Research Article

Numerical Solution of the Multiterm Time-Fractional Model for Heat Conductivity by Local Meshless Technique

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Fractional partial differential equation models are frequently used to several physical phenomena. Despite the ability to express many complex phenomena in different disciplines, researchers have found that multiterm time-fractional PDEs improve the modeling accuracy for describing diffusion processes in contrast to the results of a single term. Nowadays, it attracts the attention of the active researchers. The aim of this work is concerned with the approximate numerical solutions of the three-term time-fractional Sobolev model equation using computationally attractive and reliable technique, known as a local meshless method. Because of the meshless character and the simple application in higher dimensions, there is a growing interest in meshless techniques. To assess the reliability and accuracy of the proposed method, three test problems and two types of irregular domains are taken into account.

1. Introduction

In recent years, fractional partial differential equations (FPDEs) have drawn the consideration of numerous researchers to their applications in various fields of science and technology. Partial derivatives provide a flexible model and an extraordinary tool for description of capturing the history of the variable and genetic characteristics of various dynamic systems. Extensive research has

been carried out in the advancement of numerical and analytical solutions of linear and nonlinear FPDEs [1–6]. However, several researchers have not succeeded in deriving and modeling many complex phenomena utilizing linear or nonlinear PDEs with integer order [7]. Subsequently, the fractional is taken as account and is a good solution to this problem [8]. In the current work, three-term time-fractional Sobolev equation is considered which can be expressed as

$$\begin{aligned} & \frac{\partial^{\beta_1} \mathcal{V}(y, z, t)}{\partial t^{\beta_1}} + \frac{\partial^{\beta_2} \mathcal{V}(y, z, t)}{\partial t^{\beta_2}} + \frac{\partial^{\beta_3} \mathcal{V}(y, z, t)}{\partial t^{\beta_3}} - \frac{\partial \nabla^2 \mathcal{V}(y, z, t)}{\partial t} - \beta \nabla^2 \mathcal{V}(y, z, t) + \gamma \nabla(\mathcal{V}(y, z, t) \nabla \mathcal{V}(\bar{z}, t)) \\ & + \delta \mathcal{V}(y, z, t) = F(\bar{z}, t), \quad (y, z) \in \Omega, \quad 0 < \beta_3 \leq \beta_2 \leq \beta_1 \leq 1, \quad t > 0. \end{aligned} \quad (1)$$

With the conditions,

$$\begin{aligned} \mathcal{V}(y, z, 0) &= \mathcal{V}_0(y, z), \\ \mathcal{V}(y, z, t) &= g_1(y, z, t), \quad (y, z) \in \partial\Omega, \end{aligned} \quad (2)$$

where ∇^2 is the Laplacian and ∇ denotes gradient operators, and β , γ , and δ are known constants, whereas $(\partial^{\beta_1}/\partial t^{\beta_1})$, $(\partial^{\beta_2}/\partial t^{\beta_2})$, and $(\partial^{\beta_3}/\partial t^{\beta_3})$ represent the Caputo derivative operator of order $0 < \beta_3 \leq \beta_2 \leq \beta_1 \leq 1$ for the function $\mathcal{V}(y, z, t)$.

In recent literature, various meshless methods have been utilized for the numerical solution of various PDE models almost in every discipline of science and engineering. In particular, the RBF-based meshless methods are the mainstream of these methods. The meshless nature is one of the main reasons behind the developing interest for such approaches. The meshless methods significantly reduce the complexity of dimensionality utilizing traditional methods such as the finite element and finite difference methods. Compared to mesh-based methods, these methods do not require mesh in the domain. The meshless methods have the ability to compute the solution in regular and irregular domain utilizing scattered or uniform nodes, which increases the priority and the advantages of meshless methods. As these facts show, these methods are really workable and useful numerical methods that can be applied to real-world challenging problems [9–17].

The RBF-based meshless methods have also some deficiencies like other numerical methods, in which the most important one is the dense ill-conditioned matrices and the selection of the optimal value of the shape parameter. To avoid these drawbacks, local meshless methods are the best alternatives, suggested by the researchers which are considered to be accurate and stable for the solution of diverse integer and fractional-order PDE models [18, 19]. The local

meshless methods are less sensitive to the change in shape parameters than the global version, and it produces well-conditioned sparse matrices. Furthermore, local version of meshless methods is considered to be more effective and efficient than global ones. In recent years, the abilities of various sorts of local meshless methods in different applications have been explored [20–22].

In the current research, we have implemented the local meshless method to approximate the numerical solution of three-term time-fractional model equation (1). For this purpose, multiquadric (MQ) radial basis functions (RBFs) are used. Furthermore, two types of irregular domains are also taken in numerical examples.

2. Methodology of the Local Meshless Method

According to the local meshless method, to approximate the derivatives of $\mathcal{V}(\bar{z}, t)$ at the centers $\bar{z}_h$ by the neighborhood of $\bar{z}_h$, $\{\bar{z}_{h1}, \bar{z}_{h2}, \bar{z}_{h3}, \dots, \bar{z}_{hn_h}\} \subset \{\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{N^n}\}$, $n_h \ll N^n$, where $h = 1, 2, \dots, N^n$, we have used $\bar{z} = y$ and $\bar{z} = (y, z)$ for one-dimensional and two-dimensional cases, respectively.

Now, considering the following case for one-dimensional,

$$\mathcal{V}^{(m)}(y_h) \approx \sum_{k=1}^{n_h} \lambda_k^{(m)} \mathcal{V}(y_{hk}), \quad h = 1, 2, \dots, N. \quad (3)$$

Substituting the multiquadric RBF $\psi(\|y - y_p\|) = \sqrt{1 + (c\|y_{hk} - y_p\|)^2}$ in (3),

$$\psi^{(m)}(\|y_h - y_p\|) = \sum_{k=1}^{n_h} \lambda_{hk}^{(m)} \psi(\|y_{hk} - y_p\|), \quad p = h1, h2, \dots, hn_h. \quad (4)$$

Equation (4) in matrix form is

$$\underbrace{\begin{bmatrix} \psi_{h1}^{(m)}(y_h) \\ \psi_{h2}^{(m)}(y_h) \\ \vdots \\ \psi_{hn_h}^{(m)}(y_h) \end{bmatrix}}_{\psi_{n_h}^{(m)}} = \underbrace{\begin{bmatrix} \psi_{h1}(y_{h1}) & \psi_{h2}(y_{h1}) & \cdots & \psi_{hn_h}(y_{h1}) \\ \psi_{h1}(y_{h2}) & \psi_{h2}(y_{h2}) & \cdots & \psi_{hn_h}(y_{h2}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{h1}(y_{hn_h}) & \psi_{h2}(y_{hn_h}) & \cdots & \psi_{hn_h}(y_{hn_h}) \end{bmatrix}}_{A_{n_h}} \underbrace{\begin{bmatrix} \lambda_{h1}^{(m)} \\ \lambda_{h2}^{(m)} \\ \vdots \\ \lambda_{hn_h}^{(m)} \end{bmatrix}}_{\lambda_{n_h}^{(m)}}, \quad (5)$$

where

$$\psi_p(\|y_k\|) = \psi(\|y_k - y_p\|), \quad p = h1, h2, \dots, hn_h, \quad (6)$$

for each $k = i1, h2, \dots, hn_h$. Equation (5) in simple form is

$$\psi_{n_h}^{(m)} = A_{n_h} \lambda_{n_h}^{(m)}. \quad (7)$$

From (7), we obtain

$$\lambda_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \Psi_{n_h}^{(m)}. \quad (8)$$

(3) and (8) implies

$$\mathcal{V}^{(m)}(y_h) = (\lambda_{n_h}^{(m)})^T \mathbf{V}_{n_h}, \quad (9)$$

where

$$\mathbf{V}_{n_h} = [\mathcal{V}(y_{h1}), \mathcal{V}(y_{h2}), \dots, \mathcal{V}(y_{hn_h})]^T. \quad (10)$$

The derivatives of $\mathcal{V}(y, z, t)$ w. r. t. y and z can be found

as

$$\begin{aligned} \mathcal{V}_y^{(m)}(y_h, z_h) &\approx \sum_{k=1}^{n_h} \gamma_k^{(m)} \mathcal{V}(y_{hk}, z_{hk}), \quad h = 1, 2, \dots, N^2, \\ \mathcal{V}_z^{(m)}(y_h, z_h) &\approx \sum_{k=1}^{n_h} \eta_k^{(m)} \mathcal{V}(y_{hk}, z_{hk}), \quad h = 1, 2, \dots, N^2. \end{aligned} \quad (11)$$

For $\gamma_k^{(m)}$ and $\eta_k^{(m)}$ ($k = 1, 2, \dots, n_h$), we continue as

$$\begin{aligned} \gamma_{n_h}^{(m)} &= \mathbf{A}_{n_h}^{-1} \Phi_{n_h}^{(m)}, \\ \eta_{n_h}^{(m)} &= \mathbf{A}_{n_h}^{-1} \Phi_{n_h}^{(m)}. \end{aligned} \quad (12)$$

The time derivative $(\partial^{\beta_1} \mathcal{V}(\bar{\mathbf{z}}, t) / \partial t^{\beta_1})$ is discretized utilizing Caputo derivative [23], where $\beta_1 \in (0, 1)$ as

$$\frac{\partial^{\beta_1} \mathcal{V}(\bar{\mathbf{z}}, t)}{\partial t^{\beta_1}} = \begin{cases} \frac{1}{\Gamma(1-\beta_1)} \int_0^t \frac{\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t-\vartheta)^{-\beta_1} d\vartheta, & 0 < \beta_1 < 1, \\ \frac{\partial \mathcal{V}(\bar{\mathbf{z}}, t)}{\partial t}, & \beta_1 = 1. \end{cases} \quad (13)$$

Let τ be the time step size, and for the interval $[0, t]$, consider $t_q = q\tau$, $q = 0, 1, 2, \dots, Q$. We complete the time-fractional derivative term as

$$\begin{aligned} \frac{\partial^{\beta_1} \mathcal{V}(\bar{\mathbf{z}}, t_{q+1})}{\partial t^{\beta_1}} &= \frac{1}{\Gamma(1-\beta_1)} \int_0^{t_{q+1}} \frac{\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t_{q+1} - \vartheta)^{-\beta_1} d\vartheta, \\ &= \frac{1}{\Gamma(1-\beta_1)} \sum_{r=0}^q \int_{r\tau}^{(r+1)\tau} \frac{\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t_{r+1} - \vartheta)^{-\beta_1} d\vartheta, \\ &\approx \frac{1}{\Gamma(1-\beta_1)} \sum_{r=0}^q \int_{r\tau}^{(r+1)\tau} \frac{\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta_r)}{\partial \vartheta} (t_{r+1} - \vartheta)^{-\beta_1} d\vartheta. \end{aligned} \quad (14)$$

The term $(\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta_r) / \partial \vartheta)$ is approximated as follows:

$$\frac{\partial \mathcal{V}(\bar{\mathbf{z}}, \vartheta_r)}{\partial \vartheta} = \frac{\mathcal{V}(\bar{\mathbf{z}}, \vartheta_{r+1}) - \mathcal{V}(\bar{\mathbf{z}}, \vartheta_r)}{\vartheta} + \mathcal{O}(\tau). \quad (15)$$

Then,

$$\begin{aligned}
& \frac{\partial^{\beta_1} \mathcal{V}(\bar{z}, t_{q+1}) \approx 1/\Gamma(1-\beta_1) \sum_{r=0}^q \mathcal{V}(\bar{z}, t_{r+1}) - \mathcal{V}(\bar{z}, t_r)/\tau \int_{r\tau}^{(r+1)\tau} (t_{r+1} - \vartheta)^{-\beta_1} d\vartheta}{\partial t^{\beta_1}}, \\
& = \frac{1}{\Gamma(1-\beta_1)} \sum_{r=0}^q \frac{\mathcal{V}(\bar{z}, t_{q+1-r}) - \mathcal{V}(\bar{z}, t_{q-r})}{\tau} \int_{r\tau}^{(r+1)\tau} (t_{r+1} - \vartheta)^{-\beta_1} d\vartheta, \\
& = \begin{cases} \frac{\tau^{-\beta_1}}{\Gamma(2-\beta_1)} (\mathcal{V}^{q+1} - \mathcal{V}^q) + \frac{\tau^{-\beta_1}}{\Gamma(2-\beta_1)} \sum_{r=1}^q (\mathcal{V}^{q+1-r} - \mathcal{V}^{q-r}) [(r+1)^{1-\beta_1} - r^{1-\beta_1}], & q \geq 1, \\ \frac{\tau^{-\beta_1}}{\Gamma(2-\beta_1)} (\mathcal{V}^1 - \mathcal{V}^0), & q = 0. \end{cases}
\end{aligned} \tag{16}$$

Letting $a_0 = (\tau^{-\beta_1}/\Gamma(2-\beta_1))$ and $b_r = (r+1)^{1-\beta_1} - r^{1-\beta_1}$, $r = 0, 1, \dots, q$, we have

$$\frac{\partial^{\beta_1} \mathcal{V}(\bar{z}, t_{q+1})}{\partial t^{\beta_1}} \approx \begin{cases} a_0 (\mathcal{V}^{q+1} - \mathcal{V}^q) + a_0 \sum_{r=1}^q b_r (\mathcal{V}^{q+1-r} - \mathcal{V}^{q-r}), & q \geq 1, \\ a_0 (\mathcal{V}^1 - \mathcal{V}^0), & q = 0. \end{cases} \tag{17}$$

The fractional derivative of order β_2 and β_3 can be found as above.

3. Numerical Experiments

This section examines the accuracy and applicability of the proposed method for the three-term time-fractional model (1). In the test problems, we have considered regular and irregular domains. This computation is considered to be regular and scattered nodes with regular and irregular domains. In this article, we have used the Crank–Nicholson scheme and multiquadric (MQ) RBF with shape parameter value $c = 10$. Unless specifically stated, the spatial domain $[0, 4]$ and time step size $\tau = 0.002$ are used. Accuracy is measured as follows:

$$L_{\text{absolute}} = |\hat{\mathbb{V}} - \mathcal{V}|,$$

$$\text{Max - error} = \max(L_{\text{absolute}}), \tag{18}$$

$$\text{RMS} = \sqrt{\frac{\sum_{h=1}^{N^n} (\hat{\mathbb{V}}_h - \mathcal{V}_h)^2}{N}},$$

where $\hat{\mathbb{V}}$ is the exact solution, and $\mathcal{V}$ is the approximate solution.

Problem 1. Consider the model equation:

$$\begin{aligned}
& \frac{\partial^{\beta_1} \mathcal{V}(y, z, t)}{\partial t^{\beta_1}} + \frac{\partial^{\beta_2} \mathcal{V}(y, z, t)}{\partial t^{\beta_2}} + \frac{\partial^{\beta_3} \mathcal{V}(y, z, t)}{\partial t^{\beta_3}} - \frac{\partial \nabla^2 \mathcal{V}(y, z, t)}{\partial t} - \nabla^2 \mathcal{V}(y, z, t) = F(y, z, t), \\
& 0 < \beta_3 \leq \beta_2 \leq \beta_1 \leq 1, \\
& t > 0.
\end{aligned} \tag{19}$$

Having the exact solution,

$$\mathcal{V}(y, z, t) = e^{-t} \sin(\pi y) \sin(\pi z), \quad (y, z) \in \Omega. \tag{20}$$

The proposed meshless method is implemented for generating the required numerical results for Problem 1, which are given in Table 1. Different values of a number of

nodes N , fractional order $\beta_1 = \beta_2 = \beta_3$, and final time $t = 1$ are used, whereas the error norms stand for max – error and RMS. These results revealed the fact that the recommended meshless method is capable of better results. Showing the accurate and efficient of the method, the results are compared with the exact solution for $\beta_1 = \beta_2 = \beta_3 = 0.1$, $\beta_1 = \beta_2 = \beta_3 = 0.3$, $\beta_1 = \beta_2 = \beta_3 = 0.5$, $t = 1$, $t = 2$, and for

TABLE 1: Problem 1, approximate results for $t = 1$.

N	$\beta_1 = \beta_2 = \beta_3 = 0.2$		$\beta_1 = \beta_2 = \beta_3 = 0.5$		$\beta_1 = \beta_2 = \beta_3 = 0.8$	
	Max – error	RMS	Max – error	RMS	Max – error	RMS
8^2	$8.5869e-08$	$3.9708e-08$	$3.8841e-07$	$1.2258e-07$	$4.9568e-06$	$1.7762e-06$
10^2	$8.2120e-08$	$3.9848e-08$	$4.8466e-07$	$1.4140e-07$	$5.8961e-06$	$1.9704e-06$
12^2	$8.8415e-08$	$3.9397e-08$	$5.2865e-07$	$1.6405e-07$	$6.1379e-06$	$2.1931e-06$

TABLE 2: Problem 1, approximate results using $N = 8^2$ and $\beta = \beta_1 = \beta_2 = \beta_3$.

τ	Max – error					
	$t = 1$			$t = 2$		
	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$
0.2	$1.1044e-03$	$9.9648e-04$	$7.9676e-04$	$7.9063e-04$	$7.1256e-04$	$5.7016e-04$
0.02	$1.0790e-05$	$8.0577e-06$	$3.9095e-06$	$7.7071e-06$	$5.6947e-06$	$3.1952e-06$
0.002	$1.0500e-07$	$4.3655e-08$	$3.8841e-07$	$7.4936e-08$	$3.0435e-08$	$2.9006e-07$

various values of time step size τ . These results are computed using $N = 8^2$ and are given in Table 2. One can observe from this table that only in few iterations, the suggested meshless method produced better results, and as the number of time iteration increases, the accuracy increase and the error norm reached up to $\max - \text{error} \approx 10^{-10}$. As the condition number, stability, and accuracy of the RBF-based meshless methods heavily depend on the value of shape parameter c , a little change in shape parameter value causes instability and the results get diverge. But the suggested local meshless method is tested for Problem 1 in terms of condition

number, stability, and accuracy as shown in Figure 1 for $N = 10^2$, $\beta_1 = \beta_2 = \beta_3 = 0.5$, and $t = 1$. This figure revealed that the suggested meshless method is stable, accurate, and given ideal low condition number ≈ 1 for a long range of c up to 2000. Figure 2 shows the absolute error using $\beta_1 = \beta_2 = \beta_3 = 0.1$ and $\beta_1 = \beta_2 = \beta_3 = 0.8$ for $N = 10^2$ and $t = 1$. Better accuracy of the recommended algorithm can be seen in this figure.

Problem 2. Consider the model equation:

$$\frac{\partial^{\beta_1} \mathcal{V}(y, z, t)}{\partial t^{\beta_1}} + \frac{\partial^{\beta_2} \mathcal{V}(y, z, t)}{\partial t^{\beta_2}} + \frac{\partial^{\beta_3} \mathcal{V}(y, z, t)}{\partial t^{\beta_3}} - \frac{\partial \nabla^2 \mathcal{V}(y, z, t)}{\partial t} - \nabla^2 \mathcal{V}(y, z, t) = F(y, z, t), \quad (21)$$

$$0 < \beta_3 \leq \beta_2 \leq \beta_1 \leq 1,$$

$$t > 0.$$

Having the exact solution,

$$\mathcal{V}(y, z, t) = e^{y^2 - z^2 - t} \sin(\pi y) \sin(\pi z), \quad (y, z) \in \Omega. \quad (22)$$

In Table 3, we have implemented the suggested algorithm for generating the numerical results for Problem 2 for $N = 8^2$, $N = 10^2$, $N = 12^2$, $\beta_1 = \beta_2 = \beta_3 = 0.2$, $\beta_1 = \beta_2 = \beta_3 = 0.4$, $\beta_1 = \beta_2 = \beta_3 = 0.4$, and $t = 1$, $t = 2$. The results are assessed in term of $\max - \text{error}$ and RMS. Accurate results have been obtained in this problem as well. Showing the applicability and efficacy of the propose method, the results are compared with the exact solution for various values of $\beta_1 = \beta_2 = \beta_3$, t and τ using $N = 8^2$. These results are given in Table 4. One can observe from this table that only in few iterations, the suggested meshless method produced better results, and as the number of time iteration increases, the accuracy increase and the error norm reached up to $\text{RMS} \approx 10^{-9}$.

Just like the previous problem, the suggested method has been tested for Problem 2 in terms of condition number,

stability, and accuracy as shown in Figure 3 for $N = 10^2$, $\beta = 0.5$, and $t = 1$. It can easily be seen from the figure that the suggest meshless method is stable, accurate, and given ideal low condition number ≈ 1 for a long range of c up to 2000, whereas in Figure 4, we have shown a comparison of exact and approximate solutions for various values of time t and brilliant match of both the solutions can be found in this figure.

One of the principle advantages of the meshless techniques over mesh-based techniques is the implementation in the irregular domain with ease. In this article, two types of challenging irregular domains are taken into account, which are displayed in Figure 5. In Table 5, we have shown the numerical results obtained by the suggested meshless method corresponding to the irregular domains for Problem 1 and Problem 2. We have considered the various value of β 's, and the results are shown in form of $\max - \text{error}$ and RMS. It is observed from the table that better accuracy has been achieved in both domains.

FIGURE 1: Problem 1. (a) c and error norms, (b) c and condition number. FIGURE 2: Problem 1. Absolute error for $\beta = 0.1$ (a) and $\beta = 0.8$ (b), where $\beta = \beta_1 = \beta_2 = \beta_3$.TABLE 3: Problem 2, approximate results for $t = 1$.

N	$\beta_1 = \beta_2 = \beta_3 = 0.2$		$\beta_1 = \beta_2 = \beta_3 = 0.4$		$\beta_1 = \beta_2 = \beta_3 = 0.6$	
	Max - error	RMS	Max - error	RMS	Max - error	RMS
8^2	$1.2325e-06$	$2.0404e-07$	$2.3271e-06$	$3.3074e-07$	$1.9346e-05$	$2.5361e-06$
10^2	$1.5895e-06$	$2.1784e-07$	$3.2329e-06$	$4.5136e-07$	$2.6251e-05$	$3.2570e-06$
12^2	$1.4030e-06$	$2.1123e-07$	$4.2754e-06$	$5.8259e-07$	$3.1278e-05$	$3.9590e-06$

TABLE 4: Problem 2, approximate results for $N = 8^2$.

τ	RMS					
	$t = 1$			$t = 2$		
	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$
0.2	$2.6139e-03$	$2.2018e-03$	$1.5260e-03$	$1.8624e-03$	$1.5697e-03$	$1.0945e-03$
0.02	$2.3792e-05$	$1.2980e-05$	$4.3173e-05$	$1.6857e-05$	$9.2597e-06$	$3.3106e-05$
0.002	$2.0404e-07$	$3.3074e-07$	$2.5361e-06$	$1.4425e-07$	$2.4742e-07$	$1.9042e-06$
0.0002	$1.5812e-09$	$1.1891e-08$	$1.1014e-07$	$1.1152e-09$	$8.7439e-09$	$8.2401e-08$

(a)

(b)

FIGURE 3: Problem 2. (a) c and error norms. (b) c and condition number.

FIGURE 4: Problem 2, approximate and exact solution for indicated time.

FIGURE 5: Computational domain 1 (a) and domain 2 (b).

TABLE 5: Approximate results corresponding to the irregular domains.

	Domain	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.8$	
		Max – error	RMS	Max – error	RMS	Max – error	RMS
Problem 1	Domain 1	$2.8188e-07$	$7.0294e-08$	$3.2180e-06$	$8.2127e-07$	$3.3601e-05$	$9.0154e-06$
	Domain 2	$1.9344e-06$	$3.4442e-07$	$2.0244e-05$	$3.8603e-06$	$1.1285e-04$	$2.6506e-05$
Problem 2	Domain 1	$9.0788e-06$	$1.1575e-06$	$9.5402e-05$	$1.0099e-05$	$1.0994e-03$	$1.1639e-04$
	Domain 2	$3.3707e-06$	$6.0517e-07$	$3.2748e-05$	$6.3853e-06$	$1.2824e-04$	$3.2931e-05$

FIGURE 6: Problem 3. (a) Exact solution. (b) Numerical solution.

Problem 3. Consider the model equation:

$$\begin{aligned}
 & \frac{\partial^{\beta_1} \mathcal{V}(y, z, t)}{\partial t^{\beta_1}} + \frac{\partial^{\beta_2} \mathcal{V}(y, z, t)}{\partial t^{\beta_2}} + \frac{\partial^{\beta_3} \mathcal{V}(y, z, t)}{\partial t^{\beta_3}} - \frac{\partial \nabla^2 \mathcal{V}(y, z, t)}{\partial t} - \nabla^2 \mathcal{V}(y, z, t) + \nabla(\mathcal{V}(y, z, t) \nabla \mathcal{V}(\bar{\mathbf{z}}, t)) \\
 & + \pi^2 \mathcal{V}(y, z, t) = F(y, z, t), \quad (y, z) \in \Omega, \quad 0 < \beta_3 \leq \beta_2 \leq \beta_1 \leq 1, \quad t > 0.
 \end{aligned} \tag{23}$$

FIGURE 7: Problem 3, absolute error at (a) $t = 0.01$ and (b) $t = 0.05$.

Having the exact solution,

$$\mathcal{V}(y, z, t) = e^t \sin(\pi y) \sin(\pi z), \quad (y, z) \in \Omega. \quad (24)$$

In Figure 6, we have visualized the behavior of the exact and approximate solutions for the Problem 3 using $N = 20^2$, $\beta_1 = \beta_2 = \beta_3 = 0.5$, and $t = 0.1$, which show that the approximate solution is very compatible with the exact solution. In Figure 7, the absolute error is displayed for Problem 3.

4. Conclusion

In this study, our principle focused on the applicability and performance of the RBF-based local meshless method to approximate the numerical solution of three-term time-fractional Sobolev equations. The computed results show that the proposed technique can take care of these sorts of problems amazingly and accurately. The local procedure leads to a sparse system of linear equations, and the solution is approximated with good accuracy. Three test problems are taken into account to test the effectiveness and accuracy of the proposed meshless method utilizing rectangular and two irregular domains. The numerical results demonstrate the high accuracy and effectiveness of the method. Given the current research, the proposed technique is a surprisingly powerful and successful tool for solving numerical problems of multiterm time-fractional PDEs found in various fields of science and technology.

Data Availability

The data that support the findings of this study are openly available at <https://hindawi.com/publish-research>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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