y - x^* \rangle &\geq 0, \quad \forall y \in \Omega, \\ \langle -g(x^*), u - u^* \rangle &\geq 0, \quad \forall u \in \mathfrak{R}_+^m, \end{aligned} \quad (10)$$

where $\mathcal{J}g(x^*)$ is the Jacobian of the mapping g at x^* .

The variational inequality system (10) can be computed by

$$\begin{aligned} \langle x^* - x^* + \alpha(F(x^*) + \mathcal{J}g(x^*)^T u^*), y - x^* \rangle &\geq 0, \\ \langle u^* - u^* - \alpha g(x^*), u - u^* \rangle &\geq 0, \quad \forall u \in \mathfrak{R}_+^m, \end{aligned} \quad (11)$$

where $\alpha > 0$ is a parameter.

With the help of projection operators and Lemma 1, the above system of variational inequalities is represented in the form of operator equation as

$$\begin{aligned} x^* &= \Pi_\Omega(x^* - \alpha(F(x^*) + \mathcal{J}g(x^*)^T u^*)), \\ u^* &= \Pi_+(u^* + \alpha g(x^*)), \end{aligned} \quad (12)$$

where $\Pi_+(\cdot)$ and $\Pi_\Omega(\cdot)$ are the operators that project a vector onto the positive octant R_+^m and the set Ω , respectively. This completes the proof. $\square$

Furthermore, if $g: \mathfrak{R}^n \rightarrow \mathfrak{R}^m$ is convex, we have for the second term in the first inequality of the variational inequality (10) that

$$\langle \mathcal{J}g(x^*)^T u^*, y - x^* \rangle = \langle u^*, \mathcal{J}g(x^*)(y - x^*) \rangle \leq \langle u^*, g(y) - g(x^*) \rangle, \quad (13)$$

for all $y \in \Omega$. Thus, we can rewrite the variational inequality (10) as follows:

$$\begin{aligned} \langle F(x^*), y - x^* \rangle + \langle u^*, g(y) - g(x^*) \rangle &\geq 0, \quad \forall y \in \Omega, \\ \langle -g(x^*), u - u^* \rangle &\geq 0, \quad \forall u \in \mathfrak{R}_+^m. \end{aligned} \quad (14)$$

Remark 1. When the function g is convex and differentiable, the relations in (6)–(14) are equivalent to each other.

3. Second-Order Differential Equation System

In this section, we consider the second-order differential equation system for solving the variational inequality with constraints problem (1).

The second-order differential equation system is constructed based on equation (8) as follows:

$$\begin{aligned} \mu_1 \frac{d^2 x}{dt^2} + \beta_1 \frac{dx}{dt} + x &= \Pi_{\Omega} \left(x - \alpha \left(F(x) + \mathcal{J}g(x)^T u \right) \right), \quad x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0, \\ \mu_2 \frac{d^2 u}{dt^2} + \beta_2 \frac{du}{dt} + u &= \Pi_+ (u + \alpha g(x)), \quad u(t_0) = u_0, \dot{u}(t_0) = \dot{u}_0, \end{aligned} \quad (15)$$

where $\mu_1 > 0, \beta_1 > 0, \mu_2 > 0, \beta_2 > 0$, and $\alpha > 0$ are parameters.

In order to provide the convergence of the trajectory of the second-order differential equation system to a saddle point (x^*, u^*) of the Lagrange function $\mathcal{L}(x^*, y, u)$, we

introduce the following second-order differential equation system with the controlled process inspired by Antipin et al. [25–32] and Wang and Wang [33].

$$\begin{aligned} \mu_1 \frac{d^2 x}{dt^2} + \beta_1 \frac{dx}{dt} + x &= \Pi_{\Omega} \left(x - \alpha \left(F(\bar{x}) + \mathcal{J}g(\bar{x})^T \bar{u} \right) \right), \quad x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0, \\ \mu_2 \frac{d^2 u}{dt^2} + \beta_2 \frac{du}{dt} + u &= \Pi_+ (u + \alpha g(\bar{x})), \quad u(t_0) = u_0, \dot{u}(t_0) = \dot{u}_0, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \bar{x} &= \Pi_{\Omega} \left(x - \alpha \left(F(x) + \mathcal{J}g(x)^T \bar{u} \right) \right), \\ \bar{u} &= \Pi_+ (u + \alpha g(x)). \end{aligned} \quad (17)$$

If $\mu_1 = \mu_2 = 0$ and $\beta_1 = \beta_2 = 1$, then system (16) coincides with (8) in the work of Wang and Wang [33].

For simplicity, we denote $\ddot{x} = (d^2 x/dt^2)$ and $\ddot{u} = (d^2 u/dt^2)$. Similarly, the second-order differential equation system with the controlled process (16) can be represented in the form of variational inequalities as follows:

$$\langle \mu_1 \ddot{x} + \beta_1 \dot{x} + \alpha (F(\bar{x}) + \mathcal{J}g(\bar{x})^T \bar{u}), y - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle \geq 0, \quad \forall y \in \Omega, \quad (18)$$

$$\langle \mu_2 \ddot{u} + \beta_2 \dot{u} - \alpha g(\bar{x}), v - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle \geq 0, \quad \forall v \in \mathfrak{R}_+^m, \quad (19)$$

$$\langle \bar{x} - x + \alpha (F(x) + \mathcal{J}g(x)^T \bar{u}), y - \bar{x} \rangle \geq 0, \quad \forall y \in \Omega, \quad (20)$$

$$\langle \bar{u} - u - \alpha g(x), v - \bar{u} \rangle \geq 0, \quad \forall v \in \mathfrak{R}_+^m. \quad (21)$$

Subsequently, we assume that g, F , and $\mathcal{J}g^T$ are Lipschitz continuous with the constants $|g|, |F|$, and $|\mathcal{J}|$, respectively. In addition, we suppose that $\|\bar{u}\| \leq C_0$ and C_0 is a constant.

With regard to these conditions and (16), it is easy to show the following inequalities since the projection operator is a nonexpansive mapping.

$$\begin{aligned}\|u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u}\| &= \|\Pi_+(u + \alpha g(\bar{x})) - \Pi_+(u + \alpha g(x))\| \\ &\leq \|u + \alpha g(\bar{x}) - u - \alpha g(x)\| \leq \alpha |g| \|\bar{x} - x\|,\end{aligned}\quad (22)$$

$$\begin{aligned}\|x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x}\| &= \|\Pi_\Omega(x - \alpha(F(\bar{x}) + \mathcal{J}g(\bar{x})^T \bar{u})) - \Pi_\Omega(x - \alpha(F(x) + \mathcal{J}g(x)^T \bar{u}))\| \\ &\leq \|x - \alpha(F(\bar{x}) + \mathcal{J}g(\bar{x})^T \bar{u}) - x + \alpha(F(x) + \mathcal{J}g(x)^T \bar{u})\| \\ &= \alpha \|F(x) - F(\bar{x}) + \mathcal{J}g(x)^T \bar{u} - \mathcal{J}g(\bar{x})^T \bar{u}\| \\ &\leq \alpha \|F(x) - F(\bar{x})\| + \alpha \|\mathcal{J}g(x)^T \bar{u} - \mathcal{J}g(\bar{x})^T \bar{u}\| \\ &\leq \alpha(|F| + C_0|\mathcal{J}|)\|x - \bar{x}\| = \alpha C \|x - \bar{x}\|,\end{aligned}\quad (23)$$

where $C = |F| + C_0|\mathcal{J}|$.

Now we discuss the convergence of the trajectories $x(t)$, $u(t)$, $\dot{x}(t)$, and $\dot{u}(t)$ of the second-order differential equation system (16).

Theorem 1. Suppose that the set of solutions Ω^* of problem (1) is nonempty, F is monotone and Lipschitz continuous with the constant $|F|$, and g is convex, differentiable, and Lipschitz continuous with the constant $|g|$. Suppose that $\|u(t)\| \leq C_0$ for all $t \rightarrow \infty$, $\|\dot{u}(t)\|$ is bounded for all $t \rightarrow \infty$, $\mathcal{J}g^T$ is Lipschitz continuous with the constant $|\mathcal{J}|$, and $\Omega \subseteq \mathbb{R}^n$ is a convex closed set. Then the accumulation point of the trajectory $x(t)$ of the second-order differential equation system (16) with $1 < \beta_1 < \mu_1 < (1/2)\beta_1^2$, $0 < \mu_2 < (1/2)\beta_2^2$, and $\alpha^2(|F| + C_0|\mathcal{J}|)^2 + |g|^2 < (1/2)$ is the solution to the variational inequality with constraints problem (1).

Proof. Setting $y = x^* \in \Omega^*$ in the variational inequality (18), we have

$$\langle \mu_1 \ddot{x} + \beta_1 \dot{x} + \alpha(F(\bar{x}) + \mathcal{J}g(\bar{x})^T \bar{u}), x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle \geq 0. \quad (24)$$

Let $y = x + \mu_1 \ddot{x} + \beta_1 \dot{x}$ in the variational inequality (20); we obtain

$$\langle \bar{x} - x + \alpha(F(x) + \mathcal{J}g(x)^T \bar{u}), x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \geq 0, \quad (25)$$

which implies that

$$\begin{aligned}&\langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle + \alpha \langle F(\bar{x}), x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ &\quad - \alpha \langle F(\bar{x}) - F(x), x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle + \alpha \langle \mathcal{J}g(\bar{x})^T \bar{u}, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ &\quad - \alpha \langle \mathcal{J}g(x)^T \bar{u} - \mathcal{J}g(\bar{x})^T \bar{u}, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \geq 0.\end{aligned}\quad (26)$$

Summing inequalities (24) and (26), we infer that

$$\begin{aligned}&\langle \mu_1 \ddot{x} + \beta_1 \dot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \alpha \langle F(\bar{x}), x^* - \bar{x} \rangle + \alpha \langle \mathcal{J}g(\bar{x})^T \bar{u}, x^* - \bar{x} \rangle \\ &\quad + \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle + \alpha \langle F(x) - F(\bar{x}), x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ &\quad + \alpha \langle \mathcal{J}g(x)^T \bar{u} - \mathcal{J}g(\bar{x})^T \bar{u}, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \geq 0.\end{aligned}\quad (27)$$

By using inequality (23), we compute the above inequality as follows:

$$\begin{aligned}&\langle \mu_1 \ddot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \beta_1 \dot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ &\quad + \alpha \langle F(\bar{x}), x^* - \bar{x} \rangle + \alpha \langle \mathcal{J}g(\bar{x})^T \bar{u}, x^* - \bar{x} \rangle + (\alpha C)^2 \|\bar{x} - x\|^2 \geq 0,\end{aligned}\quad (28)$$

where $C = |F| + C_0|\mathcal{J}|$.

Since g is convex, we transform the above inequality by using inequality (14) as follows:

$$\begin{aligned} & \langle \mu_1 \ddot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \beta_1 \dot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ & + \alpha \langle F(\bar{x}), x^* - \bar{x} \rangle + \alpha \langle \bar{u}, g(x^*) - g(\bar{x}) \rangle + (\alpha C)^2 \|\bar{x} - x\|^2 \geq 0. \end{aligned} \quad (29)$$

Letting $y = \bar{x}$ in the first inequality of (14), we have

$$\alpha \langle F(x^*), \bar{x} - x^* \rangle + \alpha \langle u^*, g(\bar{x}) - g(x^*) \rangle \geq 0. \quad (30)$$

Adding the above inequality into (29), we have

$$\begin{aligned} & \langle \mu_1 \ddot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \beta_1 \dot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle \\ & + \alpha \langle F(\bar{x}) - F(x^*), x^* - \bar{x} \rangle + \alpha \langle \bar{u} - u^*, g(x^*) - g(\bar{x}) \rangle + (\alpha C)^2 \|\bar{x} - x\|^2 \geq 0 \end{aligned} \quad (31)$$

Setting $v = u^*$ in the variational inequality (19) and $v = u + \mu_2 \ddot{u} + \beta_2 \dot{u}$ in the variational inequality (21), we deduce that

$$\begin{aligned} & \langle \mu_2 \ddot{u} + \beta_2 \dot{u}, u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle - \alpha \langle g(\bar{x}), u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle \geq 0, \\ & \langle \bar{u} - u, u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle + \alpha \langle g(\bar{x}) - g(x), u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle - \alpha \langle g(\bar{x}), u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle \geq 0. \end{aligned} \quad (32)$$

Summing the above two inequalities and taking into account (22), we obtain

$$\langle \mu_2 \ddot{u} + \beta_2 \dot{u}, u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle + \langle \bar{u} - u, u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle - \alpha \langle g(\bar{x}), u^* - \bar{u} \rangle + (\alpha |g|)^2 \|\bar{x} - x\|^2 \geq 0. \quad (33)$$

In view of $\langle \bar{u}, g(x^*) \rangle \leq 0$ and $\langle u^*, g(x^*) \rangle = 0$, we conclude that

$$\begin{aligned} & \langle \mu_2 \ddot{u} + \beta_2 \dot{u}, u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle + \langle \bar{u} - u, u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle \\ & + \alpha \langle g(x^*) - g(\bar{x}), u^* - \bar{u} \rangle + (\alpha |g|)^2 \|\bar{x} - x\|^2 \geq 0. \end{aligned} \quad (34)$$

Since F is monotone, we sum (31) and (34) to obtain

$$\begin{aligned} & \langle \mu_1 \ddot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle + \langle \beta_1 \dot{x}, x^* - x - \mu_1 \ddot{x} - \beta_1 \dot{x} \rangle \\ & + \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle + \langle \mu_2 \ddot{u}, u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle + \langle \beta_2 \dot{u}, u^* - u - \mu_2 \ddot{u} - \beta_2 \dot{u} \rangle \\ & + \langle \bar{u} - u, \mu_2 \ddot{u} + \beta_2 \dot{u} + u - \bar{u} \rangle + \alpha^2 (C^2 + |g|^2) \|\bar{x} - x\|^2 \geq 0. \end{aligned} \quad (35)$$

The above inequality yields that

$$\begin{aligned}
& \mu_1 \langle \ddot{x}, x - x^* \rangle + \mu_1^2 \|\ddot{x}\|^2 + \mu_1 \beta_1 \langle \ddot{x}, \dot{x} \rangle + \beta_1 \langle \dot{x}, x - x^* \rangle + \mu_1 \beta_1 \langle \dot{x}, \ddot{x} \rangle + \beta_1^2 \|\dot{x}\|^2 \\
& - \langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle + \mu_2^2 \|\ddot{u}\|^2 + \mu_2 \langle \ddot{u}, u - u^* \rangle + \mu_2 \beta_2 \langle \ddot{u}, \dot{u} \rangle + \beta_2 \langle \dot{u}, u - u^* \rangle \\
& + \mu_2 \beta_2 \langle \dot{u}, \ddot{u} \rangle + \beta_2^2 \|\dot{u}\|^2 - \langle \bar{u} - u, u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle - \alpha^2 (C^2 + |g|^2) \|\bar{x} - x\|^2 \leq 0.
\end{aligned} \tag{36}$$

Using the relations

$$\begin{aligned}
\langle \bar{x} - x, x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x} \rangle &= \frac{1}{2} \|\mu_1 \ddot{x} + \beta_1 \dot{x}\|^2 - \frac{1}{2} \|\bar{x} - x\|^2 - \frac{1}{2} \|x + \mu_1 \ddot{x} + \beta_1 \dot{x} - \bar{x}\|^2, \\
\langle \bar{u} - u, u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u} \rangle &= \frac{1}{2} \|\mu_2 \ddot{u} + \beta_2 \dot{u}\|^2 - \frac{1}{2} \|\bar{u} - u\|^2 - \frac{1}{2} \|u + \mu_2 \ddot{u} + \beta_2 \dot{u} - \bar{u}\|^2, \\
\frac{1}{2} \|\mu_1 \ddot{x} + \beta_1 \dot{x}\|^2 &= \frac{1}{2} \mu_1^2 \|\ddot{x}\|^2 + \mu_1 \beta_1 \langle \ddot{x}, \dot{x} \rangle + \frac{1}{2} \beta_1^2 \|\dot{x}\|^2, \\
\frac{1}{2} \|\mu_2 \ddot{u} + \beta_2 \dot{u}\|^2 &= \frac{1}{2} \mu_2^2 \|\ddot{u}\|^2 + \mu_2 \beta_2 \langle \ddot{u}, \dot{u} \rangle + \frac{1}{2} \beta_2^2 \|\dot{u}\|^2,
\end{aligned} \tag{37}$$

inequality (35) can be transformed into the following:

$$\begin{aligned}
& \mu_1 \langle \ddot{x}, x - x^* \rangle + \frac{\beta_1^2}{2} \|\dot{x}\|^2 + \beta_1 \langle \dot{x}, x - x^* \rangle + \frac{\mu_1^2}{2} \|\ddot{x}\|^2 + \mu_1 \beta_1 \langle \dot{x}, \ddot{x} \rangle \\
& + \mu_2 \langle \ddot{u}, u - u^* \rangle + \frac{\beta_2^2}{2} \|\dot{u}\|^2 + \beta_2 \langle \dot{u}, u - u^* \rangle + \mu_2 \beta_2 \langle \dot{u}, \ddot{u} \rangle + \frac{\mu_2^2}{2} \|\ddot{u}\|^2 \\
& + \frac{1}{2} \|\bar{u} - u\|^2 + \left(\frac{1}{2} - \alpha^2 (C^2 + |g|^2) \right) \|\bar{x} - x\|^2 \leq 0.
\end{aligned} \tag{38}$$

According to the relations

$$\begin{aligned}
\frac{1}{2} \frac{d^2}{dt^2} \|x - x^*\|^2 &= \|\dot{x}\|^2 + \langle x - x^*, \ddot{x} \rangle, \\
\frac{1}{2} \frac{d}{dt} \|\dot{x}\|^2 &= \langle \dot{x}, \ddot{x} \rangle, \\
\frac{1}{2} \frac{d}{dt} \|x - x^*\|^2 &= \langle \dot{x}, x - x^* \rangle, \\
\frac{1}{2} \frac{d^2}{dt^2} \|u - u^*\|^2 &= \|\dot{u}\|^2 + \langle u - u^*, \ddot{u} \rangle, \\
\frac{1}{2} \frac{d}{dt} \|\dot{u}\|^2 &= \langle \dot{u}, \ddot{u} \rangle, \\
\frac{1}{2} \frac{d}{dt} \|u - u^*\|^2 &= \langle \dot{u}, u - u^* \rangle,
\end{aligned} \tag{39}$$

inequality (38) means that

$$\begin{aligned} & \frac{\mu_1}{2} \frac{d^2}{dt^2} \|x - x^*\|^2 + \left(\frac{\beta_1^2}{2} - \mu_1 \right) \|\dot{x}\|^2 + \frac{\beta_1}{2} \frac{d}{dt} \|x - x^*\|^2 + \frac{\mu_1^2}{2} \|\ddot{x}\|^2 \\ & + \frac{\mu_1 \beta_1}{2} \frac{d}{dt} \|\dot{x}\|^2 + \frac{\mu_2}{2} \frac{d^2}{dt^2} \|u - u^*\|^2 + \left(\frac{\beta_2^2}{2} - \mu_2 \right) \|\dot{u}\|^2 + \frac{\beta_2}{2} \frac{d}{dt} \|u - u^*\|^2 \\ & + \frac{\mu_2^2}{2} \|\ddot{u}\|^2 + \frac{\mu_2 \beta_2}{2} \frac{d}{dt} \|\dot{u}\|^2 + \frac{1}{2} \|u - \bar{u}\|^2 + \left(\frac{1}{2} - \alpha^2 (C^2 + |g|^2) \right) \|\bar{x} - x\|^2 \leq 0. \end{aligned} \quad (40)$$

Let $\phi(x) = (1/2)\|x - x^*\|^2$ and $\psi(u) = (1/2)\|u - u^*\|^2$; inequality (40) yields that

$$\begin{aligned} & \mu_1 \frac{d^2}{dt^2} \phi(x) + \beta_1 \frac{d}{dt} \phi(x) + \left(\frac{\beta_1^2}{2} - \mu_1 \right) \|\dot{x}\|^2 + \frac{\mu_1^2}{2} \|\ddot{x}\|^2 + \frac{\mu_1 \beta_1}{2} \frac{d}{dt} \|\dot{x}\|^2 \\ & + \mu_2 \frac{d^2}{dt^2} \psi(u) + \beta_2 \frac{d}{dt} \psi(u) + \left(\frac{\beta_2^2}{2} - \mu_2 \right) \|\dot{u}\|^2 + \frac{\mu_2^2}{2} \|\ddot{u}\|^2 + \frac{\mu_2 \beta_2}{2} \frac{d}{dt} \|\dot{u}\|^2 + \frac{1}{2} \|u - \bar{u}\|^2 + \left(\frac{1}{2} - \alpha^2 (C^2 + |g|^2) \right) \|\bar{x} - x\|^2 \leq 0. \end{aligned} \quad (41)$$

Since $\mu_1 < (1/2)\beta_1^2$, $\mu_2 < (1/2)\beta_2^2$, and $\alpha^2 (C^2 + |g|^2) < (1/2)$, inequality (41) can be integrated from t_0 to t :

$$\begin{aligned} & \mu_1 \frac{d}{dt} \phi(x) + \beta_1 \phi(x) + \left(\frac{\beta_1^2}{2} - \mu_1 \right) \int_{t_0}^t \|\dot{x}\|^2 d\tau + \frac{\mu_1^2}{2} \int_{t_0}^t \|\ddot{x}\|^2 d\tau + \frac{\mu_1 \beta_1}{2} \|\dot{x}\|^2 \\ & + \mu_2 \frac{d}{dt} \psi(u) + \beta_2 \psi(u) + \left(\frac{\beta_2^2}{2} - \mu_2 \right) \int_{t_0}^t \|\dot{u}\|^2 d\tau + \frac{\mu_2^2}{2} \int_{t_0}^t \|\ddot{u}\|^2 d\tau + \frac{\mu_2 \beta_2}{2} \|\dot{u}\|^2 \\ & + \frac{1}{2} \int_{t_0}^t \|\bar{u} - u\|^2 d\tau + \left(\frac{1}{2} - \alpha^2 (C^2 + |g|^2) \right) \int_{t_0}^t \|\bar{x} - x\|^2 d\tau \leq B, \end{aligned} \quad (42)$$

where $B = \mu_1 (d/dt)\phi(x_0) + \beta_1 \phi(x_0) + (\mu_1 \beta_1 / 2) \|\dot{x}_0\|^2 + \mu_2 (d/dt)\psi(u_0) + \beta_2 \psi(u_0) + (\mu_2 \beta_2 / 2) \|\dot{u}_0\|^2$.

Since $\|u(t)\|$ and $\|\dot{u}(t)\|$ are all bounded for all $t \rightarrow \infty$, we deduce that $|(d/dt)\psi(u)|$ has the lower bound because

$$\left| \frac{d}{dt} \psi(u) \right| = |\langle \dot{u}, u - u^* \rangle| \leq \|\dot{u}\| \|u - u^*\|. \quad (43)$$

Thus, there exists a constant B_1 such that

$$\mu_1 \frac{d}{dt} \phi(x) + \beta_1 \phi(x) \leq B_1, \quad (44)$$

which implies that

$$\mu_1 \exp\left(-\frac{\beta_1}{\mu_1} t\right) \frac{d}{dt} \left(\exp\left(\frac{\beta_1}{\mu_1} t\right) \phi(x) \right) \leq B_1. \quad (45)$$

That is,

$$\frac{d}{dt} \left(\exp\left(\frac{\beta_1}{\mu_1} t\right) \phi(x) \right) \leq B_1 \frac{1}{\mu_1} \exp\left(\frac{\beta_1}{\mu_1} t\right). \quad (46)$$

By integrating (45), we have

$$\exp\left(\frac{\beta_1}{\mu_1} t\right) \phi(x) \leq \frac{B_1}{\beta_1} \exp\left(\frac{\beta_1}{\mu_1} t\right) + C_1, \quad (47)$$

where C_1 is a constant. We conclude that

$$\phi(x) \leq \frac{B_1}{\beta_1} + C_1 \exp\left(-\frac{\beta_1}{\mu_1} t\right), \quad (48)$$

which means that $\phi(x)$ is bounded for all $t \rightarrow \infty$.

The function $\phi(x)$ is strongly convex, and it is well known that each of its Lebesgue sets is bounded. Thus, the trajectory $x(t)$ is bounded. That is, there exists a constant B_2 such that

$$\|x(t) - x^*\| \leq B_2. \quad (49)$$

Now we claim that $\int_{t_0}^t \|\dot{x}\|^2 d\tau < \infty$, $\int_{t_0}^t \|\ddot{x}\|^2 d\tau < \infty$, $\int_{t_0}^t \|\dot{u}\|^2 d\tau < \infty$, $\int_{t_0}^t \|\ddot{u}\|^2 d\tau < \infty$, $\int_{t_0}^t \|\bar{u} - u\|^2 d\tau < \infty$ and $\int_{t_0}^t \|\bar{x} - x\|^2 d\tau < \infty$.

We firstly show that $\|\dot{x}\|$ is bounded. It follows from inequality (42) that

$$\frac{d}{dt} \phi(x) + \frac{\beta_1}{\mu_1} \phi(x) + \frac{\beta_1}{2} \|\dot{x}\|^2 \leq B_3, \quad (50)$$

where B_3 is a constant. From $\phi(x) = (1/2)\|x - x^*\|^2$, the above inequality means that

$$\langle \dot{x}, x - x^* \rangle + \frac{\beta_1}{2\mu_1} \|x - x^*\|^2 + \frac{\beta_1}{2} \|\dot{x}\|^2 \leq B_3. \quad (51)$$

In the light of $\langle \dot{x}, x - x^* \rangle = - (1/2)\|\dot{x}\|^2 - (1/2)\|x - x^*\|^2 + (1/2)\|\dot{x} + x - x^*\|^2$, the above inequality infers that

$$\frac{1}{2} \left(\frac{\beta_1}{\mu_1} - 1 \right) \|x - x^*\|^2 + \frac{1}{2} (\beta_1 - 1) \|\dot{x}\|^2 \leq B_3. \quad (52)$$

Noting that $1 < \beta_1 < \mu_1$, we conclude that $\|\dot{x}\|^2$ is bounded in the following:

$$\frac{1}{2} (\beta_1 - 1) \|\dot{x}\|^2 \leq B_3 + \frac{1}{2} \left(1 - \frac{\beta_1}{\mu_1} \right) \|x - x^*\|^2 \leq B_3 + \frac{1}{2} \left(1 - \frac{\beta_1}{\mu_1} \right) B_2, \quad (53)$$

which means that $\|\dot{x}\|^2$ is bounded. It follows, from

$$\left| \frac{d}{dt} \phi(x) \right| = |\langle \dot{x}, x - x^* \rangle| \leq \|\dot{x}\| \|x - x^*\|, \quad (54)$$

that $(d/dt)\phi(x)$ also has lower bound. Thus, there exists a constant B_5 such that

$$\begin{aligned} & \left(\frac{\beta_1^2}{2} - \mu_1 \right) \int_{t_0}^t \|\dot{x}\|^2 d\tau + \frac{\mu_1^2}{2} \int_{t_0}^t \|\ddot{x}\|^2 d\tau + \left(\frac{\beta_2^2}{2} - \mu_2 \right) \int_{t_0}^t \|\dot{u}\|^2 d\tau + \frac{\mu_2^2}{2} \int_{t_0}^t \|\ddot{u}\|^2 d\tau \\ & + \frac{1}{2} \int_{t_0}^t \|\bar{u} - u\|^2 d\tau + \left(\frac{1}{2} - \alpha^2 (C^2 + |g|^2) \right) \int_{t_0}^t \|\bar{x} - x\|^2 d\tau \leq B_5, \end{aligned} \quad (55)$$

which yields that the integrals,

$$\begin{aligned} & \int_{t_0}^t \|\dot{x}\|^2 d\tau < \infty, \int_{t_0}^t \|\ddot{x}\|^2 d\tau < \infty, \int_{t_0}^t \|\dot{u}\|^2 d\tau < \infty, \int_{t_0}^t \|\ddot{u}\|^2 d\tau < \infty, \\ & \int_{t_0}^t \|\bar{u} - u\|^2 d\tau < \infty, \int_{t_0}^t \|\bar{x} - x\|^2 d\tau < \infty, \end{aligned} \quad (56)$$

converge as $t \rightarrow \infty$.

Assuming that there exists $\varepsilon > 0$ such that $\|\ddot{x}(t)\| \geq \varepsilon$, $\|\ddot{u}(t)\| \geq \varepsilon$, $\|\dot{x}(t)\| \geq \varepsilon$, $\|\dot{u}(t)\| \geq \varepsilon$, $\|\bar{u}(t) - u(t)\| \geq \varepsilon$, and $\|\bar{x}(t) - x(t)\| \geq \varepsilon$ for all $t \geq t_0$, we obtain a contradiction to the convergence of integrals. Hence, there exists a subsequence of time moments $t_i \rightarrow \infty$ such that $\|\ddot{x}(t_i)\| \rightarrow 0$, $\|\ddot{x}(t_i)\| \rightarrow 0$, $\|\ddot{u}(t_i)\| \rightarrow 0$, $\|\dot{x}(t_i)\| \rightarrow 0$, $\|\bar{u}(t_i) - u(t_i)\| \rightarrow 0$, and $\|\bar{x}(t_i) - x(t_i)\| \rightarrow 0$. Since $x(t)$ and $u(t)$ are bounded, we know that $x(t_i)$ and $u(t_i)$ are bounded. We choose the subsequences $x(t_{i_j})$ and $u(t_{i_j})$ of $x(t)$ and $u(t)$. Then there exist x' and u' such that $x(t_{i_j}) \rightarrow x'$, $u(t_{i_j}) \rightarrow u'$, $\|\ddot{x}(t_{i_j})\| \rightarrow 0$, $\|\dot{x}(t_{i_j})\| \rightarrow 0$, $\|\ddot{u}(t_{i_j})\| \rightarrow 0$, $\|\dot{u}(t_{i_j})\| \rightarrow 0$, $\|\bar{u}(t_{i_j}) - u(t_{i_j})\| \rightarrow 0$, and $\|\bar{x}(t_{i_j}) - x(t_{i_j})\| \rightarrow 0$ as $j \rightarrow \infty$.

Let us consider inequality (16) for all t_{i_j} and take the limit as $j \rightarrow \infty$; we have

$$\begin{aligned} x' &= \Pi_{\Omega} \left(x' - \alpha (F(x') + \mathcal{J}g(x')^T u') \right), \\ u' &= \Pi_{+} (u' + \alpha g(x')), \end{aligned} \quad (57)$$

which means that $x' \in \Omega^*$ is a solution of variational inequality with constraints (1) by using Lemma 2. This completes the proof. $\square$

4. Numerical Results

In this section, we test the example with three kinds of different cases by our differential equation system (16). The

transient behaviors of the proposed differential equation systems are demonstrated in each case. The numerical implementation is coded by Matlab R2019a running on a PC with Intel i5 9400F of 2.9 GHz CPU and the ordinary differential equation solver adopted is ode45, which uses a Runge-Kutta (4, 5) formula.

Example 1. Consider the variational inequality with constraints problem with $F: \mathfrak{R}^5 \rightarrow \mathfrak{R}^5$ given by $F(x) = Mx + \rho C(x) + q$ and $K = \{x \in \mathfrak{R}_+^5 | g(x) = 10 - \sum_{i=1}^5 x_i \leq 0\}$, where M is an asymmetric positive definite matrix and $C_i(x) = \arctan(x_i - 2)$, $i = 1, 2, \dots, 5$. The parameter ρ is used to vary the degree of asymmetry and nonlinearity, and the data of the example are illustrated as follows:

$$M = \begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.934 & 1.007 \\ 1.063 & 0.567 & -1.144 & 0.550 & -0.548 \\ -0.259 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix}, \quad (58)$$

and $q = (5.308, 0.008, -0.938, 1.024, -1.312)^T$, which have been considered in the work of Dang et al. [35]. Its solution is $x^* = (2, 2, 2, 2, 2)^T$.

Since the parameter ρ is used to vary the degree of asymmetry and nonlinearity, we apply the second-order differential equation system (16) to solve the variational inequalities in three cases of $\rho < 1$, $\rho = 1$, and $\rho > 1$.

Case 1. When $\rho < 1$, we take $\rho = 0.1$, which means that the degree of asymmetry of $F(x)$ is higher than that of the nonlinearity of $F(x)$. In this experiment, we set $\mu_1 = \mu_2 = 2.02$, $\beta_1 = \beta_2 = 2.01$, and $\alpha = 0.5$ in the second-order differential equation system with the controlled process (16). Figure 1 describes the convergence behaviors of $x(t)$, $u(t)$, $\dot{x}(t)$, and $\dot{u}(t)$ of the second-order differential equation system with the controlled process (16) from the given initial points for the example with $\rho = 0.1$.

Case 2. We take $\rho = 1$, which means that the degree of asymmetry of $F(x)$ is the same as the degree of the nonlinearity of $F(x)$. In this experiment, we let $\mu_1 = \mu_2 = 2.02$, $\beta_1 = \beta_2 = 2.01$, and $\alpha = 0.4$ in the second-order differential equation system with the controlled process (16). The solution trajectories $x(t)$, $u(t)$, $\dot{x}(t)$, and $\dot{u}(t)$ of the second-order differential equation system with the controlled process (16) from the given initial points for the example with $\rho = 1$ are shown in Figure 2.

Case 3. When $\rho > 1$, we take $\rho = 10$, which means that the degree of asymmetry of $F(x)$ is lower than that of the nonlinearity of $F(x)$. In this experiment, we set $\mu_1 = \mu_2 = 2.02$, $\beta_1 = \beta_2 = 2.01$, and $\alpha = 0.06$ in the second-order differential equation system with the controlled process (16). Figure 3 describes the convergence behaviors of $x(t)$, $u(t)$, $\dot{x}(t)$, and $\dot{u}(t)$ of the second-order differential equation system with the controlled process (16) from the given initial points for the example with $\rho = 10$.

FIGURE 1: Transient behavior of $x(t)$ and $u(t)$ of the second-order differential equation system (16) in the example with $\rho = 0.1$.

FIGURE 2: Transient behavior of $x(t)$ and $u(t)$ of the second-order differential equation system (16) in the example with $\rho = 1$.

Remark 2. In Theorem 1, the conditions of the parameters $\mu_1, \beta_1, \mu_2, \beta_2$, and α with $1 < \beta_1 < \mu_1 < (1/2)\beta_1^2$, $0 < \mu_2 < (1/2)\beta_2^2$, and $\alpha^2 (|F| + C_0|\mathcal{F}|)^2 + |g|^2 < (1/2)$ are sufficient but not necessary for the convergence of the trajectory of the second-order differential equation system with the controlled process (16) for solving the variational inequality with constraints problem (1). The following lists are given to illustrate the influence on the rate of convergence of trajectory of the second-order differential equation system with the controlled process (16) for solving the variational inequality with constraints problem in the above example. From the condition with $1 < \beta_1 < \mu_1 < (1/2)\beta_1^2$, we have that $2 < \beta_1 < \mu_1$. Hence, we give a discussion in the following and take the first case as an example; that is, $\rho = 0.1$ in our example.

Fixing $\mu_2 = 2.02$, $\beta_2 = 2.01$, and $\alpha = 0.5$, the convergence time will be changed as follows when μ_1 and β_1 are changed (Table 1). In this experiment, when μ_2, β_2 , and α were fixed, we found that the smaller μ_1 and β_1 are, the shorter the

FIGURE 3: Transient behavior of $x(t)$ and $u(t)$ of the second-order differential equation system (16) in the example with $\rho = 10$.

TABLE 1: Convergence time of $x(t)$ by the second-order differential equation system (16).

μ_1	β_1	Time (ms)	$x(t)$
2.02	2.01	18.3691	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.12	2.11	22.0502	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.22	2.21	22.5589	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.32	2.31	23.2263	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.42	2.41	23.5676	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$

TABLE 2: Convergence time of $x(t)$ by the second-order differential equation system (16).

μ_2	β_2	Time (ms)	$x(t)$
2.02	2.01	18.3691	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.12	2.11	18.4430	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.22	2.21	21.7473	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.32	2.31	21.7503	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
2.42	2.41	21.9388	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$

TABLE 3: Convergence time of $x(t)$ by the second-order differential equation system (16).

α	Time (ms)	$x(t)$
0.2	74.4480	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
0.3	40.5888	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
0.4	23.4882	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
0.5	18.3691	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$
0.55	39.5863	$(2.0000, 2.0000, 2.0000, 2.0000, 2.0000)^T$

convergence time is. Furthermore, we also found that the change of β_1 had a great influence on the rate of the convergence when μ_1 is fixed; meanwhile the change of μ_1 has a little effect on the rate of the convergence when β_1 is fixed.

Fixing $\mu_1 = 2.02$, $\beta_1 = 2.01$, and $\alpha = 0.5$, the convergence time will be changed as follows when μ_2 and β_2 are changed (Table 2). In this experiment, when μ_1 , β_1 , and α were fixed, we found that the smaller μ_2 and β_2 are, the shorter the convergence time is. Furthermore, we also found that the

change of β_2 had a great influence on the rate of the convergence when μ_2 is fixed; meanwhile the change of μ_2 has a little effect on the rate of the convergence when β_2 is fixed.

Fixing $\mu_1 = 2.02$, $\beta_1 = 2.01$, $\mu_2 = 2.02$, and $\beta_2 = 2.01$, the convergence time will be changed as follows when α is changed (Table 3). In this experiment, when μ_1 , β_1 , μ_2 , and β_2 were fixed, we found that the convergence time is the shortest at about $\alpha = 0.5$.

5. Conclusions

In this paper, based on the Lagrange function and projection operator, the second-order differential equation system with the controlled process is established for solving the variational inequality with constraints problem (1). The convergence of the trajectories for this kind of differential equation system is proved. Furthermore, we test one example with three kinds of different cases by the second-order differential equation system with the controlled process and illustrate the transient behaviors of the trajectories for this differential equation system in each case, which verify the effectiveness of the second-order differential equation system with the controlled process for solving the variational inequality with constraints.

It is an interesting issue that the second-order differential equation systems are used to solve the optimization problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon reasonable request.

Disclosure

Some of the results in this paper were presented at the Proceeding of the 11th World Congress on Intelligent Control and Automation, 2014 (see <https://ieeexplore.ieee.org/document/7052904>).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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