On Fault-Tolerant Resolving Sets of Some Families of Ladder Networks

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In computer networks, vertices represent hosts or servers, and edges represent the connecting medium between them. In localization, some special vertices (resolving sets) are selected to locate the position of all vertices in a computer network. If an arbitrary vertex stopped working and selected vertices still remain the resolving set, then the chosen set is called as the fault-tolerant resolving set. The least number of vertices in such resolving sets is called the fault-tolerant metric dimension of the network. Because of the variety of applications of the metric dimension in different areas of sciences, many generalizations were proposed, and fault tolerant is one of them. In this paper, we computed the fault-tolerant metric dimension of triangular snake, ladder, Mobius ladder, and hexagonal ladder networks. It is important to observe that, in all these classes of networks, the fault-tolerant metric dimension and metric dimension differ by 1.

1. Introduction

Let \( G = (V(G), E(G)) \) be a simple connected graph, where \( V(G) \) and \( E(G) \) are the set of vertices and edges, respectively. The distance \( d(u_1, u_2) \) between a pair of vertices \( u_1, u_2 \in V(G) \) is the length of the shortest path joining them. A vertex \( u \) resolves or distinguishes a pair \( u_1, u_2 \) if \( d(u_1, u) \neq d(u_2, u) \). The representation of an arbitrary vertex \( u \) corresponding to an ordered nonempty subset of vertices \( Q = \{q_1, q_2, \ldots, q_h\} \subseteq V(G) \) is the \( h \)-component vector \( r(u|Q) = (d(u, q_1), d(u, q_2), \ldots, d(u, q_h)) \). A subset \( Q \subseteq V(G) \) is named the resolving set if any pair \( u_1, u_2 \in V(G) \) possesses the condition of unique representation, that is, \( r(u_1|Q) \neq r(u_2|Q) \). The least possible cardinality of \( Q \) is named the metric dimension of the graph \( G \) and symbolized as \( \text{dim}(G) \). A resolving set containing a least possible number of elements is named as basis. A basis set \( Q_f \) of \( G \) is called the fault-tolerant basis set if, for every \( u \in Q_f \), the subset \( (Q_f/u) \) again resolves the vertices of \( G \). The least number of vertices in \( Q_f \) is termed as the fault-tolerant metric dimension of \( G \) and is denoted by \( \text{dim}_f(G) \). Chaudhry et al. [1] proved an important relationship between the fault-tolerant metric dimension and the metric dimension of a graph \( G \).

\[
\text{dim}_f(G) \geq \text{dim}(G) + 1.
\]

First time, the idea of metric dimension was studied by Slater [2] in 1975 and later by Harary and Melter [3] in 1976. Chartrand et al. [4] considered metric bases as sensors. If any of the sensors did not operate correctly, then we do not have sufficient knowledge to deal with the intruders. Hernando et al. [5] presented the idea of fault-tolerant metric dimension to overcome this kind of problems. Fault-tolerant basis set delivers accurate information despite being one of the sensors is not working.
Because of different applications, the study of fault-tolerant resolving sets of different networks is as sundry as the study of the metric dimension is.

It was proved that computing the metric dimension is NP-complete [6]. However, the idea of uniquely identifying every vertex in a graph based on distance is very useful. The application of this concept in chemical structures was presented by Chartrand el al. [4]. In robot navigation, the work by Khuller et al. [7] motivated the researchers for the theoretical investigation of the metric dimension. The resolving set has found a number of applications. For more details, see [8–10]. Generally, computing the fault-tolerant metric dimension is also an NP-complete problem. Computing the fault-tolerant metric dimension is considered as one of the interesting but difficult problems in combinatorics. So far, only few structures have been investigated. In the initial paper, Hernando et al. [5] computed the fault-tolerant basis sets of tree $T$. For a graph $G$, they derived a connection between the fault-tolerant metric dimension and the metric dimension, that is, $\dim_f(G) \leq \dim(G) (1 + 2 \times 5^{\dim(G)-1})$. Javaid et al. [11] studied this parameter of cycle graph $C_n$ and discussed some bounds on the partition dimension. For more results related to the fault-tolerant metric dimension, the interested reader can see [12–23].

The core objective of this paper is to find some classes of graphs satisfying the equation $\dim(G) = \dim_f(G) + 1$. It has been observed that if $G$ is ladder ($L_m, m \geq 3$), Möbius ladder ($ML_m, m \geq 4$), and hexagonal Möbius ladder graph ($HML_m, m \geq 2$), then the metric dimension and fault-tolerant metric dimension differ by 1. Also, for triangular snake graph $TS_n$, the above equality holds if and only if $n = 5, 7$. In order to prove these results, we need some results from the literature.

Theorem 1 (see [4]). The metric dimension of a graph is 1 iff $G \equiv P_n$.

Theorem 2 (see [24]). Let $ML_m$ be a Möbius ladder graph. Then,

$$\dim(ML_m) = \begin{cases} 3 & \text{if } m \geq 4 \text{ even,} \\ 4 & \text{if } m \geq 5 \text{ odd.} \end{cases}$$  

Theorem 3 (see [25]). Let $HML_m$ be a hexagonal Möbius ladder graph. Then,

$$\dim(HML_m) = \begin{cases} 2 & \text{if } m = 2, \\ 3 & \text{if } m \geq 3. \end{cases}$$  

Theorem 4 (see [5]). Let $L_n$ be a ladder graph with $n \geq 3$. Then,

$$\dim(L_n) = 2.$$  

2. Fault-Tolerant Metric Dimension of the Triangular Snake Graph

This section deals with the metric dimension and fault-tolerant metric dimension of the triangular snake graph. Let $P_n$ be a path with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where $n = 2s + 1$ is an odd integer. A triangular snake graph $TS_n$ is constructed from $P_n$ by connecting $v_{2i-1}$ with $v_{2i+1}$ for $1 \leq i \leq s$. Note that $s$ represents the cardinality of triangles in $TS_n$. The triangular snake graph is depicted in Figure 1. In the following Lemma 1, we calculate the metric dimension $\dim(TS_n)$ of triangular snake graph $TS_n$.

Lemma 1. Let $TS_n$ be a triangular snake graph with $s \geq 2$. Then,

$$\dim(TS_n) = 2.$$  

Proof. From Theorem 1, it follows that $\dim(TS_n) \geq 2$. To prove that $\dim(TS_n) = 2$, we prove that $\dim(TS_n) \leq 2$. Let $Q = \{v_1, v_n\}$ be a resolving set. Then, the representations of vertices $v_t$ regarding the resolving set $Q$ are as follows:

$$r(v_t|Q) = \begin{cases} \left(\frac{t-1}{2}, \frac{2s-t-1}{2}\right) & \text{if } t = 1, 3, 5, \ldots, 2s-1, \\ \left(\frac{t}{2}, \frac{2s-t+2}{2}\right) & \text{if } t = 2, 4, 6, \ldots, 2s-2, \\ \left(\frac{t}{2}, 1\right) & \text{if } t = 2s, \\ \left(\frac{t}{2}, 0\right) & \text{if } t = 2s+1. \end{cases}$$

As all the vertices have unique representations regarding the resolving set $Q$, hence, $\dim(TS_n) \leq 2$. □

Theorem 5. Let $TS_n$ be a triangular snake graph; then,

$$\dim_f(TS_n) = \begin{cases} 3 & \text{if } s = 2, 3, \\ 4 & \text{if } s \geq 4. \end{cases}$$  

Proof. First, we show that $\dim_f(TS_n) = 3$ when $s = 2, 3$. To prove this, it is enough to show that $\dim_f(TS_n) \leq 3$. The inequality $\dim_f(TS_n) \geq 3$ follows from Lemma 1 and equation (1). If we take $Q_f = \{v_1, v_4, v_{2s+1}\}$, then it satisfies the condition of the fault-tolerant resolving set. The following is the representation of any vertex $v_t$ of the graph $TS_n$ with respect to $Q_f$:

$$r(v_t|Q_f) = \left(\frac{t}{2}, \left[\frac{4-t}{2}\right] + z, \left[\frac{2s+1-t}{2}\right]\right),$$

where $z = 1$ when $t = 3, 5$ and zero, otherwise.

Now, we prove that $\dim_f(TS_n) \leq 4$ when $s \geq 4$. For this, we construct a fault-tolerant resolving set. Take
Qf = \{v_1, v_2, v_2-2, v_2\}; the following is the representation of
any vertex of graph with Qf:

\[ r(v_i|Q_f) = \left( \left\lceil \frac{t-1}{2} \right\rceil, \left\lceil \frac{t-2}{2} \right\rceil +z, \left\lceil \frac{2s-2-t}{2} \right\rceil +z, \left\lceil \frac{2s-t}{2} \right\rceil +z \right). \]  

(9)

where \( z = 1 \) for \( t = \text{even and zero, otherwise. Hence, it}
follows from the above discussion that \( \dim_f(TS_n) \leq 4 \) since
every vertex of \( TS_n \) has a unique representation regarding
resolving set \( Q_f \).

We prove the reverse inequality \( \dim_f(TS_n) \geq 4 \) by the
contradiction method. Suppose \( \dim_f(TS_n) < 4 \), it follows
from Lemma 1 that \( \dim_f(TS_n) = 3 \). Let us consider some
resolving set \( Q_f \) with cardinality 3. □

Case 1. Let \( Q_f = \{v_{1+l}, v_{2+l}, v_{3+l}\} \), where \( l \equiv 0 \text{ (mod 3)} \). If we
choose \( Q_f = \{v_1, v_2, v_3\} \), then \( r(v_i|Q_f) = r(v_j|Q_f) \). For any
other choice of \( Q_f \), we get \( r(v_i|Q_f) = r(v_j|Q_f) \).

Case 2. Let \( Q_f = \{v_1, v_3, v_5\} \) or \( Q_f = \{v_1, v_4, v_6\}\), where
\( 1 \leq i, j \leq 2 \) and \( l \geq 3 \). In other words, we are taking the first
vertex from the first triangle and the remaining vertices from
any arbitrary triangle. Then, it is easy to observe
that there exists an integer \( 6 \leq j \leq n-1 \) such that
\( r(v_j|Q_f) = r(v_{j+1}|Q_f) \).

Case 3. Let \( Q_f = \{v_i, v_j, v_k\} \), where \( 1 \leq i, j, k \leq n \). Then,
\( r(v_i|Q_f) = r(v_j|Q_f) \).

Hence, it follows from the above discussion that
\( \dim_f(TS_n) \geq 3 \) when \( s \geq 4 \). This implies that \( \dim_f(TS_n) \geq 4 \)
when \( s \geq 4 \). This completes the proof.

3. Fault-Tolerant Metric Dimension of
Ladder Graphs

A ladder graph is obtained by the Cartesian product of path
\( P_n \) with path \( P_2 \) and is denoted by \( L_m \). Observe that \( m = 2n \).
The ladder graph is shown in Figure 2. In the following
theorem, we compute the fault-tolerant metric dimension of
the ladder graph.

**Theorem 6.** Let \( L_m \) be a ladder graph with \( m \geq 3 \). Then,
\[ \dim_f(L_m) = 3. \]  

(10)

**Proof.** The inequality \( \dim_f(L_m) \geq 3 \) follows from Theorem 4
and equation (1). To prove \( \dim_f(L_m) \leq 3 \), we construct a
fault-tolerant resolving set. Let \( Q_f = \{v_1, v_3, v_{2n-1}\} \); then, the
representation of any vertex \( v_i \) of the graph with \( Q_f \) can be computed as

\[ r(v_i|Q_f) = \left( \left\lceil \frac{t-1}{2} \right\rceil, \left\lceil \frac{t-3}{2} \right\rceil +z, \left\lceil \frac{2n-1-t}{2} \right\rceil +z \right). \]  

(11)

where \( w = 1 \) for \( t = 2 \) and \( z = 1 \) for \( t = \text{even and zero, otherwise. Observe that all the vertices of} \ L_m \) have a unique representation regarding fault-tolerant set \( Q_f \). Hence, \( \dim_f(L_m) = 3 \).

The Möbius ladder was first introduced by Richard Guy
and Frank Harary [26]. It is a three-regular graph created by
a cycle \( C_n \) by joining vertices \( u, v \) of the cycle iff
d \( d(u, v) = \text{diam}(C_n) \). Möbius ladder has an even number of
vertices. It is denoted by \( M_n \), where \( n = 2m \). The vertices
which satisfy this condition are called antipodal. The graph
of the Möbius ladder is shown in Figure 3. Möbius ladders
have several uses in different fields of sciences such as
stereochemistry, computer networks, and electrotechnology.
In the next result, we find the fault-tolerant metric
dimension of the Möbius ladder. □

**Theorem 7.** Let \( M_n \) be a Möbius ladder graph. Then,
\[ \dim_f(M_n) = \begin{cases} 4 & \text{if } m \geq 4 \text{ and } m \text{ is even}, \\ 5 & \text{if } m \geq 5 \text{ and } m \text{ is odd}. \end{cases} \]  

(12)

**Proof.** Let \( m \geq 4 \) be an even integer. The inequality
\( \dim_f(M_n) \geq 4 \) follows from Theorem 2 and equation (1).
To prove \( \dim_f(M_n) \leq 4 \), we construct a set satisfying the
definition of fault-tolerant metric dimension. Let \( Q_f = \{v_1, v_2, v_3, v_{m+1}\} \); then, the distances of any vertex \( v_i \) of
the graph corresponding to \( Q_f \) can be computed as follows.

Distance of vertex \( v_i \) from vertex \( v_j \) is

\[ d(v_i, v_j) = \begin{cases} t-1 & \text{if } t = 1, 2, \ldots, \left\lceil \frac{m+2}{2} \right\rceil, \\ m-t+2 & \text{if } t = \left\lceil \frac{m+2}{2} \right\rceil +1, \ldots, m+1, \\ t-m & \text{if } t = m+2, \ldots, \left\lceil \frac{m+2}{2} \right\rceil, \\ 2m-t+1 & \text{if } t = \left\lceil \frac{m+2}{2} \right\rceil +1, \ldots, 2m. \end{cases} \]  

(13)

Distance of vertex \( v_i \) from vertex \( v_2 \) is
Now, let $m \geq 5$ be an odd integer. To prove that $\dim f(M_n) = 5$, we construct a resolving set satisfying the condition of the fault-tolerant metric dimension. Let $Q_f = \{v_1, v_2, v_{(m+5)/2}, v_{m+2}, v_{(3m+5)/2}\}$; then, the distance of any vertices $v_t$ from the vertex $v_{(m+5)/2}$, $v_{m+2}$, and $v_{(3m+5)/2}$ is as follows.

Distance of vertex $v_t$ from vertex $v_{(m+5)/2}$ is

\[
d(v_t, v_{(m+5)/2}) = \begin{cases} 
    \frac{m - 3}{2} + t & \text{if } t = 1, 2, \\
    3 - t + \frac{m - 1}{2} & \text{if } t = 3, 4, \ldots, m + 1, \\
    t - \frac{m + 5}{2} & \text{if } t = m + 2, m + 3, \\
    t - \frac{3m + 5}{2} + 1 & \text{if } t = m + 4, \ldots, 2m.
\end{cases}
\]

Distance of vertex $v_t$ from vertex $v_{m+2}$ is

\[
d(v_t, v_{m+2}) = \begin{cases} 
    \frac{m - 3}{2} + t & \text{if } t = 1, 2, \\
    \frac{m - t + 3}{2} + 1, m + 1, \\
    t + 1 & \text{if } t = 1, 2, \ldots, \frac{m}{2} - 1, \\
    t - m & \text{if } t = m + 2, \ldots, \frac{3m}{2}, \\
    2m - t + 1 & \text{if } t = \frac{3m}{2} + 1, \ldots, 2m).
\end{cases}
\]

Distance of vertex $v_t$ from vertex $v_{(3m+5)/2}$ is

\[
d(v_t, v_{(3m+5)/2}) = \begin{cases} 
    \frac{m - 5}{2} + t & \text{if } t = 1, 2, \\
    m - t + 2 & \text{if } t = \frac{3m}{2}, \ldots, m + 1, \\
    t - \frac{m - 3}{2} + 1 & \text{if } t = m + 2, m + 3, \\
    3 - t + 2m & \text{if } t = \frac{3m + 5}{2}, \ldots, 2m.
\end{cases}
\]

Since all the vertices of $M_n$ have unique representations regarding decided resolving set $Q_f$, hence, $\dim f(M_n) = 5$.

A hexagonal M"obius ladder $HML(m, n)$ is constructed by subdividing each horizontal edge of the square lattice; thus, we obtained a lattice in which each cycle has order six; after creating this lattice, twist the recently attained lattice 180° and join the utmost left and right vertices as shown in Figure 4. Hexagonal M"obius ladder $HML(m, n)$ contains $m$ horizontal and $n$ vertical cycles. Let $HML_m = HML(m, 1)$ be
a hexagonal Möbius ladder with $m$ horizontal cycles and 1 vertical cycle. In the next theorem, we compute the fault-tolerant metric dimension of HML$_m$.

**Theorem 8.** Let HML$_m$ be a hexagonal Möbius ladder graph. Then,

$$\dim_f(HML_m) = \begin{cases} 3 & \text{if } m = 2, \\ 4 & \text{if } m \geq 3. \end{cases}$$

**Proof.** Let $m = 2$; the inequality $\dim_f(HML_m) \geq 3$ follows from Theorem 3 and equation (1). To prove $\dim_f(HML_m) \leq 3$, we construct a set satisfying the definition of fault-tolerant metric dimension. Let $Q_f = \{v_1, v_2, v_3\}$; then, the distances of any vertex $v_t$ of the graph corresponding to $Q_f$ can be computed as follows.

Distance of vertex $v_t$ from vertex $v_1$ is

$$d(v_t, v_1) = \begin{cases} 1 & \text{if } t = 2, 5, 8, \\ 2 & \text{if } t = 3, 6, 7. \end{cases}$$

(21)

Distance of vertex $v_t$ from vertex $v_3$ is

$$d(v_t, v_3) = \begin{cases} 1 & \text{if } t = 3, 5, \\ 2 & \text{if } t = 1, 6, 7, \\ 3 & \text{if } t = 8. \end{cases}$$

(22)

Distance of vertex $v_t$ from vertex $v_4$ is

$$d(v_t, v_4) = \begin{cases} 1 & \text{if } t = 5, 7, \\ 2 & \text{if } t = 1, 3, 4, 8, \\ 3 & \text{if } t = 2. \end{cases}$$

(23)

It is easy to see that every vertex of HML$_m$ has its unique representation with the resolving set $Q_f$. Hence, $\dim_f(HML_m) = 2$.

Let $m \geq 3$; to prove that $\dim_f(HML_m) = 4$, it is enough to construct a fault-tolerant resolving set of four elements that satisfy the condition of fault-tolerant metric dimension. Let $Q_f = \{v_1, v_2, v_{2m}, v_{2m+2}\}$; then, the distance of any vertex $v_t$ of HML$_m$ with $Q_f$ can be computed as follows.

Distance of vertex $v_t$ from vertex $v_1$ is

$$d(v_t, v_1) = \begin{cases} |t - 2| & \text{if } t = 1, 2, \ldots, m + 1, \\ 2m + t - 3 & \text{if } t = m + 3, \ldots, 2m + 1, \\ t - 2m & \text{if } t = 2m + 2, \ldots, 3m, \\ 4m - t + 1 & \text{if } t = 3m + 1, \ldots, 4m. \end{cases}$$

(24)

Distance of vertex $v_t$ from vertex $v_2$ is

$$d(v_t, v_2) = \begin{cases} t + 1 & \text{if } t = 1, 2, \ldots, m, \\ |2m - t| & \text{if } t = m, \ldots, 2m + 1, \\ t - 2m & \text{if } t = 2m + 2, \ldots, 3m, \\ 4m - t + 1 & \text{if } t = 3m + 1, \ldots, 4m. \end{cases}$$

(25)

Distance of vertex $v_t$ from vertex $v_{2m+2}$ is

$$d(v_t, v_{2m+2}) = \begin{cases} |t - 2| + 1 & \text{if } t = 1, 3, 4, 5, \ldots, \lfloor \frac{2m + 1}{2} \rfloor, \\ 3 & \text{if } t = 2, \\ 2m + t + 2 & \text{if } t = \lceil \frac{2m + 1}{2} \rceil + 1, \ldots, 2m + 1, \\ t - 2m - 2 & \text{if } t = 2m + 2, 2m + 3, \ldots, \lfloor \frac{6m + 3}{2} \rfloor, \\ 3 - t + 4m & \text{if } t = \lceil \frac{6m + 3}{2} \rceil + 1, \ldots, 4m. \end{cases}$$

(27)

It is easy to see that all the vertices of HML$_m$ have unique representations regarding resolving set $Q_f$. Hence, $\dim_f(HML_m) = 3$.

**4. Conclusion**

In this article, we study the fault-tolerant metric dimension of the triangular snake graph, ladder, Möbius ladder, and hexagonal Möbius ladder. It is known that, for cycle graph $C_n$, we have $\dim_f(C_n) = \dim(C_n) + 1$. Therefore, it is interesting to find some classes of graph in which this equality holds. Note that, in case of the triangular snake graph, ladder, Möbius ladder, and hexagonal Möbius ladder, we have $\dim_f = \dim + 1$.

**Data Availability**

No data were used to support this study.

**Disclosure**

This research was carried out as a part of the employment of the authors.

**Conflicts of Interest**

The authors declare no conflicts of interest.
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