

Research Article

Global Well-Posedness for the d -Dimensional Magnetic Bénard Problem without Thermal Diffusion

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This paper focuses on the global existence of strong solutions to the magnetic Bénard problem with fractional dissipation and without thermal diffusion in \mathbb{R}^d with $d \geq 3$. By using the energy method and the regularization of generalized heat operators, we obtain the global regularity for this model under minimal amount dissipation.

1. Introduction

Consider the global well-posedness problem to the d -dimensional (dD) magnetic Bénard problem

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mu \Lambda^{2\alpha} u = -\nabla p + b \cdot \nabla b + \theta e_d, & x \in \mathbb{R}^d, t > 0, \\ \partial_t b + u \cdot \nabla b + \nu \Lambda^{2\beta} b = b \cdot \nabla u, & x \in \mathbb{R}^d, t > 0, \\ \partial_t \theta + u \cdot \nabla \theta = u \cdot e_d, & x \in \mathbb{R}^d, t > 0, \\ \nabla \cdot u = 0, \nabla \cdot b = 0, & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = u_0(x), b(x, 0) = b_0(x), \theta(x, 0) = \theta_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1)$$

where $u(x, t)$, $b(x, t)$, $\theta(x, t)$, and $p(x, t)$ denote the velocity field, the magnetic field, the temperature, and the pressure, respectively. $\mu \geq 0$ is kinematic viscosity and $\nu \geq 0$ is magnetic diffusion. θe_d and $u \cdot e_d$ model the acting of the buoyancy force on fluid motion and the Rayleigh–Bénard convection in a heated inviscid fluid, respectively. The parameters α and β are nonnegative, and $\Lambda^r f$ with $r \geq 0$ is defined via $\Lambda^r f(\xi) = |\xi|^r \hat{f}(\xi)$. The magnetic Bénard problem can be used to model the behavior of the thermal instability under the influence of the magnetic field. One can refer to [1–7] for more physical background.

The global regularity problem of the magnetic Bénard problem (1) has caught much attention. For the 2D case, Zhou et al. in [8] obtained the global regularity for the case $\alpha = \beta = 1$. When $\alpha = 2$ and $\nu = 0$, the global existence and uniqueness of strong solutions were established by Yamazaki in [9]. Recently, Shang in [10] showed the global well-posedness of solutions for the case $\alpha = 1/2$ and $\beta = 1$. Compared to the magnitude results on the 2D case, it appears that there are only few global regularity results on dD ($d \geq 3$) magnetic Bénard problem (1).

This paper focuses its attention on the global regularity problem of (1) with minimal amount dissipation. More

precisely, we are able to establish the following global regularity result.

Theorem 1. *Consider (1) with $\mu > 0$ and $\nu > 0$. Suppose that $(u_0, b_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$, $\nabla \cdot u_0 = 0$, and $\nabla \cdot b_0 = 0$. Assume also that*

$$(u, b, \theta) \in L^\infty(0, T; H^s(\mathbb{R}^d)), u \in L^2(0, T; H^{s+\alpha}(\mathbb{R}^d)), w \in L^2(0, T; H^{s+\beta}(\mathbb{R}^d)). \quad (3)$$

In the subsequent section, we prove Theorem 1. Moreover, the definitions of the Besov spaces are provided in the appendix. We shall separately write $L^p = L^p(\mathbb{R}^d)$, $H^s = H^s(\mathbb{R}^d)$, and $\int = \int_{\mathbb{R}^d}$ for notational convenience.

2. Proof of Theorem 1

This section proves Theorem 1. The key step is to establish global *a priori* H^s -bound for (u, b, θ) . More specifically, we shall establish the following result.

Proposition 1. *Consider (1) with $\mu > 0$ and $\nu > 0$. Suppose that $(u_0, b_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, and*

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \alpha + \beta \geq 1 + \frac{d}{2}. \quad (4)$$

Then, the corresponding solution of (1) is globally bounded in $H^s(\mathbb{R}^d)$.

We only prove Proposition 1 for the case $\alpha + \beta = 1 + d/2$. In fact, the case $\alpha + \beta > 1 + d/2$ is even simpler.

2.1. Preparations. To prove the main theorem, as preparations we give three lemmas in this section. The first contains two calculus inequalities.

Lemma 1 (see [11, 12]). Let $s > 0$. Let $1 < r < \infty$ and $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,

$$\|[\Lambda^s, f]g\|_{L^r} \leq C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \quad (5)$$

with $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$ and

$$\|\Lambda^s(fg)\|_{L^r} \leq C(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}}), \quad (6)$$

where C 's are constants.

The second is the property of the generalized heat operator.

Lemma 2 (see, e.g., [13, 14]). Let $\mathcal{A} = \{\xi \in \mathbb{R}^d: r_1 \leq |\xi| \leq r_2\}$ with $0 < r_1 < r_2$ being constants. Then, there exist two positive constants c and C such that, for

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \alpha + \beta \geq 1 + \frac{d}{2}. \quad (2)$$

Then, (1) has a unique global strong solution (u, b, θ) satisfying, for any $T > 0$,

$$f \in L^p(\mathbb{R}^d), \text{supp } \widehat{f} \subset \lambda \mathcal{A}, p \in [1, \infty], \quad (7)$$

and we have, for $\alpha > 0$, $t > 0$, and $\lambda > 0$,

$$\|e^{-(\Delta)^{\alpha}t} f\|_{L^p(\mathbb{R}^d)} \leq C e^{-c\lambda^{2\alpha}} \|f\|_{L^p(\mathbb{R}^d)}. \quad (8)$$

The last is the following logarithmic type interpolation inequality.

Lemma 3 (see [14, 15]). Let $s > 1 + d/2$ and $q \in (1, \infty)$. Then, there exists $C > 0$ such that

$$\|\nabla f\|_{L^\infty} \leq C \left(1 + \|f\|_{L^2} + \|\Lambda^{d/q} \nabla f\|_{L^q} \log(e + \|\Lambda^s f\|_{L^2}) \right). \quad (9)$$

2.2. Global H^s Bound. This section gives the proof of Proposition 1. We first prove the following global L^2 -bound.

Lemma 4. *For any $t > 0$, the solution (u, b, θ) of (1) obeys*

$$\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Lambda^\beta b(\tau)\|_{L^2}^2 d\tau \leq C. \quad (10)$$

Proof. Taking the L^2 -inner product of (1) with (u, b, θ) , together with the Young inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \mu \|\Lambda^\alpha u\|_{L^2}^2 + \nu \|\Lambda^\beta b\|_{L^2}^2 \\ &= \int \theta \mathbf{e}_d \cdot u + \int u \cdot \mathbf{e}_d \theta \end{aligned} \quad (11)$$

$$\leq \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2.$$

Then, (10) follows from this and Gronwall's inequality. The second one is a better regularity for b . \square

Lemma 5. *Let (u, b, θ) be the solution of (1). Then, for all $t > 0$ and $0 < \sigma < \alpha - 1$,*

$$\|\Lambda^\sigma b(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\beta+\sigma} b(\tau)\|_{L^2}^2 d\tau \leq C. \quad (12)$$

Proof. Applying Λ^σ to (1.1)₂ and dotting the result with $\Lambda^\sigma b$, we have

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^\sigma b\|_{L^2}^2 + \nu \|\Lambda^{\beta+\sigma} b\|_{L^2}^2 = \int \Lambda^\sigma (b \cdot \nabla u) \cdot \Lambda^\sigma b - \int \Lambda^\sigma (u \cdot \nabla b) \cdot \Lambda^\sigma b := I_1 + I_2. \quad (13)$$

Applying Lemma 1, we arrive at

$$\begin{aligned} I_1 &\leq C \left(\|\Lambda^\sigma b\|_{L^2}^{d/d-2\beta} \|\nabla u\|_{L^{d/\beta}} + \|\Lambda^\sigma \nabla u\|_{L^2}^{d/d+2+2\sigma-2\alpha} \|b\|_{L^{d/(\alpha-1-\sigma)}} \right) \|\Lambda^\sigma b\|_{L^2} \\ &\leq C \|\Lambda^{\beta+\sigma} b\|_{L^2} \|\Lambda^\alpha u\|_{L^2} \|\Lambda^\sigma b\|_{L^2} \\ &\leq \frac{\nu}{4} \|\Lambda^{\beta+\sigma} b\|_{L^2}^2 + C \|\Lambda^\alpha u\|_{L^2}^2 \|\Lambda^\sigma b\|_{L^2}^2. \end{aligned} \quad (14)$$

Note that

$$I_2 = - \int (\Lambda^\sigma (u \cdot \nabla b) - u \cdot \nabla \Lambda^\sigma b) \cdot \Lambda^\sigma b. \quad (15)$$

Then, by again applying Lemma 1, we find that I_2 obeys the same bound as I_1 . Combing the above estimates up, together with (10), Gronwall's inequality yields (12).

The last preparation is stated as follows. \square

Lemma 6. *Let (u, b, θ) be the solution of (1). Then, for all $t > 0$ and $d/2\alpha - 2 < q < \infty$,*

$$\int_0^t \|\Lambda^{d/q} \omega(\tau)\|_{L^q} d\tau < \infty, \quad (16)$$

where $\omega = \nabla \times u$.

In addition, for all $2 \leq q < \infty$,

$$\|\theta(t)\|_{L^q} < \infty. \quad (17)$$

Proof. Taking curl on the both sides of (1.1)₁, we have

$$\partial_t \omega + \mu (-\Delta)^\alpha \omega = \nabla \times (b \cdot \nabla b) - \nabla \times (u \cdot \nabla u) + \nabla \times (\theta \mathbf{e}_d), \quad (18)$$

with $\omega = \nabla \times u$. Write (18) into the integral form

$$\begin{aligned} \omega(t) &= e^{-\mu(-\Delta)^\alpha t} \omega_0 + \int_0^t e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times \nabla \cdot (b \otimes b)(\tau) d\tau \\ &\quad - \int_0^t e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times \nabla \cdot (u \otimes u)(\tau) d\tau + \int_0^t e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times (\theta \mathbf{e}_d)(\tau) d\tau. \end{aligned} \quad (19)$$

We further localize it by applying Δ_j with $j \geq -1$:

$$\begin{aligned} \Delta_j \omega(t) &= \Delta_j e^{-\mu(-\Delta)^\alpha t} \omega_0 + \int_0^t \Delta_j e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times \nabla \cdot (b \otimes b)(\tau) d\tau \\ &\quad - \int_0^t \Delta_j e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times \nabla \cdot (u \otimes u)(\tau) d\tau + \int_0^t \Delta_j e^{-\mu(-\Delta)^\alpha(t-\tau)} \nabla \times (\theta \mathbf{e}_d)(\tau) d\tau. \end{aligned} \quad (20)$$

For $q \in (1, \infty)$, taking the L^q -norm to this equation, we derive that

$$\begin{aligned} \|\Delta_j \omega(t)\|_{L^q} &\leq C e^{-c_0 \mu t 2^{2\alpha j}} \|\Delta_j \omega_0\|_{L^q} + C \int_0^t 2^{2j} e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|\Delta_j (b \otimes b)(\tau)\|_{L^q} d\tau \\ &+ C \int_0^t 2^{2j} e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|\Delta_j (u \otimes u)(\tau)\|_{L^q} d\tau + C \int_0^t 2^j e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|\Delta_j \theta(\tau)\|_{L^q} d\tau. \end{aligned} \quad (21)$$

Choose σ such that

$$0 < \sigma < \alpha - 1, \sigma + \beta = \frac{d}{2} \left(1 - \frac{1}{q}\right). \quad (22)$$

By Sobolev's inequality,

$$\|\Delta_j (b \otimes b)(\tau)\|_{L^q} \leq \|b \otimes b(\tau)\|_{L^q} \leq \|b\|_{L^{2q}}^2 \leq C \|\Lambda^{\sigma+\beta} b\|_{L^2}^2. \quad (23)$$

Similarly, we have

$$\|\Delta_j (u \otimes u)(\tau)\|_{L^q} \leq C \left(\|u\|_{L^2}^2 + \|\Lambda^{2\alpha-1} u\|_{L^2}^2 \right) = C \|u\|_{H^{2\alpha-1}}^2. \quad (24)$$

We multiply the second equation in (1) by $\theta|\theta|^{q-2}$ and integrate the result in space domain to obtain

$$\frac{1}{q} \frac{d}{dt} \|\theta\|_{L^q}^q \leq u \cdot \mathbf{e}_d \theta |\theta|^{q-2} \leq \|u\|_{L^q} \|\theta\|_{L^q}^{q-1}, \quad (25)$$

which yields

$$\|\theta\|_{L^q} \leq \|\theta_0\|_{L^q} + \int_0^t \|u(\tau)\|_{L^q} d\tau \leq C + C \int_0^t \|u(\tau)\|_{H^{2\alpha-1}} d\tau. \quad (26)$$

Note that $\|\Delta_j \theta\|_{L^q} \leq \|\theta\|_{L^q}$; then, inserting (23)–(26) into (21), we derive that

$$\begin{aligned} \|\Delta_j \omega(t)\|_{L^q} &\leq C e^{-c_0 \mu t 2^{2\alpha j}} \|\Delta_j \omega_0\|_{L^q} + C \int_0^t 2^{2j} e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|\Lambda^{\sigma+\beta} b(\tau)\|_{L^2}^2 d\tau \\ &+ C \int_0^t 2^{2j} e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|u(\tau)\|_{H^{2\alpha-1}}^2 d\tau + C \int_0^t 2^j e^{-c_0 \mu(t-\tau) 2^{2\alpha j}} \|u(\tau)\|_{H^{2\alpha-1}} d\tau. \end{aligned} \quad (27)$$

Integrating this in time, together with (12), we obtain

$$\begin{aligned} 2^{(2\alpha-2)j} \int_0^t \|\Delta_j \omega(\tau)\|_{L^q} d\tau &\leq C 2^{-2j} \|\Delta_j \omega_0\|_{L^q} + C \int_0^t \|\Lambda^{\sigma+\beta} b(\tau)\|_{L^2}^2 d\tau + C \int_0^t (\|u(\tau)\|_{H^{2\alpha-1}}^2 + \|u(\tau)\|_{H^{2\alpha-1}}) d\tau \\ &\leq C + C \int_0^t (\|u(\tau)\|_{H^{2\alpha-1}}^2 + \|u(\tau)\|_{H^{2\alpha-1}}) d\tau. \end{aligned} \quad (28)$$

Next, we apply $\Lambda^{\alpha-1}$ to the first equation of (1) to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Lambda^{\alpha-1} u\|_{L^2}^2 + \mu \|\Lambda^{2\alpha-1} u\|_{L^2}^2 \\ &= \int \Lambda^{\alpha-1} (b \cdot \nabla b) \cdot \Lambda^{\alpha-1} u - \int \Lambda^{\alpha-1} (u \cdot \nabla u) \cdot \Lambda^{\alpha-1} u + \int \Lambda^{\alpha-1} (\theta \mathbf{e}_d) \cdot \Lambda^{\alpha-1} u \\ &:= J_1 + J_2 + J_3. \end{aligned} \quad (29)$$

Using $\nabla \cdot b = 0$,

$$\begin{aligned}
J_1 &\leq C \|\Lambda^{-1} \nabla \cdot (b \otimes b)\|_{L^2} \|\Lambda^{2\alpha-1} u\|_{L^2} \\
&\leq C \|b\|_{L^4} \|\Lambda^{2\alpha-1} u\|_{L^2} \\
&\leq \frac{\mu}{8} \|\Lambda^{2\alpha-1} u\|_{L^2}^2 + C \|\Lambda^{\sigma_0+\beta} b\|_{L^2}^2,
\end{aligned} \tag{30}$$

where we have chosen σ_0 satisfying

$$0 < \sigma_0 < \alpha - 1, \sigma_0 + \beta = \frac{d}{4}. \tag{31}$$

Similarly, we have

$$\begin{aligned}
J_2 &\leq C \|\Lambda^{-1} \nabla \cdot (u \otimes u)\|_{L^2} \|\Lambda^{2\alpha-1} u\|_{L^2} \\
&\leq C \|u\|_{L^4} \|\Lambda^{2\alpha-1} u\|_{L^2} \\
&\leq \frac{\mu}{8} \|\Lambda^{2\alpha-1} u\|_{L^2}^2 + C \|\Lambda^\alpha u\|_{H^1}^2,
\end{aligned} \tag{32}$$

$$\begin{aligned}
J_3 &\leq C \|\theta e_d\|_{L^2} \|\Lambda^{2\alpha-2} u\|_{L^2} \\
&\leq C \|\theta\|_{L^2} \|u\|_{L^2}^{1/2\alpha-1} \|\Lambda^{2\alpha-1} u\|_{L^2}^{2\alpha-2/2\alpha-1} \\
&\leq \frac{\mu}{8} \|\Lambda^{2\alpha-1} u\|_{L^2}^2 + C (\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2).
\end{aligned} \tag{33}$$

Inserting (30)–(33) into (29),

$$\begin{aligned}
&\frac{d}{dt} \|\Lambda^{\alpha-1} u\|_{L^2}^2 + \mu \|\Lambda^{2\alpha-1} u\|_{L^2}^2 \\
&\leq C \left(\|\Lambda^{\sigma_0+\beta} b\|_{L^2}^2 + \|\Lambda^\alpha u\|_{H^1}^2 + \|u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right).
\end{aligned} \tag{34}$$

Then, Gronwall's inequality and (10)–(12) yields

$$\|\Lambda^{\alpha-1} u\|_{L^2}^2 + \int_0^t \|\Lambda^{2\alpha-1} u(\tau)\|_{L^2}^2 d\tau \leq C. \tag{35}$$

This together with (26) yields (17). Substituting (35) into (28), we obtain

$$\sup_{j \geq -1} 2^{(2\alpha-2)j} \int_0^t \|\Delta_j \omega\|_{L^q} d\tau < \infty. \tag{36}$$

Thus, for $q > d/2\alpha - 2$,

$$\begin{aligned}
&\|\Lambda^{d/q} \omega(\tau)\|_{L^q} d\tau \leq C \int_0^t \|\omega(\tau)\|_{B_{q,1}^{d/q}} d\tau \\
&= C \int_0^t \sum_{j \geq -1} 2^{d/qj} \|\Delta_j \omega(\tau)\|_{L^q} d\tau \\
&\leq C \int_0^t \sum_{j \geq -1} 2^{(d/q-(2\alpha-2))j} 2^{(2\alpha-2)j} \|\Delta_j \omega(\tau)\|_{L^q} d\tau \\
&= C \sum_{j \geq -1} 2^{(d/q-(2\alpha-2))j} 2^{(2\alpha-2)j} \int_0^t \|\Delta_j \omega(\tau)\|_{L^q} d\tau \\
&\leq C \sup_{j \geq -1} 2^{(2\alpha-2)j} \int_0^t \|\Delta_j \omega(\tau)\|_{L^q} d\tau \sum_{j \geq -1} 2^{(d/q-(2\alpha-2))j} \\
&= C \sup_{j \geq -1} 2^{(2\alpha-2)j} \int_0^t \|\Delta_j \omega(\tau)\|_{L^q} d\tau < \infty.
\end{aligned} \tag{37}$$

This is (16).

With Lemmas 4–6 at our disposal, we are ready to prove Proposition 1. \square

Proof. of Proposition 1. Applying Λ^s to (1) and taking the L^2 -inner product with $(\Lambda^s u, \Lambda^s b, \Lambda^s \theta)$ yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2 \right) + \mu \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \nu \|\Lambda^{s+\beta} b\|_{L^2}^2 \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7,
\end{aligned} \tag{38}$$

where

$$K_1 = - \int [\Lambda^s, u \cdot \nabla] u \cdot \Lambda^s u, \quad |K_1| \leq C \|\nabla u\|_{L^\infty} \|\Lambda^s u\|_{L^2}^2. \quad (40)$$

$$K_2 = \int \Lambda^s (\theta \mathbf{e}_d) \cdot \Lambda^s u,$$

By Young's inequality,

$$|K_2 + K_7| \leq \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2. \quad (41)$$

$$K_3 = \int \Lambda^s (b \cdot \nabla b) \cdot \Lambda^s u,$$

Using $\nabla \cdot b = 0$, we have

$$K_4 = - \int [\Lambda^s, u \cdot \nabla] b \cdot \Lambda^s b, \quad (39)$$

$$|K_3| \leq C \|\Lambda^s b\|_{L^2} \|b\| \frac{d}{L\alpha - 1} \|\Lambda^{s+1} u\|_{L^2}^{d/d+2-2\alpha}$$

$$K_5 = \int [\Lambda^s, b \cdot \nabla] u \cdot \Lambda^s b,$$

$$\leq C \|\Lambda^s b\|_{L^2} \|\Lambda^\beta b\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \quad (42)$$

$$K_6 = - \int [\Lambda^s, u \cdot \nabla] \theta \cdot \Lambda^s \theta,$$

$$\leq \frac{\mu}{8} \|\Lambda^{s+\alpha} u\|_{L^2}^2 + C \|\Lambda^\beta b\|_{L^2}^2 \|\Lambda^s b\|_{L^2}^2.$$

$$K_7 = \int \Lambda^s (u \cdot \mathbf{e}_d) \Lambda^s \theta.$$

Similarly, by Lemma 1, K_4 and K_5 are bounded by

Taking advantage of Lemma 1,

$$\begin{aligned} |K_4 + K_5| &\leq C \|\nabla u\|_{L^\infty} \|\Lambda^s b\|_{L^2}^2 + C \|\Lambda^s b\|_{L^2} \|b\| \frac{d}{L\alpha - 1} \|\Lambda^{s+1} u\|_{L^2}^{d/d+2-2\alpha} \\ &\leq \frac{\mu}{8} \|\Lambda^{s+\alpha} u\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\Lambda^s b\|_{L^2}^2 + C \|\Lambda^\beta b\|_{L^2}^2 \|\Lambda^s b\|_{L^2}^2. \end{aligned} \quad (43)$$

By Lemma 1,

Combining the above bounds with (38), we infer that

$$\begin{aligned} |K_6| &\leq C \|\nabla u\|_{L^\infty} \|\Lambda^s \theta\|_{L^2}^2 + C \|\Lambda^s \theta\|_{L^2} \|\Lambda^{1+s} u\|_{L^2}^{d/d+2-2\alpha} \|\theta\|_{L^{d/\alpha-1}} \\ &\leq C \|\nabla u\|_{L^\infty} \|\Lambda^s \theta\|_{L^2}^2 + C \|\Lambda^s \theta\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} \|\theta\|_{L^{2/\alpha-1}} \\ &\leq \frac{\mu}{8} \|\Lambda^{s+\alpha} u\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\Lambda^s \theta\|_{L^2}^2 + C \|\theta\|_{L^{d/\alpha-1}}^2 \|\Lambda^s \theta\|_{L^2}^2. \end{aligned} \quad (44)$$

$$\begin{aligned} &\frac{d}{dt} \left(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2 \right) + \mu \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \nu \|\Lambda^{s+\beta} b\|_{L^2}^2 \\ &\leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^\beta b\|_{L^2}^2 + \|\theta\|^2 \frac{d}{L\alpha - 1} \right) \left(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2 \right). \end{aligned} \quad (45)$$

We bound $\|\nabla u\|_{L^\infty}$ by Lemma 3. By Lemma 6, (10), and (17), we obtain

$$\int_0^t \|\Lambda^{d/q} \omega(\tau)\|_{L^q} d\tau < \infty, \quad \int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau < \infty, \quad \int_0^t \|\theta\|_{L^{2/\alpha-1}}^2 d\tau < \infty. \quad (46)$$

Then, the desired global H^s -bound follows from these bounds and Osgood's inequality. This completes the proof of Proposition 1. \square

Appendix

We recall the definitions of the Littlewood–Paley decomposition and Besov spaces in this appendix (see, e.g., [13, 16–19]).

Let \mathcal{S} be the usual Schwartz class and \mathcal{S}' be its dual. Write for each $j \in \mathbb{Z}$,

$$A_j = \{\xi \in \mathbb{R}^d: 2^{j-1} \leq |\xi| < 2^{j+1}\}. \quad (\text{A.1})$$

The Littlewood–Paley decomposition means that there exist functions $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$ such that

$$\text{supp} \widehat{\Phi}_j \subset A_j, \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \text{ or } \Phi_j(x) = 2^{jd}\Phi_0(2^jx),$$

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases} \quad (\text{A.2})$$

Thus, for $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) \widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}. \quad (\text{A.3})$$

Choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d. \quad (\text{A.4})$$

Then, for all $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi, \quad (\text{A.5})$$

and hence

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + j d(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^d)}, \quad (\text{A.12})$$

where C_1 and C_2 are constants depending on α , p , and q only.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f, \quad (\text{A.6})$$

in \mathcal{S}' for any $f \in \mathcal{S}'$. In addition, set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \dots \end{cases} \quad (\text{A.7})$$

Definition A.1. The inhomogeneous Besov space $B_{p,q}^s$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B_{p,q}^s} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty, \quad (\text{A.8})$$

where $\Delta_j f$ is as defined in (A.2).

An important tool in dealing with Fourier localized functions is the following Bernstein's inequality.

Proposition A.1. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

(1) *If f satisfies*

$$\text{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d: |\xi| \leq K2^j\}, \quad (\text{A.9})$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + j d(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^d)}. \quad (\text{A.10})$$

(2) *If f satisfies*

$$\text{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d: K_1 2^j \leq |\xi| \leq K_2 2^j\}, \quad (\text{A.11})$$

for some integer j and constants $0 < K_1 \leq K_2$, then

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