Research Article

Semi-Analytical Solutions for the Diffusive Kaldor–Kalecki Business Cycle Model with a Time Delay for Gross Product and Capital Stock

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This paper discusses the stability and Hopf bifurcation analysis of the diffusive Kaldor–Kalecki model with a delay included in both gross product and capital stock functions. The reaction-diffusion domain is considered, and the Galerkin analytical method is used to derive the system of ordinary differential equations. The methodology used to determine the Hopf bifurcation points is discussed in detail. Furthermore, full diagrams of the Hopf bifurcation regions considered in the stability analysis are shown, and some numerical simulations of the limit cycle are used to confirm the theoretical outcomes. The delay investment parameter and diffusion coefficient can have great impacts on the Hopf bifurcations and stability of the business cycle model. The investment parameters for the gross product and capital stock as well as the adjustment coefficient of the production market are also studied. These parameters can cause instability in, and the stabilization of, the business cycle model. In addition, we point out that, as the delay investment parameter increases, the Hopf bifurcation points for the diffusion coefficient values decrease considerably. When the delay investment parameter has a very small value, the solution of the business cycle model tends to become steady.

1. Introduction

For a long period of time, many significant nonlinear phenomena have been modelled and described via ordinary or partial differential equations (ODEs or PDEs). For example, these equations have been used to model population ecology [1–4], animals [5, 6], health [7, 8], chemicals [9, 10], and business economics [11–14]. A business cycle model is utilized to explain the working of economic laws and can also be utilized to predict investment status, yield, costs, and other important factors in the business economic model. Also, it can help practitioners avoid fluctuations [15].

One of the important economic models was created by Kaldor [13] in 1940. It uses a nonlinear model constructed with a couple of ODEs, where the nonlinearity of the investment and saving functions lead to periodic limit cycle results. Furthermore, Kalecki [14] considered a time delay between an investment decision and its impact on capital

accumulation in the business cycle model. It was shown that earnings are invested, and capital grows due to past investment decisions.

In 1999, Kaddar and Szydlowski [16] developed the Kaldor–Kalecki business cycle system by incorporating the Kalecki concept [13] into the system created by Kaldor [14]. A time delay was included in the investment function for the gross product in the capital equation. Afterwards, Kaddar and Alaoui [17] added a time delay for both gross product and capital stock due to a past decision. Therefore, the Kaldor–Kalecki system became as in [17]:

$$\begin{aligned}\frac{\partial Y}{\partial t} &= \alpha [I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{\partial K}{\partial t} &= I(Y(t - \tau), K(t - \tau)) - \delta K(t),\end{aligned}\tag{1}$$

where $Y(t)$ refers to the gross product; $K(t)$ indicates the capital stock; δ and α are the depreciation value of the capital stock and the adjustment rate of the production market, respectively; $I(Y, K)$ denotes the investment function; $S(Y, K)$ describes the savings function; and $\tau > 0$ is the time delay in investment due to the past investment decision.

The dynamic ODE system in (1) has been studied and discussed by many researchers. Recently, Jianzhi and Hongyan [18] determined that the local stability of the positive equilibrium produces the corresponding characteristic equations. The existence of the Hopf bifurcation, the direction of this bifurcation, and the stability of the limit-cycle outcomes have been studied using numerical simulations. Wu [19, 20] explored the simple-zero and double-zero singularities for the ODE equations presented in (1). They constructed bifurcation diagrams and examined the double-periodic oscillation. Kaddar and Talibi Alaoui [17] proved that Hopf bifurcation points occur when the delay parameter τ is increased. Wu and Wang [21] considered the distribution of the roots of the characteristic equation of the system in (1) at the equilibrium point. They also discussed the Hopf bifurcation and the stability of the limit cycle. Kaddar and Talibi Alaoui [12] illustrated the existence of a local Hopf bifurcation and also used an explicit algorithm to show the direction of the Hopf bifurcation and conduct a stability analysis of the system in (1).

The presence of the diffusion coefficient in both equations in system (1) is extremely important as it can strongly affect the stability. It can also change the Hopf bifurcation points and therefore change the regions used for the stability analysis [1, 22, 23]. Blanke et al. [24] investigated a diffusive Kaldor–Kalecki business cycle system with a time delay under Neumann boundary conditions. They conducted a stability analysis of the model and found that the time delay can give rise to the Hopf bifurcation when the delay stretches beyond a critical point. Furthermore, they showed that the diffusion coefficient played a key role in this model. Szydlowski and Krawiec [25] investigated the Kaldor–Kalecki system as a two-dimensional dynamic model. A time delay was considered for the capital accumulation equation. A qualitative analysis for the differential equations was considered for this model. Finally, they showed that there is a Hopf bifurcation leading to a limit-cycle result.

Based on the previous studies on system (1), it appears that there is a need for additional research on the effects of the diffusion value and the investment delay on the business cycle model. Therefore, it is of importance to study the impacts of the diffusion and delay parameters on the stability and Hopf bifurcation. Thus, this paper will focus on the Kaldor–Kalecki business model with delays in the one-dimensional (1D) domain and has several significant aims. The first aim is to show what theoretical results can be obtained by utilizing the Galerkin technique. This helpful and reliable technique can help solve, and provide an excellent prediction for, the PDE system. Moreover, in order to discuss the effects of the diffusion coefficient d and the investment delay τ on the adjustment coefficient of the production market in detail, the investment parameters for the gross product and capital stock need to be examined. In addition, we also

construct full-map diagrams of the Hopf bifurcation points and the stability analysis (stable and unstable regions) using examples of periods in limit cycle maps. All of these aims will help us explore the stability of the business cycle and predict whether policy makers' targets will be met. The knowledge gained may also help practitioners avoid large-scale business fluctuations.

This paper is arranged as follows. Section 2 explains the modelling of dynamic behaviour in the Kaldor–Kalecki business cycle model. Section 3 explains the methodology and the theoretical framework used to determine the Hopf bifurcation map. It also discusses how the Galerkin method can be used to create an ODE system from the PDE model. Section 4 constructs the maps of the Hopf bifurcation regions of stability for both the numerical simulations of the PDEs and the analytical outcome, thus showing the effects of the delay and diffusion parameters on the system. Finally, in Section 5, bifurcation diagrams and periodic oscillation limit-cycle maps showing both stable and unstable results are plotted to confirm the analytical outcomes.

2. Mathematical Model

The dynamic behaviour of the Kaldor–Kalecki business cycle model is considered by using the following replacement equations for the investment $I(Y, K)$ and saving $S(Y, K)$ variables:

$$\begin{aligned} I(Y, K) &= IY - \beta K, \\ S(Y, K) &= \gamma Y, \quad \text{where } \beta > 0 \text{ and } \gamma \in (0, 1), \end{aligned} \quad (2)$$

in the ODE equations in (1). For more details regarding the simple mathematical formulations used here, see [16, 17, 21, 26] and the references therein. Therefore, the nonlinear reaction-diffusion business cycle model with a time delay in the gross product and capital stock can be described with the following system:

$$\begin{aligned} \frac{\partial Y}{\partial t} &= dY_{xx} + \alpha[IY(t) - \beta K(t) - \gamma Y(t)], \\ \frac{\partial K}{\partial t} &= dK_{xx} + IY(t - \tau) - \beta K(t - \tau) - \delta K(t), \\ Y(x, t) &= K(x, t) = 0, \text{ at } x = \pm 1, Y(x, t) = y_s, K(x, t) \\ &= k_s, \quad -\tau < t \leq 0. \end{aligned} \quad (3)$$

Here, parameters $\alpha, \beta, \gamma, \delta$, and τ have the same meaning that they do in system (1). In addition, d denotes the diffusion coefficient of the system. This system is open, with a symmetrical pattern in the outcome in the middle of the domain for $x = 0$. In addition, $Y_s > 0$ and $K_s > 0$ represent the positive initial concentrations for times in the interval $(-\tau, 0)$. Note that $Y_s = K_s = 1$ in all the numerical simulations run in this model. The Runge–Kutta method [27, 28] is used to determine the solutions of the ODE system. The Crank–Nicholson scheme [7, 29] is considered according to the numerical simulation outcomes for PDE system (3). In

the numerical simulation, spatial and temporal discretization are applied, i.e., $(\Delta x, \Delta t) = (10 \times 10^{-3}, 10 \times 10^{-4})$.

3. Methodology and Theoretical Analytical Framework

In this section, we identify a reliable technique for solving nonlinear PDE models. The Galerkin analytical technique [30] is applied to a system of PDEs to produce ODE equations. This technique considers the orthogonality of the basic functions to convert the PDEs into an ODE system [10, 27]. Several models have employed this technique, including the Gray and Scott cubic autocatalytic system [10], logistic equations with delays [1, 2, 29], and a viral infection model [7]. In general, all of the researchers who have considered this technique have obtained significant results and validated the method.

In order to use the Galerkin technique, we consider the following trial functions:

$$\begin{aligned} Y(x, t) &= Y_1(t)\cos(\eta_1) + Y_2(t)\cos(\eta_2), \\ K(x, t) &= K_1(t)\cos(\eta_1) + K_2(t)\cos(\eta_2), \end{aligned} \quad (4)$$

where $\eta_1 = (\pi x/2)$ and $\eta_2 = (3\pi x/2)$. The trial equation functions are assumed to be $Y = \sum Y_i = Y_1 + Y_2$. Here, $K = \sum K_i = K_1 + K_2$ refers to the centre profile concentrations at $x = 0$. The trial equation expansion (4) meets the boundary conditions in the PDE model in (3). The free parameters in the system are then examined by computing the values for the delay PDEs. Next, the PDEs are weighted using two trial expansions: $\cos(\eta_1)$ and $\cos(\eta_2)$. The resulting system of four ODE equations is as follows:

$$\begin{aligned} \frac{dY_1}{dt} &= -\frac{\pi^2}{4}dY_1 + I\alpha Y_1 - \alpha\gamma Y_1 - \alpha\beta K_1, \\ \frac{dK_1}{dt} &= -\frac{\pi^2}{4}dK_1 + IY_{1\tau} - \delta K_1 - \beta K_{1\tau}, \\ \frac{dY_2}{dt} &= -\frac{9\pi^2}{4}dY_2 + I\alpha Y_2 - \alpha\gamma Y_2 - \alpha\beta K_2, \\ \frac{dK_2}{dt} &= -\frac{9\pi^2}{4}dK_2 + IY_{2\tau} - \delta K_2 - \beta K_{2\tau}, \end{aligned}$$

$$Y_{i\tau} = Y_i(x, t - \tau) \text{ and } K_{i\tau} = K_i(x, t - \tau) \text{ and } i = 1, 2. \quad (5)$$

Note that the series in (4) has been abbreviated at the two-term equation point because the two-term outcome provides sufficient accuracy. It also presents excellent outcomes compared to the PDE numerical scheme system. Moreover, the one-term system can be expressed as a couple of equations by assuming that $Y_2 = K_2 = 0$ for the ODEs in system (5).

We can utilize the equations in system (5) to explore the existence of the Hopf bifurcation points theoretically. These points are then used to determine a map showing the full stability analysis and any unstable and stable zones. The

Hopf bifurcation denotes the periodic oscillation of the limit cycle in the neighbourhood of the steady state, which can cause a transition from a stable solution to an unstable solution. For more information, see [1, 22, 23, 31]. Hence, the points of the Hopf bifurcation can be displayed by utilizing the Taylor series for the steady-state value points; see [7, 29]. This result can be examined with

$$\begin{aligned} Y_i &= Y_{is} + \epsilon\chi_1 e^{-ut}, \\ K_i &= K_{is} + \epsilon\chi_2 e^{-ut}, \quad \text{where } i = 1, 2, \epsilon \ll 1. \end{aligned} \quad (6)$$

Hence, the expressions Y_i and K_i in (6) are inserted into the ODE system in (5). Afterwards, the steady-state values are linearized. Next, the Jacobian matrix of the eigenvalues considers a small system perturbation, which demonstrates the typical growth value u by placing $u = i\omega$ in the characteristic equation via dividing the characteristic equation by the real (RE) and imaginary (IM) equations. The next conditional equation then helps to determine the points in the Hopf bifurcation:

$$\frac{dY_i}{dt} = \frac{dK_i}{dt} = \text{RE} = \text{IM} = 0, \quad \text{where } i = 1, 2. \quad (7)$$

4. Stability and the Hopf Bifurcation Maps

This section provides the Hopf bifurcation maps and the regions of stability for both the numerical simulations for the PDEs in (3) and the theoretical outcome for the ODEs in (5). The delay parameter τ and diffusion coefficient d are studied in conjunction with the adjustment coefficient for the production market and the investment parameters for the gross product and capital stock. At the end of this section, we present some numerical examples in order to show the accuracy of the analytical outcomes.

4.1. Effect of the Delay Parameter τ . Figure 1 shows the maps of two different regions of the Hopf bifurcation for the delay parameter τ versus I (upper graph), β (middle graph), and α (lower graph). The analysis of the two-term solution (dashed line) and the numerical simulation for the PDEs (black crosses) are obtained in each case. Positive parameters are utilized in each graph, $\delta = 0.15$, $\gamma = 0.10$, and $d = 0.05$, where $\beta = 1$ and $\alpha = 5$ (upper figure), $\alpha = 3$ and $I = 0.2$ (middle figure), and $I = 0.2$ and $\beta = 1$ (lower figure). There is a unique curve in each figure dividing the stability regions: the region above the curve indicates the unstable zone, whereas the region below the curve indicates the stable zone. Note that when the investment delay τ is increased, the critical values of the Hopf bifurcation points for the adjustment coefficient for the production market α increase steadily. Moreover, the investment parameters for the gross product I and the capital product β also increase with an increase in the value of τ . When the delay investment parameter has a very small value, the solution of the business cycle model tends to become stable and reaches a steady state. It appears that the analytical prediction corresponds to the numerical simulation of the PDEs, with less than 1%

FIGURE 1: The regions produced by the Hopf bifurcation curves for the delay parameter τ versus I , β , and α . The analytical outcomes are indicated by the blue dashed lines, while the numerical simulation results for the PDE model are indicated by black crosses.

error reported for all values of τ . Therefore, the delay investment value τ can have a very crucial impact on the stability regions of the business cycle in terms of the investment function and can destabilize or stabilize the model. Moreover, it potentially has a significant influence on economic equilibrium and could assume a guiding role in investment activities.

Figures 2(a) and 2(b) explore the Hopf bifurcation maps on the $\alpha - \beta$ diagrams. Figure 2(a) shows both analytical ODE (blue dashed line) and numerical simulation for the PDEs (black crosses). Positive values are used in these figures, $\delta = 0.15$, $\gamma = 0.10$, $I = 0.2$, and $d = 0.3$. The stability regions are shown. This figure shows that, as the adjustment coefficient for the production market α increases, the value of the investment in the capital product β decreases. Furthermore, Figure 2(b) provides the analytical results for the two-term solution for five different values of the delay parameter τ , namely, $\tau = 1, 2, 3, 4$, and 5 . At any selected fixed point of the adjustment coefficient for the production market α , the parameter for the investment in capital product β decreases as the delay investment parameter τ increases. Note that the resulting behaviour obtained in this figure is very similar to behaviours found in [1, 7]. Therefore, the investment delay term can also have a significant impact on Hopf bifurcation stability regions for this model.

Figures 3(a) and 3(b) present maps of the Hopf bifurcation regions for α against the investment parameter for gross production (I). Figure 3(a) shows the theoretical outcomes (blue dashed line) as well as the numerical results (black crosses). The positive values used here are $d = 0.05$,

$\delta = 0.15$, $\gamma = 0.10$, and $\beta = 1$. As the adjustment coefficient of production market α increases, the Hopf bifurcation points for the investment parameter for the production rate I decrease slowly. The matchup between the numerical scheme for the PDEs and the theoretical results for the ODE system is excellent. In Figure 3(b), the two-term analytical solutions are shown for five different values of the delay parameter, namely, $\tau = 1, 2, 3, 4$, and 5 . It can be seen that, as the adjustment coefficient for the production market parameter α increases, the unstable region becomes bigger than the stable area.

4.2. Effect of the Diffusion Coefficient Parameter d . Figure 4 presents the Hopf bifurcation maps for the diffusion coefficient in the $d - I$ plane (upper graph), $d - \beta$ plane (middle graph), and $d - \alpha$ plane (lower graph). In each figure, the two-term solution is shown with a red dashed line, while the black crosses indicate the numerical simulation results for the PDE model. The free parameters applied are $\beta = 1$ and $\alpha = 3$ (upper graph), $\alpha = 3$ and $I = 0.2$ (middle graph), and $I = 0.2$ and $\beta = 1$ (lower graph). The other parameters for all figures are $\tau = 1$, $\delta = 0.15$, and $\gamma = 0.10$. As in Figure 1, the graphs indicate the stable and unstable regions. Each graph indicates the results of the increases in the diffusion coefficient values d . As a result, the Hopf bifurcation points for the rate of the adjustment coefficient for the production market α have also increased considerably. Furthermore, the investment parameter rates I and β (for production and capital) also increase steadily

FIGURE 2: An exploration of the Hopf bifurcation maps in the α - β diagrams: (a) analytical and numerical simulation results and (b) analytical results for the two-term solution for five different values of τ .

FIGURE 3: The two regions created by the Hopf bifurcation in the α - I and β - I planes: (a) the two-term solutions and (b) the outcomes for five different values of τ .

against the increasing diffusion parameter d . Therefore, adding the diffusion coefficient d can have a significant effect on the stability of this model as it can change the stability of the model [1, 2].

Figure 5(a) presents the Hopf bifurcation maps on the α - β diagrams for the analytical (red dotted) and numerical simulation results (black crosses). The positive values used in Figure 5(a) are $\tau = 1$, $\delta = 0.15$, $\gamma = 0.10$, $I = 0.2$, and $d = 0.05$. Figure 5(b) shows plots of the analytical ODE results for five different values of the diffusion coefficient d : 0, 0.05, 0.10, 0.15, and 0.20. In both figures, the stability regions are provided. It can be shown that, at any fixed point of α , the parameter for the investment in the capital product

β increases as the diffusion coefficient d increases. Note that the resulting behaviour obtained in this figure is very similar to behaviours found in [7]. The diffusion term can also have a huge impact on the bifurcation regions in the business cycle model.

Figure 6(a) determines the Hopf bifurcation maps in the τ - I plane, while the frequency of the periodic results for ω against τ is plotted in Figure 6(b). In both figures, we see the results for two cases: no diffusion term ($d = 0$; black dashed line) and a diffusion term ($d = 0.1$; red dashed line). The positive values used are $\alpha = 5$, $\beta = 1$, and $\gamma = \delta = d = 0.15$. In Figure 6(a), the diffusion rate causes the stabilization of the system utilizing the critical value of the investment rate I ,

FIGURE 4: The Hopf bifurcation curves for $D - I$ (upper graph), $D - \beta$ (middle graph), and $D - \alpha$ (lower graph). In each case, the analytical (red dashed lines) and numerical simulation results (black crosses) are plotted.

FIGURE 5: The Hopf bifurcation maps on the $\alpha - \beta$ diagrams: (a) an exploration of the analytical and numerical simulation results and (b) plots of the analytical ODE results for five different values of the diffusion coefficient d .

which increases where there is an assumed delay parameter τ . The Hopf bifurcation points decrease in shifting from the diffusion to the nondiffusion case. The frequency of the periodic result ω in Figure 6(b) increases steadily as τ increases. The differences between the frequency values for the periodic result ω in both cases are small, and the solutions

have very similar predictions at large values of the delay parameter in this domain.

Figure 7 shows the Hopf bifurcation regions for the plot of the diffusion coefficient d versus τ (delay in investment). The two-term solution is shown as a red dashed line, while the black crosses indicate the numerical simulation results

FIGURE 6: (a) The Hopf bifurcation maps in the $\tau - I$ plane and (b) the frequency of the periodic results for ω against τ . Here, we see results for two cases: $d = 0$ (no diffusion term) and $d = 0.1$ (diffusion term).

FIGURE 7: The Hopf bifurcation curve on the $\tau - d$ map. The analytical (red dotted line) and numerical simulation (black crosses) results are given.

for the PDEs. The parameters utilized here are $\delta = 0.15$, $\gamma = 0.10$, $I = 0.2$, $\alpha = 10$, and $\beta = 1$. It can be seen that, as the delay in investment τ increases, the Hopf bifurcation points for the diffusion coefficient value d decrease considerably. Furthermore, the results indicate that the relationship between the diffusion value and investment delay has a very significant impact on the stability of the business cycle model in terms of investment activity.

Figure 8 presents a map of the Hopf bifurcation regions in the plot of the β parameter against the investment parameter for gross production (I). The two-term numerical solutions are shown. The positive values used are $d = 0.05$, $\tau = 1$, $\delta = 0.15$, $\gamma = 0.10$, and $\alpha = 3$. In this figure, it was found that, as the parameter for the investment in the capital product β increases, the Hopf bifurcation points for the investment in the gross product value I increase slowly up until $\beta = 0.75$. Beyond this value, the Hopf bifurcation points for the investment switch from a

high-conversion state to a minimum-conversion state for the gross product I and go down until $I = 0$ at $\beta \approx 1.74$. The comparisons in these figures show agreements between the analytical results for the ODEs and the simulations for the PDEs, with no more than 2% error for up to $\beta = 2$.

Lastly, a comparison is provided for the special parameter values $\tau = 15$, $\alpha = 2$, $\beta = 1$, and $\gamma = \delta = d = 0.15$. In this case, the points of the Hopf bifurcation of the investment parameter for gross production were examined for $I_c \approx 2,055, 2,101$ for the analytical one- and two-term solutions, where $I_c \approx 2,092$ is used for the numerical simulation of the PDE model. The prediction for the analytical ODE system agrees with the numerical predictions for the PDE system, with less than 1% error between them at this point. Hence, the theoretical ODE system provides reliable predictions regarding the Hopf bifurcation map as well as the stability regions.

FIGURE 8: The Hopf bifurcation curves in the $\beta - I$ plane for the analytical (red dotted line) and numerical simulation (black crosses).

FIGURE 9: (a, b) The bifurcation diagrams for the capital Y and gross product K against the adjustment coefficient for the production market α . The two-term solutions and numerical results are shown.

FIGURE 10: Continued.

FIGURE 10: The business cycle of gross product Y and capital product K versus time t is plotted. The analytical outcome of the two terms and the numerical scheme of the PDE model are shown.

5. Bifurcation Diagrams and Periodic Oscillation Maps

This section focuses on the steady-state results as well as the bifurcation diagrams, periodic results, and 2D phase-plane map. In addition, the map of the bifurcation diagrams is considered of the domain in the centre $x = 0$.

Figures 9(a) and 9(b) plot the bifurcation diagrams for capital Y and gross product K with the adjustment coefficient for the production market α . The two-term solutions are shown as a blue dashed line, while the black dots indicated numerical simulation results. The parameters used are $\tau = 1$, $\delta = 0.15$, $\gamma = 0.10$, $\beta = 1$, $I = 0.2$, and $d = 0.05$. This example shows the importance of the investment delay in changing from a steady state to an unstable one, inducing limited cyclic solutions. In both cases, the analytical Hopf bifurcation point is $\alpha_c \approx 4.33$. All of the results for $\alpha > \alpha_c \approx 4.33$ are therefore unstable. After the Hopf bifurcation, the maximum amplitude over oscillation increases with growing α , while the minimum amplitude goes down. There are good matchups between the numerical PDE results and the analytical two-term solutions over the domain of the adjustment coefficient for the production market α .

Figure 10 presents the limits of the business cycle for the gross product $Y(t)$ and capital product $K(t)$ against time. The parameters used in Figures 10(a) and 10(b) are $\tau = 1$, $\delta = 0.15$, $\gamma = 0.10$, $\alpha = 10$, $\beta = 1$, $I = 0.2$, and $d = 0.50$ (from the stable region of Figure 7), while in Figures 10(c) and 10(d), $d = 0.4$ (from the unstable zone in Figure 7). In all these figures, the two-term solution is indicated by a black line, while the red dotted line refers to the numerical simulation. Note that the Hopf bifurcation point for the analytical outcomes in this example is $d_c \approx 0.44 < 0.5$, where d is considered to be the bifurcation parameter. When $d_c \approx 0.44 < 0.5$, the results become stable, as in Figures 10(a) and 10(b). However, at $0.4 < d_c \approx 0.44$, the solution is unstable, as shown in Figures 10(c) and 10(d). The matchups between the analytical solutions and simulations in all of these figures are excellent.

6. Conclusions

This paper has provided a semianalytical outcome for the diffusive Kaldor–Kalecki model in the 1D geometry. The delay parameter was shown to exist for both gross product and capital stock functions. A system of ODEs was deolved using the Galerkin technique. The Hopf bifurcation points were found for dividing the graphs into two stability regions. Furthermore, we displayed full-map graphs for the delay parameter and diffusion coefficient values versus the parameters α , β , and I for a stability analysis of the system. The effects of these values, which can influence the stability of the model, were studied fully. The diffusion and investment delay values have different impacts on the bifurcation maps for the business cycle model. We found that the Hopf bifurcation points for the diffusion coefficient values decreased as the investment delay parameter τ increased. The results were examined by exploring several different numerical examples of the limit cycle and could help in the development of policy maker expectations and avoiding economic fluctuations. This method is therefore an extremely helpful, significant, and effective analytical technique for examining PDE models with delays. The technique provides good outcomes for all of the scenarios used in this work. In the future, we are planning to apply this method to the same model with an added delay feedback control term.

Data Availability

The data used to support the findings of this study are available upon request to the author.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The author carried out the proofs of the main results and approved the final manuscript.

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