Hindawi
Complexity
Volume 2021, Article ID 9970255, 10 pages
https://doi.org/10.1155/2021/9970255

*Research Article*

# Global Bifurcation Structure of a Predator-Prey System with a Spatial Degeneracy and B-D Functional Response

**Xiaozhou Feng**[1], **Changtong Li**[1], **Hao Sun**[1,2] **and Yuzhen Wang**[1]

[1]School of Science, Xi'an Technological University, Xi'an 710032, China
[2]Test Measuring Academy of Norinco, Huayin 714200, China

Correspondence should be addressed to Xiaozhou Feng; flxzfxz8@163.com

Received 26 March 2021; Accepted 2 June 2021; Published 21 June 2021

Academic Editor: Abdel-Haleem Abdel-Aty



In this paper, we investigate a predator-prey system with Beddington–DeAngelis (B-D) functional response in a spatially degenerate heterogeneous environment. First, for the case of the weak growth rate on the prey ($\lambda_1^{\Omega} < a < \lambda_1^{\Omega_0}$), a priori estimates on any positive steady-state solutions are established by the comparison principle; two local bifurcation solution branches depending on the bifurcation parameter are obtained by local bifurcation theory. Moreover, the demonstrated two local bifurcation solution branches can be extended to a bounded global bifurcation curve by the global bifurcation theory. Second, for the case of the strong growth rate on the prey ($a > \lambda_1^{\Omega_0}$), a priori estimates on any positive steady-state solutions are obtained by applying reduction to absurdity and the set of positive steady-state solutions forms an unbounded global bifurcation curve by the global bifurcation theory. In the end, discussions on the difference of the solution properties between the traditional predator-prey system and the predator-prey system with a spatial degeneracy and B-D functional response are addressed.

## 1. Introduction

The population dynamics system is the basic model to study the spatial and the temporal structures of the biological population. It is used to describe the dynamic distribution of population density produced by the interaction of the species in the ecosystem or the surrounding environment. In particular, the interaction of the predator-prey system is one of the basic structures in complex ecosystems, and such models have been widely studied (see [1–17]) under the uniform condition of the space, in other words, all coefficients of the model are positive. Because the natural environment of most species is spatially heterogeneous, biologists and mathematicians believe that the inhomogeneity of the spatial environment has a significant impact on the dynamic behavior of the biological population system, which is confirmed by the biologist C. B. Huffaker's biological experiment. To investigate the effect of spatial heterogeneity on the dynamic behavior of the biological population system, a natural way is to replace the constant in the model with a function containing spatial variables. Therefore, we will study a predator-prey system with spatial degeneracy and B-D-type response function [1, 2].

$$\begin{cases} -\Delta u = au - b(x)u^2 - \dfrac{cuv}{mu+kv+1}, & x \in \Omega, t > 0, \\ -\Delta v = dv - v^2 + \dfrac{euv}{mu+kv+1}, & x \in \Omega, t > 0, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is the outward unit normal vector of the boundary $\partial\Omega$ in $\mathfrak{R}^N\,(N > 1)$. $u$ and $v$ are the densities of prey and predator, respectively. $a$ and $d$ are the intrinsic growth rate of prey $u$ and predator $v$, respectively. $c$ and $e$ are capturing rate to predator and conversion rate of prey captured by a predator, respectively. $m$ denotes the saturation coefficient, and $kv$ is the inhibition of functional response function on behalf of predators. The parameters $a, c, m, k$ are assumed to be only positive constants, while $d$ can be negative. Let $\Omega_0$ be a subset of $\Omega$. Assume that $b(x) \equiv 0$ in $\Omega$, and $b(x) > 0$ in $\Omega \backslash \Omega_0$, where $b(x)$ is a function dependent on space variables

$x$, which means that two species live in a heterogeneous environment.

When the space environment is homogeneous, that is, $b(x)$ is a positive number in $\Omega$, system (1) is reduced to the constant-coefficient system, and the predator-prey system with Beddington–DeAngelis-type functional response has been studied by many scholars from different perspectives [3–5].

In this paper, the different behaviors of system (1) will be mainly discussed under the condition of spatial heterogeneity. If $b(x)$ is a positive function in $\Omega$, then there is no essential difference between the model and previous research. If $b(x)$ degenerates partially into zero within $\Omega$ and $m = k = 0$, the research results of system (1) show that the behavior of solutions has changed substantially. More concretely, when $m = k = 0$, system (1) exhibits some fixed constant $\alpha^*$ such that $a < \alpha^*$; then the behavior of the system (1) is similar to the case of $b(x)$ as the normal number. For $a \geq \alpha^*$, the behavior of system (1) has undergone an essential change, which means that even if the mortality of predators is very large, the system (1) still has a stable positive solution. Moreover, the asymptotic behavior of the coexistence solution of the equilibrium state on the system (1) was discussed in detail in references [10, 18] as $d \longrightarrow \infty$.

For the case where $m > 0, k > 0$, there is no relevant research result. Therefore, the main purpose of this paper is to investigate the change rule of solutions if $b(x)$ partially degenerates to zero in $\Omega$ and $m > 0, k > 0$. Is there a fundamental change in the behavior of system (1)? Are these changes the same as the case of $m = k = 0$? Answers to these questions are detailed in the "Conclusion" section.

This paper is organized as follows. In Section 2, some notations and important lemmas used in previous papers is given, including the generalized comparison principle and the convergence property of the semitrivial solution $u_a$. In Section 3.1, the system (1) with the weak growth rate on the prey $(\lambda_1^{\Omega} < a < \lambda_1^{\Omega_0})$ is studied; a priori estimates on any positive steady-state solutions are established by the comparison principle; two local bifurcation solution branches depending on the bifurcation parameter $d > 0$ are obtained by local bifurcation theory, and the demonstrated two local bifurcation solution branches can be extended to a bounded global bifurcation curve by the global bifurcation theory. In Section 3.2, the system (1) with the strong growth rate on the prey $(a > \lambda_1^{\Omega_0})$ is investigated; a priori estimates on any positive steady-state solutions are obtained by applying reduction to absurdity; and then we prove that the set of positive steady-state solutions forms an unbounded global bifurcation curve by the global bifurcation theory. In Section 4, some discussions on the difference of the solution properties between the traditional predator-prey system and the predator-prey system with a spatial degeneracy and B-D functional response are listed.

## 2. Preliminaries

The main purpose of this section is to give some important lemmas and notations used in later papers.

In this paper, we always suppose that $b(x)$ is a nonnegative function in $\Omega$; moreover, there exists a subset $\Omega_0 \subset \Omega$ such that $b(x) \equiv 0$, $x \in \overline{\Omega}_0$ and $b(x) > 0$, $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. The research of this paper is dependent on the above assumption and is not suitable for the case $\partial\Omega_0 \cap \partial\Omega \neq 0$.

Obviously, $v = 0$ satisfies the second equation of system (1). For this case, $u$ satisfies the following logistic equation:

$$-\Delta u = au - b(x)u^2, \quad x \in \Omega, u = 0, x \in \partial\Omega. \tag{2}$$

According to the result of [19], we know that Equation (2) has a unique zero solution if $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$; Equation (2) exhibits the unique zero solution denoted by $u_a$ if $a \notin (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$. Moreover, the mapping $a \longrightarrow u_a(x)$ is strictly monotone increasing with $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$.

Next, let $\lambda_1^{O}$ be the principal eigenvalues of the operator $-\Delta$ under the Dirichlet boundary condition on region $O$. For convenience, we introduce the notation $\lambda_1^{O}(\phi)$, which stands for the principal eigenvalue of the following eigenvalue problem:

$$-\Delta u + \phi u = \lambda u, \quad x \in O, u = 0, x \in \partial O. \tag{3}$$

Under the sense of these signs, $\lambda_1^{O} = \lambda_1^{O}(0)$, By the conclusion of [19], in the sense of the norm $L^{\infty}(\Omega)$, we obtain that if $a \longrightarrow \lambda_1^{\Omega}$, then $u_a \longrightarrow 0$. If $a \longrightarrow \lambda_1^{\Omega_0}$,

$$\begin{cases} u_a \longrightarrow \infty, x \in \overline{\Omega}_0, \\ u_a \longrightarrow U_{\lambda_1^{\Omega_0}}, x \in \overline{\Omega} \setminus \overline{\Omega}_0, \end{cases} \tag{4}$$

where $u_a$ represents the minimal positive solution of the following boundary value blow-up problem:

$$-\Delta U = aU - b(x)U^2, \quad x \in \Omega \setminus \overline{\Omega}_0, U|_{\partial\Omega} = 0, U|_{\partial\Omega_0} = \infty, \tag{5}$$

where $U|_{\partial\Omega_0} = \infty$ implies $\lim_{d(x,\partial\Omega_0) \longrightarrow 0} u(x) = \infty$.

In a word, if $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, then system (1) exhibits a unique semitrivial solution $(u_a, 0)$. For $a \notin (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, system (1) does not exhibit such a semitrivial solution.

If $u = 0$, then $v$ satisfies the following logistic equation:

$$-\Delta v = dv - v^2, \quad x \in \Omega, v = 0, x \in \partial\Omega. \tag{6}$$

By [20], if $d \leq \lambda_1^{\Omega}$, this equation does not have a positive solution. If $d > \lambda_1^{\Omega}$, this equation has a unique positive solution denoted by $\theta_d$. Hence, if $d > \lambda_1^{\Omega}$, then system (1) has a unique semitrivial solution $(0, \theta_d)$.

To end this paper, we need to introduce some results. First, we introduce the following generalized comparison principle.

**Lemma 1** (see [19]). *If $u_1, u_2$ are two order continuous derivable positive functions in $\Omega \setminus \overline{\Omega}_0$, and satisfy*

$$\begin{aligned} &\Delta u_1 + au_1 - b(x)u_1^p \leq 0 \leq \Delta u_2 + au_2 - b(x)u_2^p, \quad x \in \Omega \setminus \overline{\Omega}_0, \\ &Bu_1 \geq Bu_2, \quad x \in \partial\Omega, \lim_{d(x,\partial\Omega_0) \longrightarrow 0} (u_2 - u_1) \leq 0, \end{aligned} \tag{7}$$

*then $u_1, u_2$ satisfy $u_1 \geq u_2$ in $\overline{\Omega} \setminus \overline{\Omega}_0$. Next, let's introduce a very useful lemma, which plays a key role in proving a priori estimates of positive solutions.*

**Lemma 2** (see [10]). *Suppose $u_n \in C^2(\Omega)$ satisfies*

$$-\Delta u_n \leq a u_n, \quad x \in \Omega, u_n|_{\partial\Omega} = 0, u_n \geq 0, \left\|u_n\right\|_\infty = 1, \tag{8}$$

where $a$ is a positive constant. Then there exists a function $u_\infty \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that $u_n$ weakly converges to $u_\infty$ with the sense of $H_0^1(\Omega)$ norm; $u_n$ strongly converges to $u_\infty$ with the sense of norm $L^p(\Omega)$, $\|u_\infty\|_\infty = 1$.

In a paper [21], López-Gómez and Sabina de Lis have proved that $du_a/da$ converges uniformly to $\infty$ with $a \longrightarrow \lambda_1^{\Omega_0}$ on any compact subset of $\Omega_0$. Next, we demonstrate the following result for the case of $a \longrightarrow \lambda_1^{\Omega}$.

**Lemma 3.** *Let $\Phi_1(x)$ be the corresponding eigenfunction to $\lambda_1^\Omega$ and $\max_{\overline{\Omega}}\Phi_1(x) = 1$. If $a \longrightarrow \lambda_1^\Omega$ in $\overline{\Omega}$, then*

$$\frac{u_a(x)}{a - \lambda_1^\Omega} \longrightarrow \frac{\int_\Omega \Phi_1^2 dx}{\int_\Omega b(x)\Phi_1^3 dx}\Phi_1(x), \quad x \in \overline{\Omega}. \tag{9}$$

*Proof* □

Let $\widehat{u}_a = \overline{\Omega}_a/\|u_a\|_2$, then $\|\widehat{u}_a\|_2 = 1$ and $\widehat{u}_a$ satisfies

$$-\Delta\widehat{u}_a = a\widehat{u}_a - b(x)u_a\widehat{u}_a, \quad x \in \Omega, \widehat{u}_a = 0, x \in \partial\Omega. \tag{10}$$

By multiplying two sides of Equation (10) by $\widehat{u}_a$ and integrating over $\Omega$ by parts, we get

$$\int_\Omega |\nabla\widehat{u}|^2 dx = a\int_\Omega \widehat{u}_a^2 dx - \int_\Omega b(x)u_a\widehat{u}_a^2 dx \leq a\int_\Omega \widehat{u}_a^2 dx. \tag{11}$$

Hence, as $a \longrightarrow \lambda_1^\Omega$, $\{\widehat{u}_a\}$ is uniformly bounded in $H_0^1(\Omega)$; it follows that $\widehat{u}_a$ weakly converges to $\widehat{u}$ depending on the norm of $H_0^1(\Omega)$, and $\widehat{u}_a$ strongly converges to $\widehat{u}$ depending on the norm of $L^2(\Omega)$. Since $\|\widehat{u}_a\|_2 = 1$, we know that $\|\widehat{u}\|_2 = 1$, then $\widehat{u} \geq 0 (\neq 0)$. We chose any function $\phi \in H_0^1(\Omega)$, and by multiplying two sides of Equation (10) by $\phi$ and integrating over $\Omega$ by parts, we obtain

$$\int_\Omega \nabla\widehat{u}_a\nabla\phi dx = a\int_\Omega \widehat{u}_a\phi dx - \int_\Omega b(x)u_a\widehat{u}_a\phi dx. \tag{12}$$

Let $a \longrightarrow \lambda_1^\Omega$, then $\int_\Omega \nabla\widehat{u}\nabla\phi dx = \lambda_1^\Omega\int_\Omega \widehat{u}\phi dx$, which implies that $\widehat{u}$ is a weak solution of the following equation:

$$-\Delta\widehat{u}_a = \lambda_1^\Omega\widehat{u}, \quad x \in \Omega, \widehat{u} = 0, x \in \partial\Omega. \tag{13}$$

According to Harnack's inequality, $\widehat{u} > 0, x \in \Omega$, then $\widehat{u} = \Phi_1$. Thanks to the regularization theory, $\widehat{u}_a \longrightarrow \Phi_1$ with $a \longrightarrow \lambda_1^\Omega$ under the norm of $C^1(\overline{\Omega})$. Equation (10) can be rewritten as follows:

$$-\Delta\widehat{u}_a = a\widehat{u}_a - b(x)\left\|u_a\right\|_2\widehat{u}_a^2, x \in \Omega, \quad \widehat{u}_a = 0, x \in \partial\Omega. \tag{14}$$

By multiplying two sides of the above equation by $\Phi_1$ and integrating over $\Omega$ by Green's formula, we obtain

$$\int_\Omega \left(a - \lambda_1^\Omega\right)\widehat{u}\Phi_1 dx = \int_\Omega b(x)\left\|u_a\right\|_2\widehat{u}_a^2\Phi_1 dx. \tag{15}$$

Let $a \longrightarrow \lambda_1^\Omega$, then $\|u_a\|_2/a - \lambda_1^\Omega = \int_\Omega \Phi_1^{2dx}/\int_\Omega b(x)\Phi_1^3$ $dx$. Thus, as $a \longrightarrow \lambda_1^\Omega$, we have

$$\frac{u_a(x)}{a - \lambda_1^\Omega} = \frac{u_a(x)}{\left\|u_a\right\|_2}\frac{\left\|u_a\right\|_2}{a - \lambda_1^\Omega} \longrightarrow \frac{\int_\Omega \Phi_1^2 dx}{\int_\Omega b(x)\Phi_1^3 dx}\Phi_1(x), \quad x \in \overline{\Omega}. \tag{16}$$

This completes the proof.

## 3. Global Bifurcation Structure

In this section, we will discuss the global bifurcation structure of system (1) and give sufficient conditions for the coexistence of the two species. Let $a$ be fixed, we will investigate the global bifurcation of system (1) by dividing two different cases as follows:

$$\text{(i)}\ \lambda_1^\Omega < a < \lambda_1^{\Omega_0}, \ \text{(ii)}\ a > \lambda_1^{\Omega_0}. \tag{17}$$

At the same time, unless specifically explained, parameters $c, e, m, k > 0$ are fixed. The parameter $d$ will be considered as the bifurcation parameter. Next, we will discuss two cases depending on the value range of $a$ by two subsections.

### *3.1. Weak Growth Rate on the Prey ($\lambda_1^\Omega < a < \lambda_1^{\Omega_0}$).*

For any $d > 0$, system (1) has two semitrivial solutions $(u_a, 0)$ and $(0, \theta_d)$. Hence, system (1) has two semitrivial positive curves as follows:

$$\Gamma_u = \left\{(d, u_a, 0):\ -\infty < d < \infty\right\}, \Gamma_v = \left\{(d, u_a, \theta_d):\ \lambda_1^\Omega < d < \infty\right\}. \tag{18}$$

To get our conclusion, we first have to give a priori estimates on any positive steady-state solutions by the comparison principle as follows.

**Lemma 4.** *Suppose that $\lambda_1^\Omega < a < \lambda_1^{\Omega_0}$. Any positive solution $(u, v)$ of system (1) satisfies*

$$0 < u < u_a, \theta_d < v < \theta_{[d+e/m]} < d + \frac{e}{m}. \tag{19}$$

where $u_a$ is the unique solution of Equation (2). Moreover, if system (1) has a positive solution, then the parameter $d$ satisfies

$$d > \lambda_1^\Omega - \frac{e}{m}, a > \lambda_1^\Omega\left(\frac{c\theta_d}{m\left\|u_a(x)\right\|_\infty + k\theta_d + 1}\right). \tag{20}$$

*Proof.* Notice that the first equation of system (1), it is easy to get

$$-\Delta u = au - b(x)u^2 - \frac{cuv}{mu+kv+1} < au - b(x)u^2. \quad (21)$$

It follows from the comparison principle that $0 < u < u_a$, where $u_a$ is the unique solution of (2). Similarly, according to the second equation of (1), we directly get

$$-\Delta v = dv - v^2 + \frac{euv}{mu+kv+1} < \left(d + \frac{e}{m}\right)v - v^2, \quad (22)$$

$$-\Delta v = dv - v^2 + \frac{euv}{mu+kv+1} > dv - v^2. \quad (23)$$

Combining with the comparison principle, we obtain

$$\theta_d < v < \theta_{[d+e/m]} < d + \frac{e}{m}. \quad (24)$$

If system (1) has a positive solution, we easily get

$$d = \lambda_1^{\Omega}\left(v - \frac{eu}{mu+kv+1}\right) > \lambda_1^{\Omega}\left(-\frac{eu}{mu+kv+1}\right) > \lambda_1^{\Omega}\left(-\frac{e}{m}\right), \quad (25)$$

$$a = \lambda_1^{\Omega}\left(b(x)u + \frac{cv}{mu+kv+1}\right) > \lambda_1^{\Omega}\left(\frac{cv}{mu+kv+1}\right) > \lambda_1^{\Omega}\left(\frac{c\theta_d}{m\,\|u_a(x)\|_\infty + k\theta_d + 1}\right). \quad (26)$$

Next, two local bifurcation solution branches depending on the bifurcation parameter $d > 0$ are obtained by local bifurcation theory as follows. Let $w = u_a - u$, $X = \{u \in W^{2,p}(\Omega): u = 0, x \in \partial\Omega\}, Y = L^p(\Omega)$. The operator $F$: $\mathfrak{R} \times X \times X \longrightarrow Y \times Y$ can be defined as follows:

$$F(d,w,v) = \begin{pmatrix} \Delta w + aw - 2b(x)u_a w + b(x)w^2 + \dfrac{c(u_a - w)v}{m(u_a - w) + kv + 1} \\ \Delta v + dv - v^2 + \dfrac{e(u_a - w)v}{m(u_a - w) + kv + 1} \end{pmatrix}. \quad (27)$$

Next, we prove that $(d,w,v) = (\widehat{d},0,0)$ is a local bifurcation point of system (1), where $\widehat{d} = \lambda_1^{\Omega}(-eu_a/mu_a + 1)$. For simplicity, let $A = m(u_a - w) + kv + 1, B = v(kv+1)$, $C = (u_a - w)[m(u_a - w) + 1], D = m(u_a - w)(2kv + 1) + kv + 1$. By direct calculation, we obtain

$$\begin{aligned}
F_{(w,v)}(d,w,v) &= \begin{pmatrix} \Delta + a - 2b(x)u_a + 2b(x)w - \dfrac{cB}{A^2} & \dfrac{cC}{A^2} \\ -\dfrac{eB}{A^2} & \Delta + d - 2v + \dfrac{eC}{A^2} \end{pmatrix}, \\
F_{(w,v)}(d,w,v)\begin{pmatrix}\phi \\ \varphi\end{pmatrix} &= \begin{pmatrix} \Delta\phi + \phi(a - 2b(x)u_a + 2b(x)w) + c\dfrac{C\varphi - B\phi}{A^2} \\ \Delta\varphi + \varphi(d - 2v) + e\dfrac{C\varphi - B\phi}{A^2} \end{pmatrix}, \\
F_d(d,w,v) &= \begin{pmatrix} 0 \\ v \end{pmatrix}, F_{d(w,v)}(d,w,v)\begin{pmatrix}\phi \\ \varphi\end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \\
F_{(w,v)(w,v)}(d,w,v)\begin{pmatrix}\phi \\ \varphi\end{pmatrix}^2 &= \begin{pmatrix} 2b(x)\phi^2 - 2c\dfrac{D\phi\varphi - mB\phi^2 + kC\varphi^2}{A^3} \\ -2\varphi^2 - 2e\dfrac{D\phi\varphi - mB\phi^2 + cC\varphi^2}{A^3} \end{pmatrix}.
\end{aligned} \quad (28)$$

For $(d, w, v) = (\widehat{d}, 0, 0)$, it is easy to get $N(F_{(w,v)}(\widehat{d}, 0, 0)) = \operatorname{span}\{(\phi_1, \varphi_1)\}$, where $(\phi_1, \varphi_1)$ satisfies

$$\begin{cases} -\Delta\phi + a\phi - 2b(x)u_a\phi - \dfrac{cu_a\phi}{mu_a + 1} = 0, & x \in \Omega, \\ -\Delta\varphi + \widehat{d}\varphi + v^2 + \dfrac{eu_a\varphi}{mu_a + 1} = 0, & x \in \Omega, \\ \phi = \varphi = 0, & x \in \partial\Omega. \end{cases} \tag{29}$$

Since $\widehat{d} = \lambda_1^{\Omega}(-eu_a/mu_a + 1)$, we can choose $\varphi_1 > 0$. On the other hand, $u_a$ is a positive solution of Equation (2), which demonstrates that $-\Delta - a + 2b(x)u_a$ is a positive operator, and $\phi_1 = (-\Delta - a + 2b(x)u_a)^{-1}(-eu_a\varphi_1/mu_a + 1) > 0$. The range can be represented as $R(F_{(w,v)}(u_1, 0, 0)) = \left\{(f, g) \in Y^2\colon \int_{\Omega} g(x)\varphi_1 \mathrm{d}x = 0\right\}$, since $\int_{\Omega}\varphi_1^2 \mathrm{d}x > 0$, we have $F_{d(w,v)}(u_1, 0, 0)[\phi_1, \psi_1] = [0, \psi_1] \notin R(F_{(w,v)}(d, 0, 0))$. Hence, by the local bifurcation theory [20], we obtain the following result on the local bifurcation solution. □

**Theorem 1.** *Suppose that $\lambda_1^{\Omega} < a < \lambda_1^{\Omega_0}$, the local positive bifurcation solution set that bifurcates from the point $(\widehat{d}, u_a, 0)$ of system (1) forms a smooth curve*

$$\Gamma_1 = \left\{\widehat{d}(s), u_a - u_1(s), v_1(s)\colon s \in [0, \delta]\right\}, \tag{30}$$

where $\widehat{d}(0) = \lambda_1^{\Omega}(-eu_a/mu_a + 1), u_1(s) = s\phi_1(x) + o(|s|), v_1(s) = s\varphi_1(x) + o(|s|)$. By calculating, we get

$$\widehat{d}'(0) = -\frac{\langle F_{(w,v)(w,v)}(\widehat{d}, 0, 0)[\phi_1, \varphi_1]^2, l_1\rangle}{2\langle F_{d(w,v)}(\widehat{d}, 0, 0)[\phi_1, \varphi_1], l_1\rangle} = \frac{\int_{\Omega}\varphi_1^3 \mathrm{d}x + e\int_{\Omega}\phi_1\varphi_1^2 + cu_a\varphi_1^3/(mu_a + 1)^2 \mathrm{d}x}{\int_{\Omega}\varphi_1^2 \mathrm{d}x} > 0, \tag{31}$$

where $l_1$ is a linear functional in space $Y^2$, defined as follows.

Similarly, we can prove that

$$\langle [f, g], l_1\rangle = \int_{\Omega} g(x)\varphi_1(x)\mathrm{d}x, \tag{32}$$

is also a $(\widetilde{d}, 0, \theta_{\widetilde{d}})$ local bifurcation point of system (1), where $\widetilde{d}$ is determined uniquely by $a = \lambda_1^{\Omega}(c\theta_{\widetilde{d}}/k\theta_{\widetilde{d}} + 1)$.

Set $\chi = v - \theta_d$, define the operator $G\colon \mathfrak{R} \times X \times y \longrightarrow Y \times Y$ by

$$G(d, u, \chi) = \begin{pmatrix} \Delta u + au - b(x)u^2 - \dfrac{cu(\chi + \theta_d)}{mu + k(\chi + \theta_d) + 1} \\ \Delta\chi + d\chi - \chi^2 - 2\chi\theta_d + \dfrac{eu(\chi + \theta_d)}{mu + k(\chi + \theta_d) + 1} \end{pmatrix}. \tag{33}$$

Let $A_1 = mu + k(\chi + \theta_d) + 1, B_1 = u(mu + 1), C_1 = (\chi + \theta_d)[k(\chi + \theta_d) + 1], D_1 = k(\chi + \theta_d)(2mu + 1) + mu + 1$. By directly calculating the Fréchet derivative of the operator, we get

$$G_{(u,w)}(\mu, u, w) = \begin{pmatrix} \Delta + a - 2b(x)u - \dfrac{cC_1}{A_1^2} & -\dfrac{cB_1}{A_1^2} \\ \dfrac{eC_1}{A_1^2} & \Delta + d - 2\chi - 2\theta_d + \dfrac{eB_1}{A_1^2} \end{pmatrix}. \tag{34}$$

It follows that

$$G_{(u,\chi)}(d,u,\chi)\begin{pmatrix}\phi\\ \varphi\end{pmatrix}=\begin{pmatrix}\Delta\phi+a\phi-2b(x)u\phi-c\dfrac{C_1\phi+B_1\varphi}{A_1^2}\\ \\ \Delta\varphi+d\varphi-2\chi\varphi-2\theta_d\varphi+e\dfrac{C_1\phi+B_1\varphi}{A_1^2}\end{pmatrix},$$

$$G_d(d,u,\chi)=\begin{pmatrix}-\dfrac{c\theta_d' B_1}{A_1^2}\\ \\ \chi-2\chi\theta_d'+\dfrac{e\theta_d' B_1}{A_1^2}\end{pmatrix},$$

$$G_{d(u,\chi)}(d,u,\chi)\begin{pmatrix}\phi\\ \varphi\end{pmatrix}=\begin{pmatrix}c\theta_d'\dfrac{-D_1\phi+2kB_1\varphi}{A_1^3}\\ \\ \varphi-2\theta_d'\varphi+e\theta_d'\dfrac{-D_1\phi+2kB_1\varphi}{A_1^3}\end{pmatrix}, \tag{35}$$

$$G_{(u,\chi)^2}(d,u,\chi)\begin{pmatrix}\phi\\ \varphi\end{pmatrix}^2=\begin{pmatrix}-2b(x)\phi^2+2c\dfrac{m\phi^2C_1-\phi\varphi D_1+k\varphi^2B_1}{A_1^3}\\ \\ 2e\dfrac{\phi\varphi D_1-m\phi^2C_1-kB_1\varphi^2}{A_1^3}\end{pmatrix}.$$

For $(d,u,\chi)=(\widetilde{d},0,0)$, it is easy to get $N(G_{(u,\chi)}(\widetilde{d},0,0))=\operatorname{span}\{(\phi_2,\varphi_2)\}$, where $(\phi_2,\varphi_2)$ satisfies

$$\begin{cases}\Delta\phi+a\phi-\dfrac{c\theta_{\widetilde{d}}}{k\theta_{\widetilde{d}}+1}=0, & x\in\Omega,\\ \\ \Delta\varphi+\widetilde{d}\varphi-2\theta_{\widetilde{d}}\varphi+\dfrac{e\theta_{\widetilde{d}}}{k\theta_{\widetilde{d}}+1}=0, & x\in\Omega,\\ \\ \phi=\varphi=0, & x\in\partial\Omega.\end{cases} \tag{36}$$

Since $a=\lambda_1^{\Omega}(c\theta_{\widetilde{d}}/k\theta_{\widetilde{d}}+1)$, we can choose $\varphi_2>0$. On the other hand, $-\Delta-\widetilde{d}+2\theta_{\widetilde{d}}$ is a positive operator, and $\varphi_2=(-\Delta-u_2+2\theta_{\widetilde{d}})^{-1}(c\theta_{\widetilde{d}}\phi_2/k\theta_{\widetilde{d}}+1)>0$. The range can be represented as $R(G_{(u,\chi)}(\widetilde{d},0,0))=\{(f,g)\in Y^2\colon \int_\Omega f(x)\phi_2(x)\mathrm{d}x=0\}$, since $\int_\Omega -c\theta_{\widetilde{d}}\phi_2^2/k\theta_{\widetilde{d}}+1\mathrm{d}x<0$, it follows that

$$G_{d(u,\chi)}(\widetilde{d},0,0)[\phi_2,\varphi_2]=\left[-\frac{c\theta_{\widetilde{d}}'\phi_2}{\left(k\theta_{\widetilde{d}}+1\right)^2},\varphi_2-2\theta_{\widetilde{d}}'\varphi_2-\frac{e\theta_{\widetilde{d}}'\varphi_2}{\left(k\theta_{\widetilde{d}}+1\right)^2}\right]\notin R\left(G_{(u,\chi)}(u_2,0,0)\right). \tag{37}$$

Thus, according to the local bifurcation theory [22], we obtain similar result as follows.

**Theorem 2.** *Suppose that $\lambda_1^{\Omega}<a<\lambda_1^{\Omega_0}$, the local bifurcation branch near $(\widetilde{d},0,\theta\widetilde{d})$ of system (1) forms a smooth curve*

$$\Gamma_2=\left\{\widetilde{d}(s),u_2(s),\theta_d+v_2(s)\colon s\in[0,\delta]\right\}, \tag{38}$$

where $\widetilde{d}(0)=\widetilde{d}$, $u_2(s)=s\phi_2(x)+o(|s|)$, $v_2(s)=s\varphi_2(x)+o(|s|)$. By calculating, we get

$$\widetilde{d}(0) = -\frac{\langle G_{(u,\chi)(u,\chi)}(\widetilde{d},0,0)[\phi_2,\varphi_2]^2, l_2\rangle}{2\langle G_{d(u,\chi)}(\widetilde{d},0,0)[\phi_2,\varphi_2], l_2\rangle} = \frac{\int_\Omega \left[b(x)\phi_2 + c\phi_2/(k\theta_d+1)^2\right]\phi_2^2 \mathrm{d}x - \int_\Omega cm\theta_d\phi_2^3/(k\theta_d+1)^2\mathrm{d}x}{-\int_\Omega cm\theta_d'\phi_2^2/(k\theta_d+1)^2\mathrm{d}x}, \tag{39}$$

where $l_2$ is a linear functional in space $Y \times Y$, defined as follows:

$$\langle [f,g], l_2\rangle = \int_\Omega f(x)\phi_2(x)\mathrm{d}x. \tag{40}$$

Next, we use the modified global bifurcation theorem [23] to prove the global bifurcation structure of the system (1) under the condition of the weak growth rate on the prey.

**Theorem 3.** *Suppose that $\lambda_1^\Omega < a < \lambda_1^{\Omega_0}$, then the positive solution set of system (1) forms a bounded smooth curve $\Gamma$ that connects $\Gamma_1$ and $\Gamma_2$ and satisfies*

$$\mathrm{proj}_u\Gamma = (d_*, d^*) \text{ or } (d_*, d^*], \tag{41}$$

where $d_* = \widehat{d}, \widetilde{d} \le d^* > \infty$. Moreover, the bifurcation direction of $\Gamma_1$ at point $(\lambda_1^\Omega(-eu_a/mu_a+1), u_a, 0)$ is supercritical $(\widehat{d}'(0) > 0)$. If $0 \le m < m_0$, then the bifurcation direction of $\Gamma_2$ at point $(\widetilde{d}, 0, \theta\widetilde{d})$ is supercritical $(\widetilde{d}'(0) > 0)$; if $m > m_0$, then the bifurcation direction of $\Gamma_2$ at point $(\widetilde{d}, 0, \theta\widetilde{d})$ is subcritical $(\widetilde{d}'(0) < 0)$, where $m_0$ is determined by

$$m_0 = \frac{\int_\Omega \left[b(x)\phi_2 + c\phi_2/(k\theta_d+1)^2\right]\phi_2^2\mathrm{d}x}{\int_\Omega c\theta_d\phi_2^3/(k\theta_d+1)^2\mathrm{d}x}. \tag{42}$$

*Proof.* The proof of this theorem is similar to Theorem 4 below; the detailed proofs are omitted here. □

*3.2. Strong Growth Rate on the Prey ($a > \lambda_1^{\Omega_0}$).* Comparing with the weak growth rate on the prey, there exists only one semitrivial solution curve $\Gamma_2$ for the system (1) under the strong growth rate on the prey. In this case, $(d,u,v) = (\widetilde{d}, 0, \theta\widetilde{d})$ is still a local bifurcation point. The positive solution near $(\widetilde{d}, 0, \theta\widetilde{d})$ of system (1) forms a smooth curve $\Gamma_2$. By the method of Lemma 4, we can prove that $d > \lambda_1^\Omega - e/m$ if system (1) has positive solutions. Next, we establish a boundary result of any positive solutions of system (1) when $d$ is bounded.

**Lemma 5.** *If $a > \lambda_1^{\Omega_0}, d_n \le M$, then there exists a positive constant C that does not depend on n such that any positive solution of system (1) satisfies*

$$\|u_n\|_\infty + \|v_n\|_\infty \le C. \tag{43}$$

*Proof.* Since $d_n \le M$, it follows that

$$-\Delta v_n \le \left(M + \frac{e}{m}\right)v_n - v_n^2. \tag{44}$$

Hence, $\|u_n\|_\infty \le \theta_{[M+e/m]} \le M + e/m$. Suppose the conclusion of the lemma does not hold. Then for $d = d_n$, there exists some positive solution sequence $\{(u_n, v_n)\}$ of system (1) such that $\|u_n\|_\infty \longrightarrow \infty$ with $n \longrightarrow \infty$.

Set $\widehat{u}_n = u_n/\|u_n\|_\infty$. The first equation of $u_n$ system (1) implies $-\Delta\widehat{u}_n \le \lambda\widehat{u}_n$. Following Lemma 2, $\widehat{u}_n$ weakly converges to $\widehat{u}$ in $H_0^1(\Omega)$ and strongly converges to $\widehat{u}$ in $L^p(\Omega)$, and $\|\widehat{v}_n\|_\infty = 1$. Similarly, let $\widehat{v}_n = v_n/\|v_n\|_\infty$. The first equation of system (1) implies

$$-\Delta\widehat{v}_n \le \left(M + \frac{e}{m}\right)\widehat{v}_n. \tag{45}$$

Hence, $\widehat{v}_n$ weakly converges to $\widehat{v}$ in $H_0^1(\Omega)$ and strongly converges to $\widehat{v}$ in $L^p(\Omega)$, and $\|\widehat{v}_n\|_\infty = 1$.

Next, it turns out that $\widehat{u}$ is almost zero in $\Omega\setminus\Omega_0$. According to Lemma 1, it is easy to see that $u_n \le U_a$ in $\Omega\setminus\Omega_0$. Hence, $\widehat{u}_n$ is uniformly bounded on an arbitrary subset of $\overline{\Omega}\setminus\overline{\Omega}_0$. Thanks to $\|u_n\|_\infty \longrightarrow \infty$, so $\widehat{u}_n$ is uniformly convergent to 0 on any subset of $\overline{\Omega}\setminus\overline{\Omega}_0$. Thus, $\widehat{u}$ is almost zero in the $\overline{\Omega}\setminus\overline{\Omega}_0$. And because $\partial\Omega_0$ is smooth enough, then $\widehat{u} \in H_0^1(\Omega)$.

According to the first equation $u_n$ of system (1), we have

$$-\Delta\widehat{u}_n = a\widehat{u}_n - b(x)\|u_n\|_\infty\widehat{u}_n^2 - \frac{c\widehat{u}_n}{m\|\widehat{u}_n\|_\infty + kv_n + 1}, \quad x \in \Omega, \widehat{u}_n = 0, x \in \partial\Omega. \tag{46}$$

By multiplying two sides of the above equation by $\phi$ and integrating over $\Omega$, where the support set of the function $\phi$ is $\Omega_0$, we obtain

$$\int_{\Omega_0}\nabla\widehat{u}_n\nabla\phi\mathrm{d}x = a\int_{\Omega_0}\widehat{u}_n\phi\mathrm{d}x - \int_{\Omega_0}\frac{c\widehat{u}_n v_n\phi}{m\|\widehat{u}_n\|_\infty v_n + kv_n + 1}\mathrm{d}x. \tag{47}$$

Letting $n \longrightarrow \infty$, we get

$$\int_{\Omega_0}\nabla\widehat{u}\nabla\phi\mathrm{d}x = \lambda\int_{\Omega_0}\widehat{u}\phi\mathrm{d}x. \tag{48}$$

But as $n \longrightarrow \infty$,

$$\left|\int_\Omega \frac{\widehat{u}_n v_n\phi}{m\|u_n\|_\infty\widehat{u}_n + kv_n + 1}\mathrm{d}x\right| \le \frac{\|v_n\|_\infty}{m\|u_n\|_\infty}\|\phi\|_{L^1_{\Omega_0}} \longrightarrow 0. \tag{49}$$

Hence, $\widehat{u} \ge 0$ is a weak solution to the following problem:

$$-\Delta\hat{u} = \lambda\hat{u}, \quad x \in \Omega_0, \hat{u} = 0, x \in \partial\Omega_0. \tag{50}$$

Thus, by using Harnack's inequality, we know that $\hat{u} > 0$ or $\hat{u} \equiv 0$ in $\Omega_0$. If $\hat{u} > 0$, then $a = \lambda_1^{\Omega_0}$; this is in contradiction with $a > \lambda_1^{\Omega_0}$. If $\hat{u} \equiv 0$, then $\hat{u} \equiv 0$ in $\Omega$; this is in contradiction with $\|\hat{u}_n\|_\infty = 1$. Therefore, it follows that $\|v_n\|_\infty$ is uniformly bounded by the antievidence method. □

**Theorem 4.** *Suppose that $a > \lambda_1^{\Omega_0}$, then the positive solution set of system (1) forms an unbounded smooth curve $\Gamma$ that extends $\Gamma_2$ to $\infty$ by $d$. If $0 \le m < m_0$, then the bifurcation direction of $\Gamma_2$ at point $(\tilde{d}, 0, \theta\tilde{d})$ is supercritical ($\tilde{d}'(0) > 0$); if $m > m_0$, then the bifurcation direction of $\Gamma_2$ at point $(\tilde{d}, 0, \theta\tilde{d})$ is subcritical ($\tilde{d}'(0) < 0$), where $m_0$ is defined in Theorem 3.*

*Proof.* In order to apply the global bifurcation theorem of [23] better, we define the following mapping $H$: $\mathfrak{R} \times R \times R \longrightarrow Y \times Y$ by

$$H(d,u,v) = \begin{pmatrix} u \\ v \end{pmatrix} - (-\Delta)^{-1} \begin{pmatrix} au - b(x)u^2 - \dfrac{cuv}{mu + kv + 1} \\ dv - v^2 + \dfrac{euv}{mu + kv + 1} \end{pmatrix}. \tag{51}$$

According to the regularization theory of elliptic equations and the Sobolev embedding theorem, the second term of the mapping $H$ is a compact mapping. Moreover, the nonnegative solution of the system (1) is equivalent to the zero point of the mapping $H(d,u,v)$.

The Fréchet derivative of the mapping $H(d,u,v)$ at $(u,v) = (0,\theta_d)$ is given as follows:

$$H_{(u,v)}(d,0,\theta_d) = I - (-\Delta)^{-1} \begin{pmatrix} a - \dfrac{c\theta_d}{k\theta_d + 1} & 0 \\ \dfrac{e\theta_d}{k\theta_d + 1} & d - 2\theta_d \end{pmatrix}. \tag{52}$$

The corresponding adjoint operator is recorded as $H^*_{(u,v)}(d,0,\theta_d)$, that is,

$$H^*_{(u,v)}(d,0,\theta_d) = I - \begin{pmatrix} \left(a - \dfrac{c\theta_d}{k\theta_d + 1}\right)(-\Delta)^{-1} & \dfrac{e\theta_d}{k\theta_d + 1}(-\Delta)^{-1} \\ 0 & (d - 2\theta_d)(-\Delta)^{-1} \end{pmatrix}. \tag{53}$$

By standard calculation, we obtain

$$N\left(H_{(u,v)}(\tilde{d},0,\theta\tilde{d})\right) = \operatorname{span}\{(\phi_2,\varphi_2)\}, N\left(H^*_{(u,v)}\left(\tilde{d},0,\theta_{\tilde{d}}\right)\right) = \operatorname{span}\{(-\Delta\phi_2, 0)\}. \tag{54}$$

Thanks to the global bifurcation theorem, the local bifurcation curve $\Gamma_2$ can be extended to a smooth global curve $\Gamma$, and

$$\Gamma_2 \subset \Gamma \subset \left\{(d,u,v) \in (\mathfrak{R} \times X \times X) \setminus \left\{\left(\tilde{d},0,\theta_{\tilde{d}}\right)\right\} : H(d,u,v) = 0\right\}. \tag{55}$$

Moreover, according to Theorem 6.4.3 in document [22], the global curve $\Gamma$ must satisfy one of the following three alternatives:

($a$) $\Gamma$ is unbounded in $\mathfrak{R} \times X \times X$

($b$) There exists a constant $d \ne \tilde{d}$ such that $d(d, 0, \theta_d) \in \Gamma$

($c$) There exists a function $(d,\phi,\varphi) \in \mathfrak{R} \times (Z \setminus \{(0,\theta_d)\})$ such that $(d,\phi,\varphi) \in \Gamma$

where $Z$ is the complementary space of $N(H_{(u,v)}(\tilde{d},0,\theta_{\tilde{d}}))$ and satisfies

$$Z = \left\{(f,g) \in X \times X : \int_\Omega f\phi_2 \mathrm{d}x = 0\right\}. \tag{56}$$

The positive one can be defined as follows:

$$P = \left\{ w \in W^{2,P}(\Omega) \colon w > 0, \quad x \in \Omega, \partial w/\partial n < 0, x \in \partial\Omega \right\}. \tag{57}$$

Next, we prove $\Gamma \setminus \left\{ (\tilde{d}, 0, \theta_{\tilde{d}}) \subset \mathfrak{R} \times P \times P \right\}$. Applying reduction to absurdity, we assume that $\Gamma \setminus \left\{ (\tilde{d}, 0, \theta_{\tilde{d}}) \nsubseteq (\mathfrak{R} \times P \times P) \right\}$. There exists a sequence

$$\left\{ \left( d_j, u_j, v_j \right) \right\}_{j=1}^{\infty} \subset \left( \Gamma \setminus \left\{ \left( \tilde{d}, 0, \theta_{\tilde{d}} \right) \right\} \right) \cap (\mathfrak{R} \times P \times P), \tag{58}$$

such that

$$\lim_{j \longrightarrow \infty} \left( d_j, u_j, v_j \right) = \left( d_\infty, u_\infty, v_\infty \right), \tag{59}$$

in $\mathfrak{R} \times X \times X$, where $(d_\infty, u_\infty, v_\infty) \in ((\Gamma \setminus \left\{ (\tilde{d}, 0, \theta_{\tilde{d}}) \right\} \cap (\mathfrak{R} \times \partial (P \times P)))$, and $(u_\infty, v_\infty)$ is a nonnegative solution of system (1) corresponding to $d = d_\infty$. According to the strong maximum principle [22, 24], we know that $(u_\infty, v_\infty)$ satisfies one of the three following alternatives:

(i) $u_\infty \equiv 0, x \in \Omega, v_\infty \equiv 0, x \in \Omega$

(ii) $u_\infty > 0, x \in \Omega, v_\infty \equiv 0, x \in \Omega$

(iii) $u_\infty \equiv 0, x \in \Omega, v_\infty > 0, x \in \Omega$

Similar to the method to Theorem 1 in [24], we can find a contradiction to each of these cases. Therefore, we have

$$\Gamma \setminus \left\{ \left( \tilde{d}, 0, \theta_{\tilde{d}} \right) \right\} \subset \mathfrak{R} \times P \times P. \tag{60}$$

Since equation (60) holds, it is obviously impossible for case (b) to occur. Because $\phi_2 > 0$ in $\overline{\Omega}$; obviously, case (c) does not impossible hold. Therefore, the global curve $\Gamma$ just belongs to case (a), combined with Lemma 3.2; the projection of the global curve $\Gamma$ on the $d$ axis contains $(0, \infty)$.

The specific analysis and calculation of the direction of local bifurcation branches have been given in the previous section, so we have completed the proof of this theorem. □

## 4. Conclusion

In this work, the effect of spatial degradation on the steady-state problem of a predator-prey system with B-D functional response has been investigated. By studying the bifurcation structure of the system, sufficient conditions for the coexistence of two species are obtained. However, due to the phenomenon of space degradation, there are some phenomena that are different from the traditional predator-prey system (i.e., all the coefficients of the system are normal or positive). Specifically, for the traditional predator-prey system, the two species do not coexist when the predator's own growth rate $d$ is large, but for the system (1), the growth rate $d$ of the predator is very large, and the two species can still coexist in the common habitat $\Omega$, even if the predator growth rate $d$ is very large. The results show that spatial degradation has a significant effect on the steady-state behavior of the system. On the other hand, our results are essentially different from the results of the Lotka–Volterra predator-prey system ($m = k = 0$). Specifically, under the case of the strong growth rate on the prey, for any large $d$, the two species of the system (1) can coexist, but for the smaller $d\,(\leq \lambda_1^{\Omega} - e/m)$, the two species cannot coexist. However, for the Lotka–Volterra predator-prey system, no matter how small the predator's self-growth rate $d$, the two species can continue to coexist, but for the larger $d$, the two species cannot coexist. The result reveals that the B-D reaction function also has a significant effect on the steady-state behavior of the system.

## Data Availability

This article belongs to the qualitative analysis of the dynamic system and no data were involved.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors participated in every phase of research conducted for this paper. All authors read and approved the final manuscript.

## Acknowledgments

The work was partially supported by the National Natural Science Youth Fund of China (12001425), the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2020JM-569), the Shaanxi Province Department of Education Fund (18JK0393), the Project of Improving Public Scientific Quality in Shaanxi Province (No. 2020PSL(Y)073), and Test & Measuring Academy of Norinco, Huayin, Shaanxi Province.

## References

[1] J. R. Beddington, "Mutual interference between parasites or predators and its effect on searching efficiency," *The Journal of Animal Ecology*, vol. 44, no. 1, pp. 331–340, 1975.

[2] D. L. Deangelis, R. A. Goldstein, and R. V. O'Neill, "A model for tropic interaction," *Ecology*, vol. 56, no. 4, pp. 881–892, 1975.

[3] X. Z. Feng and S. P. Sun, "The effect of parameters on positive solutions and asymptotic behavior of an unstirred chemostat model with B-D functional response," *Advances In Difference Equations*, vol. 181, pp. 1–23, 2018.

[4] H. Jiang, L. Wang, and R. Yao, "Numerical simulation and qualitative analysis for a predator-prey model with B-D functional response," *Mathematics And Computers In Simulation*, vol. 117, pp. 39–53, 2015.

[5] G. Guo and J. Wu, "Multiplicity and uniqueness of positive solutions for A predator-prey model with B-D functional response," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1632–1646, 2010.

[6] C. B. Huffaker, "Experimental studies on predation: dispersion factors and predator-prey oscillations," *Hilgardia*, vol. 27, no. 14, pp. 343–383, 1958.

[7] J. Blat and K. J. Brown, "Global bifurcation of positive solutions in some systems of elliptic equations," *SIAM Journal on Mathematical Analysis*, vol. 17, no. 6, pp. 1339–1353, 1986.

[8] A. Casai, J. C. Eilbeck, and J. Lopez-Gomez, "Existence and uniqueness of coexistence states for A predator-prey model with diffusion," *Differential Integral Equations*, vol. 7, pp. 411–439, 1994.

[9] Y. Du and Y. Lou, "S-shaped global solution curve and Hopf bifurcation of positive solutions to A predator-prey model," *Journal of Differential Equations*, vol. 144, no. 2, pp. 390–440, 1998.

[10] E. N. Dancer and Y. Du, "Effects of certain degeneracies in the predator-prey model," *SIAM Journal on Mathematical Analysis*, vol. 34, no. 2, pp. 292–314, 2002.

[11] M. Das, A. Maiti, and G. P. Samanta, "Stability analysis of a prey-predator fractional order model incorporating prey refuge," *Ecological Genetics and Genomics*, vol. 7-8, pp. 33–46, 2018.

[12] D. Pal and G. P. Samanta, "Effects of dispersal speed and strong Allee effect on stability of a two-patch predator-prey model," *International Journal of Dynamics and Control*, vol. 6, no. 4, pp. 1484–1495, 2018.

[13] S. Saha and G. P. Samanta, "Influence of dispersal and strong Allee effect on a two-patch predator-prey model," *International Journal of Dynamics and Control*, vol. 7, no. 4, pp. 1321–1349, 2019.

[14] D. Manna, A. Maiti, and G. P. Samanta, "A Michaelis-Menten type food chain model with strong Allee effect on the prey," *Applied Mathematics and Computation*, vol. 311, pp. 390–409, 2017.

[15] S. Mondal and G. P. Samanta, "Pelican-Tilapia interaction in Salton sea: an eco-epidemiological model with strong Allee effect and additional food," *Modeling Earth Systems and Environment*, vol. 136, 2021.

[16] S. Mondal and G. P. Samanta, "Dynamics of an additional food provided predator-prey system with prey refuge dependent on both species and constant harvest in predator," *Physica A: Statistical Mechanics and Its Applications*, vol. 534, p. 122301, 2019.

[17] S. Mondal and G. P. Samanta, "Dynamics of a delayed predator-prey interaction incorporating nonlinear prey refuge under the influence of fear effect and additional food," *Journal of Physics A: Mathematical and Theoretical*, vol. 53, no. 29, p. 295601, 2020.

[18] E. N. Dancer and Y. Du, "On A free boundary problem arising from population biology," *Indiana University Mathematics Journal*, vol. 52, no. 1, pp. 51–68, 2003.

[19] Y. Du and Q. Huang, "Blow-up solutions for A class of semilinear elliptic and parabolic equations," *SIAM Journal on Mathematical Analysis*, vol. 31, no. 1, pp. 1–18, 1999.

[20] M. G. Crandall and P. H. Rabinowitz, "Bifurcation from simple eigenvalues," *Journal of Functional Analysis*, vol. 8, no. 2, pp. 321–340, 1971.

[21] J. López-Gómez and J. C. Sabina de Lis, "First variations of principal eigenvalues with respect to the domain and pointwise growth of positive solutions for problems where bifurcation from infinity occurs," *Journal of Differential Equations*, vol. 148, no. 1, pp. 47–64, 1998.

[22] Q. X. Ye and Z. Y. Li, *Introduction to Reaction-Diffussion Equations*, Science Press, Beijing, China, 1990, In Chinese.

[23] J. Lopez-Gomez, "Spectral theory and nonlinear functional analysis," *Research Notes In Mathematics*, vol. 426, 2001.

[24] S. Li, J. Wu, and S. Liu, "Effect of cross-diffusion on the stationary problem of a Leslie prey-predator model with a protection zone," *Calculus of Variations and Partial Differential Equations*, vol. 56, no. 3, p. 82, 2017.