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Research Article

Asymptotic Behavior of the Kirchhoff Type Stochastic Plate Equation on Unbounded Domains

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In this paper, we study the asymptotic behavior of solutions to the Kirchhoff type stochastic plate equation driven by additive noise defined on unbounded domains. We first prove the uniform estimates of solutions and then establish the existence and upper semicontinuity of random attractors.

1. Introduction

Plate equations can be found in many fields such as certain physical areas as to vibration and elasticity theories of solid mechanics. In this paper, we consider the following Kirchhoff type stochastic plate equation with additive noise defined on

$$\begin{cases} u_{tt} + \alpha u_t + \Delta^2 u_t + \Delta^2 u + \lambda u - M\left(\|\nabla u\|^2\right)\Delta u \\ + f(x,u) = g(x,t) + \beta h(x)\dfrac{dW}{dt}, \\ u(x,\tau) = u_0(x), \\ u_t(x,\tau) = u_1(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, $\alpha, \lambda > 0$ and β are positive constants, f is a nonlinearity satisfying certain growth and dissipative conditions, $g(x,\cdot)$ and h are given functions in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n)$, respectively, and $W(t)$ is a two-sided real-valued Wiener process on a probability space.

The function $M(\cdot)$ satisfies the following conditions:

(1) $M \in C^1(\mathbb{R})$, such that

$$M_1 \le M(s) \le M_2, \tag{2}$$

where M_1 and M_2 are some positive real constants.

(2) Let $\widehat{M}(z) = \int_0^z M(r)dr$; for $\forall z \ge 0$,

$$M(z)z \ge \widehat{M}(z) \ge 0. \tag{3}$$

For the nonlinear function $f(x,u)$, we presume $f(x,\cdot) \in C^2(\mathbb{R})$ and let $F(x,u) = \int_0^u f(x,s)\,ds$; for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, there exist positive constants $c_i\,(i = 1,2,3)$, such that

$$|f(x,u)| \le c_1|u|^k + \eta_1(x),\ \eta_1 \in L^2(\mathbb{R}^n), \tag{4}$$

$$f(x,u)u - c_2F(x,u) \ge \eta_2(x),\ \eta_2 \in L^1(\mathbb{R}^n), \tag{5}$$

$$F(x,u) \ge c_3|u|^{k+1} - \eta_3(x),\ \eta_3 \in L^1(\mathbb{R}^n), \tag{6}$$

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \le \varpi, \tag{7}$$

where $\varpi > 0$, $1 \le k \le (n+4/n-4)$. Note that (4) and (5) imply

$$F(x,u) \le c\left(|u|^2 + |u|^{k+1} + \eta_1^2 + \eta_2\right). \tag{8}$$

As for deterministic plate equations, many authors have showed the existence of global attractors (see [1–9]). For the stochastic case, the existence of random attractors for plate equations has been investigated in [10, 11, 12] on bounded domains. In addition, there are results about the existence of random attractors and asymptotic compactness for plate equations on unbounded domains in [13–18].

When $M(s) \equiv 0$ in (1), we have investigated the existence of a random attractor for plate equations with additive noise and nonlinear damping defined on $\mathbb{R}^n$ (see [14]). However, when equation (1) is Kirchhoff type, the problem is not yet considered by any predecessors.

To overcome the noncompactness of Sobolev embeddings on $\mathbb{R}^n$, we will apply the idea of uniform estimates on the tails of solutions as in [19, 20] as well as the compactness methods introduced in [21]. More precisely, we first show that the tails of the solutions of (1) are uniformly small outside a bounded domain for large time, and then we derive the asymptotic compactness of solutions in bounded domains by splitting the solutions.

This paper is organized as follows. In Section 2, we present some notations and proposition about random dynamical systems. In Section 3, we define a continuous cocycle for equation (1) in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. In Section 4, we obtain all necessary uniform estimates of solutions. In Section 5, we show the existence and uniqueness of a random attractor for (1) defined on $\mathbb{R}^n$.

2. Notations

In this section, we present some basic notations and known results on nonautonomous random dynamical systems which can be found in [22, 20].

Let $(X, \|\cdot\|_X)$ be a complete separable metric space and be $(\Omega, F, P, \{\theta_t\}_{t\in\mathbb{R}})$ an ergodic metric dynamical system (see [23]).

Proposition 1. *Let $\mathscr{D}$ be an inclusion closed collection of some families of nonempty subsets of X and Φ be a continuous cocycle on X over $(\Omega, F, P, \{\theta_t\}_{t\in\mathbb{R}})$. Then, Φ has a unique $\mathscr{D}$-pullback random attractor $\mathscr{A}$ in $\mathscr{D}$ if Φ is $\mathscr{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathscr{D}$-pullback absorbing set K in $\mathscr{D}$.*

Next, we present criteria concerning upper semicontinuity of nonautonomous random attractors with respect to a parameter.

Theorem 1. *Let $(X, \|\cdot\|_X)$ be a separable Banach space and Φ_0 be an autonomous dynamical system with the global attractor $\mathscr{A}_0$ in X. Given $\beta > 0$, suppose that Φ_β is the perturbed random dynamical system with a random attractor $\mathscr{A}_\beta \in \mathscr{D}$ and a random absorbing set $E_\beta \in \mathscr{D}$. Then, for $P-a.e.\tau \in \mathbb{R}$, $\omega \in \Omega$,*

$$d_H\left(\mathscr{A}_\beta(\tau,\omega), \mathscr{A}_0\right) \longrightarrow 0, \text{ as } \beta \longrightarrow 0, \tag{9}$$

if the following conditions are satisfied:

(i) There exists some deterministic constant c such that, for $P-a.e.\tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\beta\longrightarrow 0} \left\|E_\beta(\tau,\omega)\right\|_X \le c. \tag{10}$$

(ii) There exists $\beta_0 > 0$, such that for $P-a.e.\tau \in \mathbb{R}, \omega \in \Omega$,

$$\bigcup_{0<\beta\le\beta_0} \mathscr{A}_\beta(\tau,\omega) \text{ is precompact in } X. \tag{11}$$

(iii) For $P-a.e.\tau \in \mathbb{R}, \omega \in \Omega$, $t \ge 0$, $\beta_n \longrightarrow 0$, and $x_n, x \in X$ with $x_n \longrightarrow x$, it holds that

$$\lim_{n\longrightarrow\infty} \Phi_{\beta_n}(t,\tau,\omega)x_n = \Phi_0(t)x, \tag{12}$$

where $\|E_\beta(\tau,\omega)\|_X = sup_{x\in E_\beta(\tau,\omega)}\|x\|_X$.

3. Cocycles for Stochastic Plate Equation

In this section, we present some basic settings about (1) and prove that it generates a continuous cocycle in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$, with the domain $D(A) = H^4(\mathbb{R}^n)$. We can define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A)^{\nu/4}$ is a Hilbert space with the following inner product and norm.

$$\begin{aligned} (u,v)_\nu &= \left(A^{\nu/4}u, A^{\nu/4}v\right), \\ \|\cdot\|_\nu &= \left\|(A)^{\nu/4}\right\|. \end{aligned} \tag{13}$$

Set $\mathscr{H} = H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ with norm

$$\|Y\|_{\mathscr{H}} = \left(\|v\|^2 + \left(\delta^2 + \lambda - \delta\alpha\right)\|u\|^2 + (1-\delta)\|\Delta u\|^2\right)^{1/2}, \tag{14}$$

for $Y = (u,v)^\top \in \mathscr{H}$, where $\top$ stands for the transposition.

Let $\xi = u_t + \delta u$, where δ is a small positive constant whose value will be determined later; then, (1) is equivalent to

$$\begin{cases} \dfrac{du}{dt} = \xi - \delta u, \\ \dfrac{d\xi}{dt} = [\delta(\alpha + A - \delta) - A]u - (\alpha + A - \delta)\xi \\ -\lambda u + M\left(\|\nabla u\|^2\right)\Delta u - f(x,u) + g(x,t) + \beta h(x)\dfrac{dW}{dt}, \\ u(x,\tau) = u_0(x), \xi(x,\tau) = \xi_0(x), \end{cases} \tag{15}$$

where $\xi_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$.

For g: we assume that there exists a positive constant σ such that

$$\int_{-\infty}^{\tau} e^{\sigma s}\|g(\cdot,s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \tag{16}$$

and

$$\lim_{r\longrightarrow\infty} \int_{-\infty}^{\tau}\int_{|x|\ge r} e^{\sigma s}|g(\cdot,s)|^2 dx ds = 0, \quad \forall \tau \in \mathbb{R}, \tag{17}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

Denote $\omega(t) = W(t) = W(t,x)$, $t \in \mathbb{R}$; then, we consider the Ornstein–Uhlenbeck equation $\mathrm{d}y + y\mathrm{d}t = \mathrm{d}W(t)$ and the Ornstein–Uhlenbeck process

$$y(\theta_t\omega) = -\int_{-\infty}^{0} e^{s}(\theta_t\omega)(s)\mathrm{d}s, \quad t \in \mathbb{R}. \tag{18}$$

From [24], it is known that the random variable $|y(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full P measure such that $y(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$. Put

$$\begin{aligned} z(\theta_t\omega) &= z(x,\theta_t\omega) \\ &= h(x)y(\theta_t\omega), \end{aligned} \tag{19}$$

which solves

$$\mathrm{d}z + z\mathrm{d}t = h dW. \tag{20}$$

Lemma 1 (see [25]). *For any $\varepsilon > 0$, there exists a tempered random variable $\gamma\colon \Omega \longrightarrow \mathbb{R}^+$, such that for all $t \in \mathbb{R}$, $\omega \in \Omega$,*

$$\begin{aligned} \|z(\theta_t\omega)\| &\le e^{\varepsilon|t|}\gamma(\omega)\|h\|, \\ \|\nabla z(\theta_t\omega)\| &\le e^{\varepsilon|t|}\gamma(\omega)\|\nabla h\|, \\ \|\Delta z(\theta_t\omega)\| &\le e^{\varepsilon|t|}\gamma(\omega)\|\Delta h\|, \end{aligned} \tag{21}$$

where $\gamma(\omega)$ satisfies

$$e^{-\varepsilon|t|}\gamma(\omega) \le \gamma(\theta_t\omega) \le e^{\varepsilon|t|}\gamma(\omega). \tag{22}$$

Now, let $v(t,\tau,\omega) = \xi(t,\tau,\omega) - \beta z(\theta_t\omega)$, and we have

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = v - \delta u + \beta z(\theta_t\omega), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = (\delta - \alpha - A)v + [\delta(-\delta + \alpha + A) - \lambda - A]u \\ \quad +\beta[1 - (\alpha + A - \delta)]z(\theta_t\omega) + M\left(\|\nabla u\|^2\right)\Delta u \\ \quad -f(x,u) + g(x,t), \\ u(x,\tau,\tau) = u_0(x), v(x,\tau,\tau) = v_0(x), \end{cases} \tag{23}$$

where $v_0(x) = \xi_0(x) - z(\theta_\tau\omega)$, $x \in \mathbb{R}^n$. We will consider (23) for $\omega \in \widetilde{\Omega}$ and write $\widetilde{\Omega}$ as Ω from now on.

The well-posedness of the deterministic problem (23) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ can be established by standard methods as in [13, 26–28]. If (2)–(7) are fulfilled, let $\varphi^{(\beta)}(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0^{(\beta)}) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0),\ v(t+\tau,\tau,\theta_{-\tau}\omega,v_0))^{\top}$, where $\varphi_0^{(\beta)} = (u_0,v_0)^{\top}$. Then, for every $\omega \in \Omega, \tau \in \mathbb{R}$ and $\varphi_0^{(\beta)} \in \mathscr{H}(\mathbb{R}^n)$, problem (23) has a unique $(\mathscr{F},\mathscr{B}(H^2(\mathbb{R}^n)) \times \mathscr{B}(L^2(\mathbb{R}^n)))$-measurable solution $\varphi^{(\beta)}(\cdot,\tau,\omega,\varphi_0^{(\beta)}) \in C([\tau,\infty),\mathscr{H}(\mathbb{R}^n))$ with $\varphi^{(\beta)}(\tau,\tau,\omega,\varphi_0^{(\beta)}) = \varphi_0^{(\beta)}$, $\varphi^{(\beta)}(t,\tau,\omega,\varphi_0^{(\beta)}) \in \mathscr{H}_0(\mathbb{R}^n)$ being continuous in $\varphi_0^{(\beta)}$ for each $t > \tau$. Moreover, for every $(t,\tau,\omega,\varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathscr{H}(\mathbb{R}^n)$, the mapping

$$\Phi_\beta\left(t,\tau,\omega,\varphi_0^{(\beta)}\right) = \varphi^{(\beta)}\left(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0^{(\beta)}\right), \tag{24}$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n)$ over $\mathbb{R}$ and $(\Omega,F,P,\{\theta_t\}_{t\in\mathbb{R}})$.

4. Uniform Estimates of Solutions

In this section, we derive uniform estimates on the solutions of problem (23) and construct a tempered pullback absorbing set.

Let $\delta \in (0,1)$ be small enough such that

$$\delta^2 + \lambda - \delta\alpha > 0, 1 - \delta > 0, \tag{25}$$

and define σ appearing in (17) by

$$\sigma = \min\left\{\alpha - \delta, \delta, \frac{c_2\delta}{2}\right\}. \tag{26}$$

Lemma 2. *Assume that (2)–(7) and (16) hold. Then, for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau,\omega)\colon \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \ge T$,*

$$\begin{aligned} &\left\|\varphi^{(\beta)}(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\right\|_{\mathscr{H}}^2 + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2\mathrm{d}s \\ &\quad + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\mathrm{d}s + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\mathrm{d}s \\ &\quad + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\mathrm{d}s \\ &\quad < c + c\beta^2\int_{-\infty}^{0} e^{\sigma s}\left(1 + \left\|\Delta z(\theta_s\omega)\right\|^2 + \left\|\nabla z(\theta_s\omega)\right\|^2 + \left\|z(\theta_s\omega)\right\|^2 + \left\|z(\theta_s\omega)\right\|_{H^2}^{k+1}\right)\mathrm{d}s, \end{aligned} \tag{27}$$

where $\varphi_0^{(\beta)} = (u_0,v_0)^{\top} \in D(\tau-t,\theta_{-t}\omega)$ and c is a positive constant depending on λ, σ, α, and δ but independent of τ, ω, and D.

Proof. Taking the inner product of $(23)_2$ with v in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -(\alpha-\delta)(v,v) - \left(\lambda+\delta^2-\delta\alpha\right)(u,v) - (1-\delta)(Au,v) - (Av,v) + \beta(1-\alpha+\delta)(z(\theta_t\omega),v) - \beta(Az(\theta_t\omega),v) + (g(x,t),v) + M\left(\|\nabla u\|^2\right)\Delta u - (f(x,u),v). \tag{28}$$

By $(23)_1$, we get

$$v = \frac{du}{dt} + \delta u - \beta z(\theta_t\omega). \tag{29}$$

Next, we estimate some terms of (28).

$$\begin{aligned}(u,v) &= \left(u, \frac{du}{dt} + \delta u - \beta z(\theta_t\omega)\right) \\ &= \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \beta(u, z(\theta_t\omega)) \\ &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \beta\|z(\theta_t\omega)\|\|u\| \\ &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{3\delta}{4}\|u\|^2 - \frac{\beta^2}{3\delta}\|z(\theta_t\omega)\|^2,\end{aligned} \tag{30}$$

$$\begin{aligned}-(Au,v) &= -\left(\Delta^2 u, \frac{du}{dt} + \delta u - \beta z(\theta_t\omega)\right) \\ &= -\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + -\delta\|\Delta u\|^2 + \beta(\Delta u, \Delta z(\theta_t\omega)) \\ &\leq -\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + -\delta\|\Delta u\|^2 + +\beta\|\Delta z(\theta_t\omega)\|\|\Delta u\| \\ &\leq -\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + -\frac{3\delta}{4}\|\Delta u\|^2 + +\frac{\beta^2}{3\delta}\|\Delta z(\theta_t\omega)\|^2,\end{aligned} \tag{31}$$

$$\begin{aligned}(f(x,u),v) &= \left(f(x,u), \frac{du}{dt} + \delta u - \beta z(\theta_t\omega)\right) \\ &= \frac{d}{dt}\int_{\mathbb{R}^n} F(x,u)dx + \delta(f(x,u),u) \\ &\quad - \beta(f(x,u), z(\theta_t\omega)).\end{aligned} \tag{32}$$

By (5), we get

$$(f(x,u),u) \geq c_2 \int_{\mathbb{R}^n} F(x,u)dx + \int_{\mathbb{R}^n} \eta_2(x)dx. \tag{33}$$

By conditions (4) and (6), we obtain

$$\begin{aligned}&\beta(f(x,u), z(\theta_t\omega)) \\ &\leq \beta\int_{\mathbb{R}^n}(c_1|u|^k + \eta_1(x))|z(\theta_t\omega)|dx \\ &\leq \beta\|\eta_1(x)\|\|z(\theta_t\omega)\| + c_1\beta\left(\int_{\mathbb{R}^n}|u|^{k+1}dx\right)^{k/k+1} z\|(\theta_t\omega)\|_{k+1} \\ &\leq \beta\|\eta_1(x)\|\|z(\theta_t\omega)\| \\ &\quad + c_1\beta\left(\int_{\mathbb{R}^n} F(x,u) + \eta_3(x)dx\right)^{k/k+1}\|z(\theta_t\omega)\|_{k+1} \\ &\leq \frac{1}{2}\|\eta_1(x)\|^2 + \frac{\beta^2}{2}\|z(\theta_t\omega)\|^2 + \frac{\delta c_2}{2}\int_{\mathbb{R}^n} F(x,u)dx \\ &\quad + \frac{\delta c_2}{2}\int_{\mathbb{R}^n}\eta_3(x)dx + c\beta^2\|z(\theta_t\omega)\|_{H^2}^{k+1}.\end{aligned} \tag{34}$$

Using the Cauchy–Schwarz inequality and the Young inequality, there holds

$$\beta(1-\alpha+\delta)(z(\theta_t\omega),v) \leq \frac{2(1-\alpha+\delta)^2\beta^2}{\alpha-\delta}\|z(\theta_t\omega)\|^2 + \frac{\alpha-\delta}{8}\|v\|^2, \tag{35}$$

$$-\beta(Az(\theta_t\omega),v) = -\beta(\Delta z(\theta_t\omega),\Delta v) \leq \frac{\beta^2}{2}\|\Delta z(\theta_t\omega)\|^2 + \frac{1}{2}\|\Delta v\|^2, \tag{36}$$

$$(g(x,t),v) \leq \|g(x,t)\|\|v\| \leq \frac{2}{\alpha-\delta}\|g(x,t)\|^2 + \frac{\alpha-\delta}{8}\|v\|^2. \tag{37}$$

By (2) and (3) we have

$$\begin{aligned}&\left(M\left(\|\nabla u\|^2\right)\Delta u, v\right) \\ &= \left(M\left(\|\nabla u\|^2\right)\Delta u, u_t + \delta u - \beta z(\theta_t\omega)\right) \\ &= -\frac{1}{2}\frac{d}{dt}\widehat{M}\left(\|\nabla u\|^2\right) + \delta M\left(\|\nabla u\|^2\right)\left(\|\nabla u\|^2\right) \\ &\quad - \beta\left(M\left(\|\nabla u\|^2\right)\right)\nabla u, \nabla z(\theta_t\omega) \\ &\leq -\frac{1}{2}\frac{d}{dt}\widehat{M}\left(\|\nabla u\|^2\right) - \delta M\left(\|\nabla u\|^2\right)\left(\|\nabla u\|^2\right) \\ &\quad + \frac{\delta}{4}M\left(\|\nabla u\|^2\right)\left(\|\nabla u\|^2\right) + c\beta^2 M\left(\|\nabla u\|^2\right)\|\nabla z(\theta_t\omega)\|^2 \\ &\leq -\frac{1}{2}\frac{d}{dt}\widehat{M}\left(\|\nabla u\|^2\right) - \frac{3\delta}{4}M\left(\|\nabla u\|^2\right)\left(\|\nabla u\|^2\right) \\ &\quad + c\beta^2\|\nabla z(\theta_t\omega)\|^2 \\ &\leq -\frac{1}{2}\frac{d}{dt}\widehat{M}\left(\|\nabla u\|^2\right) - \frac{3\delta}{4}\widehat{M}\left(\|\nabla u\|^2\right) + c\beta^2\|\nabla z(\theta_t\omega)\|^2.\end{aligned} \tag{38}$$

By (30)–(38), it follows from (28) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v\|^2+\left(\delta^2+\lambda-\delta\alpha\right)\|u\|^2+(1-\delta)\|\Delta u\|^2+\widehat{M}\left(\|\nabla u\|^2\right)+2\int_{\mathbb{R}^n}F(x,u)\mathrm{d}x\right)\\
&\leq-\frac{3}{4}(\alpha-\delta)\|v\|^2-\frac{3}{4}\delta\left(\delta^2+\lambda-\delta\alpha\right)\|u\|^2-\frac{3}{4}\delta(1-\delta)\left(\|\nabla u\|^2\right)-\frac{3\delta}{4}\widehat{M}\left(\|\nabla u\|^2\right)\\
&\quad-\frac{\delta c_2}{2}\int_{\mathbb{R}^n}F(x,u)dx-\frac{1}{2}\left(\|\nabla v\|^2\right)+c\beta^2\left(1+\left\|\Delta z(\theta_t\omega)\right\|^2+\left\|\Delta z(\theta_t\omega)\right\|^2+\left\|\Delta z(\theta_t\omega)\right\|^2+\left\|\Delta z(\theta_t\omega)^2\right\|_{H^2}^{k+1}\right)+c\|g(x,t)\|^2.
\end{aligned}\tag{39}$$

By (14) and (26), we get from (39) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\varphi\|_{\mathscr{H}}^2+2\int_{\mathbb{R}^n}F(x,u)\mathrm{d}x\right)+\sigma\left(\|\varphi\|_{\mathscr{H}}^2+2\int_{\mathbb{R}^n}F(x,u)\mathrm{d}x\right)\\
&\quad+\frac{1}{2}(\alpha-\delta)\|v\|^2+\frac{1}{2}\delta\left(\delta^2+\lambda-\delta\alpha\right)\|u\|^2+\frac{1}{2}\delta(1-\delta)\|\Delta u\|^2+\|\Delta v\|^2+\frac{\delta}{2}\widehat{M}\left(\|\nabla u\|^2\right)\\
&\leq c\beta^2\left(1+\left\|\Delta z(\theta_t\omega)\right\|^2+\left\|\nabla z(\theta_t\omega)\right\|^2+\left\|z(\theta_t\omega)\right\|^2+\left\|z(\theta_t\omega)\right\|_{H^2}^{k+1}\right)+c\|g(x,t)\|^2.
\end{aligned}\tag{40}$$

Multiplying (40) by $e^{\sigma t}$ and integrating over $(\tau-t,\tau)$ and then replacing ω by $\theta_{-\tau}\omega$, we get

$$\begin{aligned}
&\left(\left\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\right\|_{\mathscr{H}}^2+2\int_{\mathbb{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\mathrm{d}x\right)\\
&\quad+\frac{1}{2}(\alpha-\delta)\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2\mathrm{d}s\\
&\quad+\frac{1}{2}\delta\left(\delta^2+\lambda-\delta\alpha\right)\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\mathrm{d}s\\
&\quad+\frac{1}{2}\delta(1-\delta)\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\mathrm{d}s+\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2\mathrm{d}s\\
&\quad+\frac{\delta}{2}\int_{\tau-t}^{\tau}e^{\sigma s}\widehat{M}\left(\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\right)\mathrm{d}s\\
&\leq e^{-\sigma t}\left(\left\|\varphi_0\right\|_{\mathscr{H}}^2+2\int_{\mathbb{R}^n}F(x,u_0)\mathrm{d}x\right)\\
&\quad+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}1+\left\|\Delta z(\theta_{s-\tau}\omega)\right\|^2+\left\|\Delta z(\theta_{s-\tau}\omega)\right\|^2+\left\|\Delta z(\theta_{s-\tau}\omega)\right\|^2+\left\|z(\theta_{s-\tau}\omega)\right\|_{H^2}^{k+1}\mathrm{d}s\\
&\quad+\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|g(x,s)\|^2\\
&\leq e^{-\sigma t}\left(\left\|\varphi_0\right\|_{\mathscr{H}}^2+2\int_{\mathbb{R}^n}F(x,u_0)\mathrm{d}x\right)+c\beta^2\int_{-t}^{0}e^{\sigma s}1+\left\|\Delta z(\theta_s\omega)\right\|^2+\left\|z(\theta_s\omega)\right\|^2+\left\|\nabla z(\theta_s\omega)\right\|^2+\left\|z(\theta_s\omega)\right\|_{H^2}^{k+1}\mathrm{d}s\\
&\quad+\int_{-\infty}^{\tau}e^{\sigma(s-\tau)}\|g(x,s)\|^2.
\end{aligned}\tag{41}$$

It follows from Lemma 1 that

$$\begin{aligned}&\int_{-t}^{0} e^{\sigma s}\|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|_{H^2}^{k+1} \mathrm{d}s \\ &\leq \int_{-\infty}^{0} e^{\sigma s}\|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|^2 + \|\Delta z(\theta_s\omega)\|_{H^2}^{k+1} \mathrm{d}s \\ &\leq \int_{-\infty}^{0} e^{\sigma s/2}\left(\gamma^2(\omega)\left(\|\Delta h\|^2 + \|\nabla h\|^2 + \|h\|^2\right) + \gamma^{k+1}(\omega)\left(\|\Delta h\|^{k+1} + \|\nabla h\|^{k+1} + \|h\|^{k+1}\right)\right)\mathrm{d}s < +\infty.\end{aligned} \tag{42}$$

From (8),

$$\int_{\mathbb{R}^n} F(x,u_0)\mathrm{d}x \leq c\left(1 + \|u_0\|^2 + \|u_0\|_{H^2}^{k+1}\right). \tag{43}$$

Because $\varphi_0 = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$ and $D \in \mathscr{D}$, we get from (43) that

$$\lim_{t\longrightarrow+\infty} e^{-\sigma t}\left(\|\varphi_0\|_{\mathscr{H}}^2 + 2\int_{\mathbb{R}^n} F(x,u_0)dx\right) = 0. \tag{44}$$

Therefore, (17), (42), and (43) deduce the desired result (27). □

Lemma 3. *Assume that (2)–(7) and (16) hold. Then, there exists a random ball $\left\{E_\beta(\tau,\omega)\colon \tau\in\mathbb{R}, \omega\in\Omega\right\}\in\mathscr{D}$ centered at 0 with random radius*

$$\varrho(\tau,\omega) = c + c\beta^2\int_{-\infty}^{0} e^{\sigma s}\left(1 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_{H^2}^{k+1}\right)ds, \tag{45}$$

such that $\left\{E_\beta(\tau,\omega)\colon \tau\in\mathbb{R}, \omega\in\Omega\right\}\in\mathscr{D}$ is a closed measurable $\mathscr{D}$-pullback absorbing set for the continuous cocycle associated with problem (23) in $\mathscr{D}$, that is, for every $\tau\in\mathbb{R}, \omega\in\Omega$, and $D = \{D(\tau,\omega)\colon \tau\in\mathbb{R}, \omega\in\Omega\}\in\mathscr{D}$, there exists $T = T(\tau,\omega,D) > 0$, such that for all $t\geq T$,

$$\Phi_\beta(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega))\subseteq A(\tau,\omega). \tag{46}$$

Proof. This is an immediate consequence of (24) and Lemma 2.

Let $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function such that $0\leq\rho(x)\leq 1$ for all $x\in\mathbb{R}^n$, and

$$\rho(x) = 0 \text{ for } 0\leq|x|\leq 1; \text{ and } \rho(x) = 1 \text{ for } |x|\geq 2. \tag{47}$$

For every $r\in\mathbb{N}$, let

$$\rho_r(x) = \rho\left(\frac{x}{r}\right), \quad x\in\mathbb{R}^n. \tag{48}$$

Then, there exist positive constants c_4, c_5, c_6, and c_7 independent of k such that $|\Delta\rho_r(x)|\leq 1/rc_4$, $|\Delta\rho_r(x)|\leq 1/rc_5$, $|\Delta\rho_r(x)|\leq 1/rc_6$, $|\Delta^2\rho_r(x)|\leq 1/rc_7$ for all $x\in\mathbb{R}^n$ and $r\in\mathbb{N}$.

Given$r\geq 1$, denote $\mathbb{H}_r = \{x\in\mathbb{R}^n\colon |x| < r\}$ and $\mathbb{R}^n\smallsetminus\mathbb{H}_r$ be the complement of$\mathbb{H}_r$. To prove asymptotic compactness of solution on $\mathbb{R}^n$, we prove the following lemma. □

Lemma 4. *Assume that (2)–(7) and (16) hold. Then, for every $\tau\in\mathbb{R}, \omega\in\Omega$, and $D = \{D(\tau,\omega)\colon \tau\in\mathbb{R}, \omega\in\Omega\}\in\mathscr{D}$, there exist $T = T(\tau,\omega,D,\varepsilon) > 0$ and $\widetilde{R} = \widetilde{R}(\tau,\omega,\varepsilon)\geq 1$, such that for all $t\geq T, r\geq\widetilde{R}$,*

$$\left\|\varphi^{(\beta)}(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\right\|_{\mathscr{H}(\mathbb{R}^n\smallsetminus\mathbb{H}_r)}^2 \leq \varepsilon, \tag{49}$$

where $\varphi_0^{(\beta)} = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Taking the inner product of $(3.7)_2$ with $\rho_r(x)v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|v|^2\mathrm{d}x = &-(\alpha-\delta)\int_{\mathbb{R}^n}\rho_r(x)|v|^2\mathrm{d}x - \left(\lambda + \delta^2 - \delta\alpha\right)\int_{\mathbb{R}^n}\rho_r(x)uv\mathrm{d}x \\ &- (1-\delta)\int_{\mathbb{R}^n}\rho_r(x)(Au)v\mathrm{d}x - \int_{\mathbb{R}^n}\rho_r(x)(Av)v\mathrm{d}x + \beta(1-\alpha+\delta)\int_{\mathbb{R}^n}\rho_r(x)z(\theta_t\omega)v\mathrm{d}x \\ &- \beta\int_{\mathbb{R}^n}\rho_r((x)Az(\theta_t\omega))v\mathrm{d}x + \int_{\mathbb{R}^n}\rho_r(x)g(x,t)v\mathrm{d}x + \int_{\mathbb{R}^n}\rho_r(x)M\left(\|\nabla u\|^2\right)\Delta uv\mathrm{d}x \\ &- \int_{\mathbb{R}^n}\rho_r(x)f(x,u)v\mathrm{d}x.\end{aligned} \tag{50}$$

Using the Young inequality and the Sobolev interpolation inequality,

$$\|\nabla v\| \le \varsigma\|v\| + C_\varsigma\|\Delta v\|, \forall \varsigma > 0. \tag{51}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^n}\rho_r(x)uvdx &= \int_{\mathbb{R}^n}\rho_r(x)u\left(\frac{du}{dt} + \delta u - \beta z(\theta_t\omega)\right)dx\\ &= \int_{\mathbb{R}^n}\rho_r(x)\left(\frac{1}{2}\frac{d}{dt}u^2 + \delta u^2 - \beta z(\theta_t\omega)u\right)dx\\ &\ge \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|u|^2dx + \frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)|u|^2dx - \frac{\beta^2}{2\delta}\int_{\mathbb{R}^n}\rho_r(x)\left|z(\theta_t\omega)\right|^2dx,\end{aligned} \tag{52}$$

$$\begin{aligned}&-\int_{\mathbb{R}^n}\rho_r(x)\cdot Au\cdot vdx\\ &= -\int_{\mathbb{R}^n}\Delta u\cdot\Delta\left(\rho_r(x)\left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t\omega)\right)\right)dx\\ &= -\int_{\mathbb{R}^n}\Delta u\cdot\left(\Delta\rho_r(x)\cdot tvn + q2h\nabla\rho_r x(x)7\cdot C\nabla; v + \rho_r(x)\cdot\Delta\left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t\omega)\right)\right)dx\\ &\le \frac{c_5}{r}\int_{\mathbb{R}^n}|\Delta u\cdot v|dx + \frac{2c_4}{r}\int_{\mathbb{R}^n}|\Delta u\cdot\nabla v|dx - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx - \delta\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2\,dx\\ &\quad + \epsilon\int_{\mathbb{R}^n}\rho_r(x)\|\Delta u\|\Delta z(\theta_t\omega)dx\\ &\le \frac{c_5}{2r}\left(|\Delta u|^2d + |v|^2d\right) + \frac{2c_4}{r}\|\Delta u\|\left(\varsigma\|v\| + C_\varsigma\|\Delta v\|\right) - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx\\ &\quad - \delta\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx + \frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx + \frac{\epsilon^2}{2\delta}\int_{\mathbb{R}^n}\rho_r(x)\left|\Delta z(\theta_t\omega)\right|^2dx\\ &\le \frac{c_5}{r}\left(\|\Delta u\|^2 + \|v\|^2\right) + \frac{c_4}{r}\|\Delta u\|^2 + 2\varsigma^2\|v\|^2 + 2C_\varsigma^2\|\Delta v\|^2\\ &\quad - \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx - \frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx + \frac{\epsilon^2}{2\delta}\int_{\mathbb{R}^n}\rho_r(x)\left|\Delta z(\theta_t\omega)\right|^2dx,\end{aligned} \tag{53}$$

$$\begin{aligned}\int_{\mathbb{R}^n}\rho_r(x)f(x,u)vdx &= \int_{\mathbb{R}^n}\rho_r(x)f(x,u)\left(\frac{du}{dt} + \delta u - \beta z(\theta_t\omega)\right)dx\\ &= \frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)F(x,u)dx + \delta\int_{\mathbb{R}^n}\rho_r(x)f(x,u)udx - \beta\int_{\mathbb{R}^n}\rho_r(x)f(x,u)z(\theta_t\omega)dx.\end{aligned} \tag{54}$$

By (6), we get

$$\delta\int_{\mathbb{R}^n}\rho_r(x)f(x,u)udx \ge c_2\delta\int_{\mathbb{R}^n}\rho_r(x)F(x,u)dx + \delta\int_{\mathbb{R}^n}\rho_r(x)\eta_2(x)dx. \tag{55}$$

By (4) and (6), we get

$$\begin{aligned}\beta\int_{\mathbb{R}^n}\rho_r(x)f(x,u)z(\theta_t\omega)dx &\le \beta\int_{\mathbb{R}^n}\rho_r(x)\left(c_1|u|^k+\eta_1(x)\right)\left|z(\theta_t\omega)\right|dx\\ &\le \frac{1}{2}\int_{\mathbb{R}^n}\rho_r(x)\left|\eta_1(x)\right|^2dx+\frac{\beta^2}{2}\int_{\mathbb{R}^n}\rho_r(x)\left|z(\theta_t\omega)\right|^2dx\\ &\quad + c\beta^2\int_{\mathbb{R}^n}\rho_r(x)\left|z(\theta_t\omega)\right|^{k+1}dx+\frac{c_2\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)\left(F(x,u)+\eta_3(x)\right)dx.\end{aligned} \tag{56}$$

For the remainder terms, using the Cauchy–Schwarz inequality and the Young inequality, we have

$$\begin{aligned}-\int_{\mathbb{R}^n}\rho_r(x)\cdot Av\cdot vdx &= -\int_{\mathbb{R}^n}\Delta^2v\cdot\rho_r(x)\cdot vdx = -\int_{\mathbb{R}^n}\Delta v\cdot\Delta\left(\rho_r(x)\cdot v\right)dx\\ &= -\int_{\mathbb{R}^n}\Delta v\cdot\left(\Delta\rho_r(x)\cdot v+2\nabla\rho_r(x)\cdot\nabla v+\rho_r(x)\cdot\Delta v\right)dx\\ &\le \frac{c_5}{2r}\left(\|\Delta v\|^2+\|v\|^2\right)+\frac{2c_4}{r}\|\Delta v\|\|\Delta v\|-\int_{\mathbb{R}^n}\rho_k(x)|\Delta v|^2dx\\ &\le \frac{c_5}{2r}\left(\|\Delta v\|^2+\|v\|^2\right)+\frac{2c_4}{r}\|\Delta v\|\left(\varsigma\|v\|+C_\varsigma\|\Delta v\|\right)-\int_{\mathbb{R}^n}\rho_r(x)|\Delta v|^2dx\\ &\le \frac{c_5}{2r}\left(\|\Delta v\|^2+\|v\|^2\right)+\frac{c_4}{r}\left(\|\Delta v\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2\right)-\int_{\mathbb{R}^n}\rho_r(x)|\Delta v|^2dx,\end{aligned} \tag{57}$$

$$\begin{aligned}-\beta\int_{\mathbb{R}^n}\rho_r(x)Az(\theta_t\omega)\cdot vdx &= -\beta\int_{\mathbb{R}^n}\Delta^2z(\theta_t\omega)\cdot\rho_r(x)\cdot vdx = -\beta\int_{\mathbb{R}^n}\Delta z(\theta_t\omega)\cdot\Delta\left(\rho_r(x)\cdot v\right)dx\\ &= -\beta\int_{\mathbb{R}^n}\Delta z(\theta_t\omega)\cdot\left(\Delta\rho_r(x)\cdot v+2\nabla\rho_r(x)\cdot\nabla v+\rho_r(x)\cdot\Delta v\right)dx\\ &\le \frac{c_5\beta}{r}\int_{\mathbb{R}^n}\left|\Delta z(\theta_t\omega)\cdot v\right|dx+\frac{2c_4\beta}{r}\int_{\mathbb{R}^n}\left|\Delta z(\theta_t\omega)\cdot\nabla v\right|dx+\beta\int_{\mathbb{R}^n}\rho_r(x)\left|\Delta z(\theta_t\omega)\right|\cdot|\Delta v|dx\\ &\le \frac{c_5\beta}{2r}\left(\left\|\Delta z(\theta_t\omega)\right\|^2+\|v\|^2\right)+\frac{2c_4\beta}{r}\left\|\Delta z(\theta_t\omega)\right\|\left(\varsigma\|v\|+C_\varsigma\|\Delta v\|\right)+\beta\int_{\mathbb{R}^n}\rho_r(x)\left|\Delta z(\theta_t\omega)\right|\cdot|\Delta v|dx\\ &\le \frac{c_5\beta}{2r}\left(\left\|\Delta z(\theta_t\omega)\right\|+\|v\|^2\right)+\frac{c_4\beta}{r}\left(\left\|\Delta z(\theta_t\omega)\right\|^2+2\varsigma^2\|v\|^2+2C_\varsigma^2\|\Delta v\|^2\right)\\ &\quad+\int_{\mathbb{R}^n}\rho_r(x)|\Delta v|^2dx+\frac{\beta^2}{4}\int_{\mathbb{R}^n}\rho_r(x)\left|\Delta z(\theta_t\omega)\right|^2dx,\end{aligned} \tag{58}$$

$$\begin{aligned}&\beta(1-\alpha+\delta)\int_{\mathbb{R}^n}\rho_r(x)z(\theta_t\omega)vdx\\ &\le \frac{(1-\alpha+\delta)^2\beta^2}{\alpha-\delta}\int_{\mathbb{R}^n}\rho_r(x)\left|z(\theta_t\omega)\right|^2dx+\frac{\alpha-\delta}{4}\int_{\mathbb{R}^n}\rho_r(x)|v|^2dx,\end{aligned} \tag{59}$$

$$\int_{\mathbb{R}^n}\rho_r(x)g(x,t)vdx \le \frac{1}{\alpha-\delta}\int_{\mathbb{R}^n}\rho_r(x)|g(x,t)|^2dx+\frac{\alpha-\delta}{4}\int_{\mathbb{R}^n}\rho_r(x)|v|^2dx. \tag{60}$$

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho_r(x)M\left(\|\nabla u\|^2\right)\Delta uv dx &= \int_{\mathbb{R}^n}\rho_r(x)M\left(\|\nabla u\|^2\right)\Delta u\left(u_t+\delta u-\beta z(\theta_t\omega)\right)dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)\widehat{M}\left(\|\nabla u\|^2\right)dx-\delta\int_{\mathbb{R}^n}\rho_r(x)M\left(\|\nabla u\|^2\right)|\nabla u|^2dx \\
&\quad+\beta\int_{\mathbb{R}^n}\rho_r(x)M\left(\|\nabla u\|^2\right)\nabla u\nabla z(\theta_t\omega)dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)\left(\widehat{M}\|\nabla u\|^2\right)dx-\delta\int_{\mathbb{R}^n}\rho_r(x)\left(M\|\nabla u\|^2\right)|\nabla u|^2dx \\
&\quad+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)\left(M\|\nabla u\|^2\right)|\nabla u|^2dx+c\beta^2\int_{\mathbb{R}^n}\rho_r(x)M\|\nabla u\|^2\left|\nabla z(\theta_t\omega)\right|^2dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)\left(M\|\nabla u\|^2\right)dx-\frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)\left(M\|\nabla u\|^2\right)|\nabla u|^2dx+c\beta^2\int_{\mathbb{R}^n}\rho_r(x)\left|\nabla z(\theta_t\omega)\right|^2dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)\widehat{M}\left(\|\nabla u\|^2\right)dx-\frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)\widehat{M}\left(\|\nabla u\|^2\right)dx+c\beta^2\int_{\mathbb{R}^n}\rho_r(x)\left|\nabla z(\theta_t\omega)\right|^2dx.
\end{aligned} \tag{61}$$

It follows from (52)–(61) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)|v|^2+\left\|\delta^2+\lambda-\delta\alpha\right\||u|^2+(1-\delta)|\Delta u|^2+\widehat{M}\left(\|\nabla u\|^2\right)+2F(x,u)dx \\
&\le\frac{c}{r}\left\|\|\Delta v\|^2+\|v\|^2+\|\Delta u\|^2+\left\|\Delta z\|\theta_t\omega\|\right\|^2\right\|-\frac{\alpha-\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)|v|^2dx-\frac{\delta\left(\delta^2+\lambda-\delta\alpha\right)}{2} \\
&\times\int_{\mathbb{R}^n}\rho_r(x)|u|^2dx-\frac{\delta\|1-\delta\|}{2}\int_{\mathbb{R}^n}\rho_r(x)|\Delta u|^2dx-\frac{c_2\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)F(x,u)dx \\
&-\frac{\delta}{2}\int_{\mathbb{R}^n}\rho_r(x)\widehat{M}\left(\|\nabla u\|^2\right)dx+c\beta^2\int_{\mathbb{R}^n}\rho_r(x)1+\left|\Delta z(\theta_t\omega)\right|^2 \\
&+\left|\nabla z(\theta_t\omega)\right|^2+\left|\nabla z(\theta_t\omega)\right|^2+\left|\nabla z(\theta_t\omega)\right|^{k+1}+|g(x,t)|^2dx.
\end{aligned} \tag{62}$$

Denote

$$X=|v|^2+\left(\delta^2+\lambda-\delta\alpha\right)|u|^2+(1-\delta)|\Delta u|^2+\widehat{M}\|\nabla u\|^2. \tag{63}$$

By (26), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_r(x)(X+2F(x,u))dx+\sigma\int_{\mathbb{R}^n}\rho_r(x)(X+2F(x,u))dx \\
&\le\frac{c}{r}\|\Delta v\|^2+\|v\|^2+\|\Delta u\|^2+\left\|\Delta z\|\theta_t\omega\|\right\|^2+c\beta^2\int_{\mathbb{R}^n}\rho_r(x)1+\left|\Delta z\|\theta_t\omega\|\right|^2 \\
&+\left|\nabla z(\theta_t\omega)\right|^2+\left|\nabla z(\theta_t\omega)\right|^2+\left|\nabla z(\theta_t\omega)\right|^{k+1}+|g(x,t)|^2dx.
\end{aligned} \tag{64}$$

Multiplying (64) by $e^{\sigma t}$ and integrating over $(\tau-t,\tau)$ and then replacing ω by $\theta_{-\tau}\omega$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_r(x)\left(X\left(\tau,\tau-t,\theta_{-\tau}\omega,X_0\right)+2F\left(x,u\left(\tau,\tau-t,\theta_{-\tau}\omega,u_0\right)\right)\right)dx\\
&\le e^{-\sigma t}\int_{\mathbb{R}^n}\rho_r(x)\left(X_0+2F\left(x,u_0\right)\right)dx+\frac{c}{r}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\\
&\quad+\left\|\Delta v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2+\left\|\Delta u\left(s,\tau-t,\theta_{-\tau}\omega,u_0\right)\right\|^2+\left\|\Delta z\left(\theta_{s-\tau}\omega\right)\right\|^2ds\\
&\quad+c\beta^2\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\int_{\mathbb{R}^n}\rho_r(x)1+\left|\Delta z\left(\theta_{s-\tau}\omega\right)\right|^2+\left|\nabla z\left(\theta_{s-\tau}\omega\right)\right|^2+\left|\nabla z\left(\theta_{s-\tau}\omega\right)\right|^2\\
&\quad+\left|\nabla z\left(\theta_{s-\tau}\omega\right)\right|^{k+1}+|g(x,s)|^2dx\,ds\\
&\le e^{-\sigma t}\int_{\mathbb{R}^n}\rho_r(x)\left(X_0+2F\left(x,u_0\right)\right)dx+\frac{c}{r}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\\
&\quad+\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2+\left\|\Delta u\left(s,\tau-t,\theta_{-\tau}\omega,u_0\right)\right\|^2ds+\frac{c}{r}\int_{-t}^{0}e^{\sigma s}\left\|\Delta z\left(\theta_s\omega\right)\right\|^2ds\\
&\quad+c\int_{\tau-t}^{\tau}e^{\sigma s}\int_{|x|\ge r}|g(x,s)|^2dx\,ds+c\beta^2\int_{-t}^{0}e^{\sigma s}\int_{|x|\ge r}1+\left|\Delta z\left(\theta_s\omega\right)\right|^2+\left|\nabla z\left(\theta_s\omega\right)\right|^2+\left|z\left(\theta_s\omega\right)\right|^2\\
&\quad+\left|z\left(\theta_s\omega\right)\right|^{k+1}dx\,ds\\
&\le e^{-\sigma t}\int_{\mathbb{R}^n}\rho_r(x)\left(X_0+2F\left(x,u_0\right)\right)dx+\frac{c}{r}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\\
&\quad+\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2+\left\|\Delta u\left(s,\tau-t,\theta_{-\tau}\omega,u_0\right)\right\|^2ds+\frac{c}{r}\int_{-\infty}^{0}e^{\sigma s}\left|\nabla z\left(\theta_s\omega\right)\right|^2ds\\
&\quad+c\int_{-\infty}^{\tau}e^{\sigma s}\int_{|x|\ge r}\left|g(x,s)^2\right|dx\,ds+c\beta^2\int_{-\infty}^{0}e^{\sigma s}\int_{|x|\ge r}1+\left|\Delta z\left(\theta_s\omega\right)\right|^2\\
&\quad+\left|\nabla z\left(\theta_s\omega\right)\right|^2+\left|z\left(\theta_s\omega\right)\right|^2+\left|z\left(\theta_s\omega\right)\right|^{k+1}dx\,ds.
\end{aligned}
\tag{65}
$$

Due to $\varphi_0^{(\beta)}\in D(\tau-t,\theta_{-t}\omega)\in\mathscr{D}$ and (43), it is easy to obtain that there exists $\widetilde{T}_1=\widetilde{T}_1(\tau,\varepsilon,\omega,D)>0$, such that for all $t>\widetilde{T}_1$,

$$
e^{-\sigma t}\int_{\mathbb{R}^n}\rho_r(x)X_0+2F\left(x,u_0\right)dx\le\varepsilon. \tag{66}
$$

By Lemma 2, there are $\widetilde{T}_2=\widetilde{T}_2(\tau,\varepsilon,\omega,D)>0$ and $\widetilde{R}_1=\widetilde{R}_1(\varepsilon,\omega,D)>1$, such that for all $t>\widetilde{T}_2$ and $r>\widetilde{R}_1$,

$$
\begin{aligned}
&\frac{c}{r^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\left\|\Delta v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\\
&\quad+\left\|v\left(s,\tau-t,\theta_{-\tau}\omega,v_0\right)\right\|^2\\
&\quad+\left\|\Delta us,\tau-t,\theta_{-\tau}\omega,u_0\right\|^2ds\le\varepsilon.
\end{aligned}
\tag{67}
$$

By Lemma 1, there are $\widetilde{T}_3=\widetilde{T}_3(\varepsilon,\omega)>0$ and $\widetilde{R}_2=\widetilde{R}_2(\varepsilon,\omega)>1$, such that for all $t>\widetilde{T}_3$ and $r>\widetilde{R}_2$,

$$
\begin{aligned}
&c\beta^2\int_{-\infty}^{0}e^{\sigma s}\int_{|x|\ge r}\left(1+\left|\Delta z\left(\theta_s\omega\right)\right|^2\right)+\left|\nabla z\left(\theta_s\omega\right)\right|^2+\left|z\left(\theta_s\omega\right)\right|^2+\left|z\left(\theta_s\omega\right)\right|^{k+1}dxds\\
&\quad+\frac{c}{r^2}\int_{-\infty}^{0}e^{\sigma s}\left\|\Delta z\left(\theta_s\omega\right)\right\|^2ds\le\varepsilon.
\end{aligned}
\tag{68}
$$

By equation (17), there is $\widetilde{R}_3=\widetilde{R}_3(\tau,\varepsilon)>1$, such that for all $r>\widetilde{R}_3$,

$$
c\int_{-\infty}^{\tau}e^{\sigma s}\int_{|x|\ge r}|g(x,s)|^2dx\,ds\le\varepsilon. \tag{69}
$$

Denote $\widetilde{T}=\max\left\{\widetilde{T}_1,\widetilde{T}_2,\widetilde{T}_3\right\}$, $\widetilde{R}=\max\{\widetilde{R}_1,\widetilde{R}_2,\widetilde{R}_3\}$; by (65)–(69), for all $t>\widetilde{T}$ and $r>\widetilde{R}$, we have

$$\int_{\mathbb{R}^n} \rho_r(x)\big(X(\tau,\tau-t,\theta_{-\tau}\omega,X_0) + 2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\big)dx \le 4\varepsilon, \tag{70}$$

which implies

$$\left\|\varphi^{(\beta)}\left(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0^{(\beta)}\right)\right\|^2_{\mathscr{H}(\mathbb{R}^n\setminus\mathbb{H}_r)} \le 4\varepsilon. \tag{71}$$

Let $\widehat{\rho} = 1-\rho$ with ρ given by (48). Fix $r \ge 1$ and set

$$\begin{cases} \widehat{u}(t,\tau,\omega,\widehat{u_0}) = \widehat{\rho_r}(x)u(t,\tau,\omega,u_0), \\ \widehat{v}(t,\tau,\omega,\widehat{v_0}) = \widehat{\rho_r}(x)v(t,\tau,\omega,v_0), \end{cases} \tag{72}$$

and then $\widehat{\varphi}^{(\beta)}(t,\tau,\omega,\widehat{\varphi_0}^{(\beta)}) = (\widehat{u}(t,\tau,\omega,\widehat{u_0}),\widehat{v}(t,\tau,\omega,\widehat{v_0}))^\top$ is the solution of problem equation (23) on the bounded domain $\mathbb{H}_{2r}$, where $\widehat{\varphi_0}^{(\beta)} = \widehat{\rho_r}(x)\varphi_0^{(\beta)} \in \mathscr{H}(\mathbb{H}_{2r})$.

Multiplying (23) by $\widehat{\rho_r}(x)$ and using (72), we find that

$$\begin{cases} \dfrac{d\widehat{u}}{dt} = \widehat{v} - \delta\widehat{u} + \beta\widehat{\rho_r}(x)z(\theta_t\omega), \\ \dfrac{d\widehat{v}}{dt} = -(\alpha-\delta)\widehat{v} - \left(\lambda+\delta^2-\delta\alpha\right)\widehat{u} - (1-\delta)A\widehat{u} - A\widehat{v} \\ +\beta(1-\alpha+\delta)\widehat{\rho_r}(x)z(\theta_t\omega) \\ -\beta\widehat{\rho_r}(x)Az(\theta_t\omega) + \widehat{\rho_r}(x)g(x,t) + M\left(\|\nabla u\|^2\right)\Delta\widehat{u} \\ -M\left(\|\nabla u\|^2\right)\Delta\widehat{\rho_r}(x)u \\ -2M\left(\|\nabla u\|^2\right)\nabla\widehat{\rho_r}(x)\nabla u - \widehat{\rho_r}(x)f(x,u) \\ +4(1-\delta)\Delta\nabla\widehat{\rho_r}(x)\nabla u \\ +6(1-\delta)\Delta\widehat{\rho_r}(x)\Delta u + 4(1-\delta)\nabla\widehat{\rho_r}(x)\Delta\nabla u \\ +(1-\delta)uA\widehat{\rho_r}(x) \\ +4\Delta\nabla\widehat{\rho_r}(x)\nabla v + 6\Delta\widehat{\rho_r}(x)\Delta v + 4\nabla\widehat{\rho_r}(x)\Delta\nabla v + vA\widehat{\rho_r}(x). \end{cases} \tag{73}$$

Consider the eigenvalue problem.

$$A\widehat{u} = \lambda\widehat{u} \text{ in } \mathbb{H}_{2r}, \text{ with } \widehat{u} = \frac{\partial\widehat{u}}{\partial n} = 0 \text{ on } \partial\mathbb{H}_{2r}. \tag{74}$$

Problem (74) has a family of eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ with the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \ldots, \lambda_i \longrightarrow +\infty\,(i \longrightarrow +\infty), \tag{75}$$

such that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{H}_{2r})$. Given n, let $X_n = \operatorname{span}\{e_1,\ldots,e_n\}$ and P_n: $L^2(\mathbb{H}_{2r}) \longrightarrow X_n$ be the projection operator. □

Lemma 5. *Assume that (2)–(7) and (16) hold. Then, for every $\tau\in\mathbb{R}, \omega\in\Omega$, and $D = \{D(\tau,\omega): \tau\in\mathbb{R}, \omega\in\Omega\}\in\mathscr{D}$, there exist $\widehat{T} = \widehat{T}(\tau,\omega,D,\varepsilon) > 0$ and $\widehat{R} = \widehat{R}(\tau,\omega,\varepsilon) \ge 1$ and $N = N(\tau,\omega,\varepsilon) > 0$, such that for all $t \ge \widehat{T}, r \ge \widehat{R}$, and $n \ge N$,*

$$\left\|(I-P_n)\widehat{\varphi}^{(\beta)}\left(\tau,\tau-t,\theta_{-\tau}\omega,\widehat{\varphi}_0^{(\beta)}\right)\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} \le \varepsilon, \tag{76}$$

where $\widehat{\varphi}_0^{(\beta)} = \widehat{\rho_r}(x)\varphi_0^{(\beta)}, \varphi_0^{(\beta)} = (u_0,v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$.

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u}, \widehat{u}_{n,2} = (I-P_n)\widehat{u}, \widehat{v}_{n,1} = P_n\widehat{v}, \widehat{v}_{n,2} = (I-P_n)\widehat{v}$. Applying $I-P_n$ to $(73)_1$, we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \beta(I-P_n)\widehat{\rho_r}(x)z(\theta_t\omega). \tag{77}$$

Then, applying $I-P_n$ to $(73)_2$ and taking the inner product of the resulting equation with $\widehat{v}_{n,2}$ in $L^2(\mathbb{H}_{2r})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 \\ &= -(\alpha-\delta)\|\widehat{v}_{n,2}\|^2 - \left(\lambda+\delta^2-\delta\alpha\right)\left(\widehat{u}_{n,2},\widehat{v}_{n,2}\right) - (1-\delta)\left(A\widehat{u}_{n,2},\widehat{v}_{n,2}\right) - \left(A\widehat{v}_{n,2},\widehat{v}_{n,2}\right) \\ &\quad + \beta(1-\alpha+\delta)\left(\widehat{\rho_r}(x)z(\theta_t\omega),\widehat{v}_{n,2}\right) - \beta\left(\widehat{\rho_r}(x)Az(\theta_t\omega),\widehat{v}_{n,2}\right) + \left(\widehat{\rho_r}(x)g(x,t),\widehat{v}_{n,2}\right) \\ &\quad + \left(M\left(\|\nabla u\|^2\right)\Delta\widehat{u}_{n,2},\widehat{v}_{n,2}\right) - \left(M\left(\|\nabla u\|^2\right)\Delta\widehat{\rho_r}(x)u + 2M\left(\|\nabla u\|^2\right)\nabla\widehat{\rho_r}(x)\nabla u,\widehat{v}_{n,2}\right) \\ &\quad - \left(\widehat{\rho_r}(x)f(x,u),\widehat{v}_{n,2}\right) + \big(4(1-\delta)\Delta\nabla\widehat{\rho_r}(x)\nabla u + 6(1-\delta)\Delta\widehat{\rho_r}(x)\Delta u \\ &\quad + 4(1-\delta)\nabla\widehat{\rho_r}(x)\Delta\nabla u + (1-\delta)uA\widehat{\rho_r}(x),\widehat{v}_{n,2}\big) \\ &\quad + \left(4\Delta\nabla\widehat{\rho_r}(x)\nabla v + 6\Delta\widehat{\rho_r}(x)\Delta v + 4\nabla\widehat{\rho_r}(x)\Delta\nabla v + vA\widehat{\rho_r}(x),\widehat{v}_{n,2}\right). \end{aligned} \tag{78}$$

Next, we estimate some terms of (78).

$$\begin{aligned}\left(\hat{u}_{n,2}, \hat{v}_{n,2}\right) &= \left(\hat{u}_{n,2}, \frac{d\hat{u}_{n,2}}{dt} + \delta\hat{u}_{n,2} - \beta(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right)\\ &\geq \frac{1}{2}\frac{d}{dt}\left\|\hat{u}_{n,2}\right\|^2 + \delta\left\|\hat{u}_{n,2}\right\|^2 - \beta\left\|\hat{u}_{n,2}\right\| \cdot \left\|(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right\|\\ &\geq \frac{1}{2}\frac{d}{dt}\left\|\hat{u}_{n,2}\right\|^2 + \frac{\delta}{2}\left\|\hat{u}_{n,2}\right\|^2 - c\beta^2\left\|(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right\|^2,\end{aligned} \tag{79}$$

$$\begin{aligned}-\left(A\hat{u}_{n,2}, \hat{v}_{n,2}\right) &= -\left(\Delta\hat{u}_{n,2}, \Delta\left(\frac{d\hat{u}_{n,2}}{dt} + \delta\hat{u}_{n,2} - \beta(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right)\right)\\ &\leq -\frac{1}{2}\frac{d}{dt}\left\|\Delta\hat{u}_{n,2}\right\|^2 - \delta\left\|\Delta\hat{u}_{n,2}\right\|^2 + \beta\left\|\Delta\hat{u}_{n,2}\right\| \cdot \left\|(I-P_n)\Delta\left(\hat{\rho_r}(x)z(\theta_t\omega)\right)\right\|\\ &\leq -\frac{1}{2}\frac{d}{dt}\left\|\Delta\hat{u}_{n,2}\right\|^2 - \left(\frac{\delta}{2} + \frac{7}{\alpha-\delta}\right)\left\|\Delta\hat{u}_{n,2}\right\|^2 + c\beta^2\left\|(I-P_n)\Delta\left(\hat{\rho_r}(x)z(\theta_t\omega)\right)\right\|^2.\end{aligned} \tag{80}$$

Denote $\theta = n(k-1)/4(k+1)$; by (4), we get

$$\begin{aligned}&\left(\hat{\rho_r}(x)f(x,u), \hat{v}_{n,2}\right) \leq c_1\int_{\mathbb{R}^n}\hat{\rho_r}(x)|u|^k\left|\hat{v}_{n,2}\right|dx + \int_{\mathbb{R}^n}\hat{\rho_r}(x)\left|\eta_1(x)\right|\left|\hat{v}_{n,2}\right|dx\\ &\leq c_1\|u\|_{k+1}^k\left\|\hat{v}_{n,2}\right\|_{k+1} + \left\|\eta_1\right\|\left\|\hat{v}_{n,2}\right\|\\ &\leq c_1\|u\|_{k+1}^k\left\|\Delta\hat{v}_{n,2}\right\|^{\theta}\left\|\hat{v}_{n,2}\right\|^{1-\theta} + \lambda_{n+1}^{-1/2}\left\|\eta_1\right\|\left\|\Delta\hat{v}_{n,2}\right\|\\ &\leq c_1\lambda_{n+1}^{\theta-1/2}\|u\|_{H^2}^k\left\|\Delta\hat{v}_{n,2}\right\| + \lambda_{n+1}^{-1/2}\left\|\eta_1\right\|\left\|\Delta\hat{v}_{n,2}\right\|\\ &\leq \lambda_{n+1}^{-1/2}\left\|\Delta\hat{v}_{n,2}\right\|\left(c_1\lambda_{n+1}^{\frac{\theta}{2}}\|u\|_{H^2}^k + \left\|\eta_1\right\|\right)\\ &\leq \frac{1}{6}\left\|\Delta\hat{v}_{n,2}\right\|^2 + \frac{3}{2}\lambda_{n+1}^{-1}\left(c_1\lambda_{n+1}^{\theta/2}\|u\|_{H^2}^k + \left\|\eta_1\right\|\right)^2.\end{aligned} \tag{81}$$

For the remainder terms on the right-hand side of (74), we have

$$\begin{aligned}&-\beta\left(\hat{\rho_r}(x)Az(\theta_t\omega), \hat{v}_{n,2}\right) \leq \beta\left\|(I-P_n)\hat{\rho_r}(x)\Delta z(\theta_t\omega)\right\| \cdot \left\|\Delta\hat{v}_{n,2}\right\|\\ &\leq \frac{3\beta^2}{2}\left\|(I-P_n)\hat{\rho_r}(x)\Delta z(\theta_t\omega)\right\|^2 + \frac{1}{6}\left\|\Delta\hat{v}_{n,2}\right\|^2,\end{aligned} \tag{82}$$

$$\begin{aligned}\beta(1-\alpha+\delta)\left(\hat{\rho_r}(x)z(\theta_t\omega), \hat{v}_{n,2}\right) &\leq \beta(1-\alpha+\delta)\left\|(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right\| \cdot \left\|\hat{v}_{n,2}\right\|\\ &\leq c\beta^2\left\|(I-P_n)\hat{\rho_r}(x)z(\theta_t\omega)\right\|^2 + \frac{\alpha-\delta}{28}\left\|\hat{v}_{n,2}\right\|^2,\end{aligned} \tag{83}$$

$$\begin{aligned}\left(\widehat{\rho}_r(x)g(x,t),\widehat{v}_{n,2}\right) &\le \left\|(I-P_n)\widehat{\rho}_r(x)g(x,t)\right\|\cdot\left\|\widehat{v}_{n,2}\right\|\\ &\le \frac{7}{2(\alpha-\delta)}\left\|(I-P_n)\widehat{\rho}_r(x)g(x,t)\right\|^2+\frac{\alpha-\delta}{14}\left\|\widehat{v}_{n,2}\right\|^2,\end{aligned} \tag{84}$$

$$\begin{aligned}&(1-\delta)\left(4\Delta\nabla\widehat{\rho}_r(x)\cdot\nabla u+6\Delta\widehat{\rho}_r(x)\cdot\Delta u+4\nabla\widehat{\rho}_r(x)\cdot\Delta\nabla u+uA\widehat{\rho}_r(x),\widehat{v}_{n,2}\right)\\ &\le \frac{4c_6(1-\delta)}{r}\lambda_{n+1}^{-1/4}\|\Delta u\|\cdot\left\|\widehat{v}_{n,2}\right\|+\frac{6c_5(1-\delta)}{r}\|\Delta u\|\cdot\left\|\widehat{v}_{n,2}\right\|\\ &\quad+\frac{4c_4(1-\delta)}{r}\lambda_{n+1}^{-1/4}\|\Delta u\|\cdot\left\|\Delta\widehat{v}_{n,2}\right\|+\frac{c_7(1-\delta)}{r}\|u\|\cdot\left\|\widehat{v}_{n,2}\right\|\\ &\le c\lambda_{n+1}^{--1/2}\|\Delta u\|^2+\frac{1}{3}\left\|\Delta\widehat{v}_{n,2}\right\|^2+\frac{c}{r}\left(\|u\|^2+\|\Delta u\|^2\right)+\frac{5(\alpha-\delta)}{28}\left\|\widehat{v}_{n,2}\right\|^2,\end{aligned} \tag{85}$$

$$\begin{aligned}&\left(4\Delta\nabla\widehat{\rho}_r(x)\cdot\nabla v+6\Delta\widehat{\rho}_r(x)\cdot\Delta v+4\nabla\widehat{\rho}_r(x)\cdot\Delta\nabla v+vA\widehat{\rho}_r(x),\widehat{v}_{n,2}\right)\\ &\le \frac{4c_6}{r}\lambda_{n+1}^{-1/4}\|\Delta v\|\cdot\left\|\widehat{v}_{n,2}\right\|+\frac{6c_5}{r}\|\Delta v\|\cdot\left\|\widehat{v}_{n,2}\right\|+\frac{4c_4}{r}\lambda_{n+1}^{-1/4}\|\Delta v\|\cdot\left\|\widehat{v}_{n,2}\right\|+\frac{c_7}{r}\|\Delta v\|\cdot\left\|\widehat{v}_{n,2}\right\|\\ &\le c\lambda_{n+1}^{-1/2}\|\Delta v\|^2+\frac{1}{3}\left\|\Delta\widehat{v}_{n,2}\right\|^2+\frac{c}{r}\left(\|v\|^2+\|\Delta v\|^2\right)+\frac{5(\alpha-\delta)}{28}\left\|\widehat{v}_{n,2}\right\|^2,\end{aligned} \tag{86}$$

$$\left(M\left(\|\nabla u\|^2\right)\Delta\widehat{u}_{n,2},\widehat{v}_{n,2}\right)\le\frac{7}{\alpha-\delta}\left\|\Delta\widehat{u}_{n,2}\right\|^2+\frac{\alpha-\delta}{56}\left\|\widehat{v}_{n,2}\right\|^2, \tag{87}$$

$$\left(M\left(\|\nabla u\|^2\right)\Delta\widehat{\rho}_r(x)u+2M\left(\|\nabla u\|^2\right)\nabla\widehat{\rho}_r(x)\nabla u,\widehat{v}_{n,2}\right)\le\frac{c}{r}\|\Delta u\|^2+\frac{\alpha-\delta}{56}\left\|\widehat{v}_{n,2}\right\|^2. \tag{88}$$

By (30)–(39), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\widehat{v}_{n,2}\right\|^2+\left(\delta^2+\lambda-\delta\alpha\right)\left\|\widehat{u}_{n,2}\right\|^2+(1-\delta)\left\|\Delta\widehat{u}_{n,2}\right\|^2\right)\\ &\le-\frac{\alpha-\delta}{2}\left\|\widehat{v}_{n,2}\right\|^2-\frac{\delta}{2}\left(\delta^2+\lambda-\delta\alpha\right)\left\|\widehat{u}_{n,2}\right\|^2-\frac{\delta}{2}(1-\delta)\left\|\Delta\widehat{u}_{n,2}\right\|^2\\ &\quad+c\beta^2\left(\left\|(I-P_n)\widehat{\rho}_r(x)z(\theta_t\omega)\right\|^2+(I-P_n)\left\|\Delta\left(\widehat{\rho}_r(x)z(\theta_t\omega)\right)\right\|^2+\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\Delta z(\theta_t\omega)\right\|^2\right)\\ &\quad+\left\|c(I-P_n)\widehat{\rho}_r(x)g(x,t)\right\|^2+c\lambda_{n+1}^{-1/2}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)+\frac{c}{r}\left(\|u\|^2+\|v\|^2+\|\Delta u\|^2+\|\Delta v\|^2\right)\\ &\quad+\frac{3}{2}\lambda_{n+1}^{-1}\left(c_1\lambda_{n+1}^{\theta/2}\|u\|_{H^2}^k+\left\|\eta_1\right\|\right)^2.\end{aligned} \tag{89}$$

By (13), (25), and (89), we get

$$\begin{aligned}\frac{d}{dt}\left(\left\|\widehat{\varphi}_{n,2}^{(\beta)}\right\|_{\mathscr{H}(\mathbb{H}_{2r})}^2\right)&\le-\sigma\left\|\widehat{\varphi}_{n,2}^{(\beta)}\right\|_{\mathscr{H}(\mathbb{H}_{2r})}^2+c\beta^2\left(\left\|(I-P_n)\widehat{\rho}_r(x)z(\theta_t\omega)\right\|^2\right.\\ &\quad\left.+\left\|(I-P_n)\Delta\left(\widehat{\rho}_r(x)z(\theta_t\omega)\right)\right\|^2+\left\|(I-P_n)\widehat{\rho}_r(x)\Delta z(\theta_t\omega)\right\|^2\right)\\ &\quad+c\left\|(I-P_n)\widehat{\rho}_r(x)g(x,t)\right\|^2+c\lambda_{n+1}^{-1/2}\left(\|\Delta u\|^2+\|\Delta v\|^2\right)+\frac{c}{r}\left(\|u\|^2+\|v\|^2+\|\Delta u\|^2+\|\Delta v\|^2\right)\\ &\quad+\frac{3}{2}\lambda_{n+1}^{-1}\left(c_1\lambda_{n+1}^{\theta/2}\|u\|_{H^2}^k+\left\|\eta_1\right\|\right)^2.\end{aligned} \tag{90}$$

As $\eta_1 \in L^2(\mathbb{R}^n)$, $\lambda_n \longrightarrow \infty$, there exist $\widehat{N}_1 = \widehat{N}_1(\varepsilon) > 0$ and $\widehat{R}_1 = \widehat{R}_1(\varepsilon) > 0$ such that for all $n > \widehat{N}_1$ and $r > \widehat{R}_1$,

$$\begin{aligned}\frac{d}{dt}\left(\left\|\widehat{\varphi}_{n,2}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})}\right) \leq & -\sigma\left\|\widehat{\varphi}_{n,2}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} + c\left\|(I-P_n)\widehat{\rho}_r(x)g(x,t)\right\|^2 \\ & + \left(\frac{c}{r}+\varepsilon\right)\left(\|u\|^2 + \|v\|^2 + \|\Delta u\|^2 + \|\Delta v\|^2\right) \\ & + \varepsilon\left(1 + \|u\|^{2k}_{H^2} + \left|y(\theta_t\omega)\right|^2\right).\end{aligned} \tag{91}$$

Multiplying (91) by $e^{\sigma t}$ and integrating over $(\tau - t, \tau)$ and then substituting ω by $\theta_{-\tau}\omega$, for all $n > \widehat{N}_1$ and $r > \widehat{R}_1$, we have

$$\begin{aligned}\left\|\widehat{\varphi}_{n,2}^{(\beta)}\tau, \tau - t, \theta_{-\tau}\omega, \widehat{\varphi}_{n,2,0}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} \leq & e^{-\sigma t}\left\|\widehat{\varphi}_{n,2,0}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} + c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}(I-P_n)\left\|\widehat{\rho}_r(x)g(x,s)\right\|^2 ds \\ & + \left(\frac{c}{r}+\varepsilon\right)\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\ & + \left\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2 + \left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\ & + \left\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2 ds \\ & + \varepsilon\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(1 + \left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^{2k}_{H^2} + \left|y(\theta_{s-\tau}\omega)\right|^2\right) ds.\end{aligned} \tag{92}$$

By (4), $\varphi_0^{(\beta)} \in D(\tau - t, \theta_{-t}\omega)$, and $D(\tau - t, \theta_{-t}\omega) \in \mathscr{D}$, there exist $\widehat{T}_1 = \widehat{T}_1(\tau, \varepsilon, D, \omega) > 0$ and $\widehat{R}_1 = \widehat{R}_1(\tau, \varepsilon, \omega) > 1$, such that if $t > \widehat{T}_1$ and $r > \widehat{R}_1$, then

$$e^{-\sigma t}\left\|\widehat{\varphi}_{n,2,0}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} \leq \varepsilon. \tag{93}$$

For the second term on the right-hand side of (92), by (16), we know that there is $\widehat{N} = \widehat{N}(\tau, \varepsilon, \omega) > 0$, such that for all $n > \widehat{N}$,

$$c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\left\|(I-P_n)\widehat{\rho}_r(x)g(x,s)\right\|^2 ds \leq \varepsilon. \tag{94}$$

For the third and fourth terms on the right-hand side of (93), by Lemma 2, there exist $\widehat{T}_2 = \widehat{T}_2(\tau, \varepsilon, D, \omega) > 0$ and $\widehat{R}_2(\tau, \varepsilon, \omega) > 1$, such that for all $t > \widehat{T}_2$ and $r > \widehat{R}_2$, there holds

$$\begin{aligned}\left(\frac{c}{r}+\varepsilon\right)\int_{\tau-t}^{\tau} & e^{\sigma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\ & + \left\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2 \\ & + \left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\ & + \left\|\Delta v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2 ds \leq \varepsilon.\end{aligned} \tag{95}$$

For the last term on the right-hand side of (93), by Lemma 2, there is $\widehat{T}_3 = \widehat{T}_3(\tau, \varepsilon, D, \omega) > 0$, such that for all $t > \widehat{T}_3$, it follows that

$$\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(1 + \left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^{2k}_{H^2} + \left|y(\theta_{s-\tau}\omega)\right|^2\right) ds < \infty. \tag{96}$$

Denote $\widehat{T} = \max\{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3\}$ and $\widehat{R} = \max\{\widehat{R}_1, \widehat{R}_2\}$. Then, by (92)–(96), for all $t > \widehat{T}$, $r > \widehat{R}$, and $n > \widehat{N}$, we get

$$\left\|\widehat{\varphi}_{n,2}^{(\beta)}\tau, \tau - t, \theta_{-\tau}\omega, \widehat{\varphi}_{n,2,0}^{(\beta)}\right\|^2_{\mathscr{H}(\mathbb{H}_{2r})} \leq c\varepsilon, \tag{97}$$

which completes the proof. □

5. Random Attractors

In this section, we prove the existence of $\mathscr{D}$-pullback attractors for stochastic problem (23).

Lemma 6. *Assume that (2)–(7) and (16) hold. Then, the solution of problem (23) is asymptotic compactness in $\mathscr{H}(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $B = \{B(\tau, \omega)$: $\tau \in \mathbb{R}, \omega \in \Omega\} \in \mathscr{D}$, the sequence $\left\{\varphi^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\right\}$*

has a convergent subsequence in $\mathscr{H}(\mathbb{R}^n)$ *provided* $t_m \longrightarrow \infty$ *and* $\varphi_{0,m}^{(\beta)} \in B(\tau - t_m, \theta_{-t_m}\omega)$.

Proof. We first let $t_m \longrightarrow \infty$, $B \in \mathscr{D}$, and $\varphi_{0,m}^{(\beta)} \in B(\tau - t_m, \theta_{-t_m}\omega)$. By Lemma 2, $\{\varphi^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\}$ is bounded in $\mathscr{H}(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, there exists $M_1 = M_1(\tau, \omega, B) > 0$ such for all $m > M_1$,

$$\left\|\varphi^{(\beta)}\left(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)}\right)\right\|_{\mathscr{H}(\mathbb{R}^n)}^2 \leq \varrho^2(\tau, \omega). \tag{98}$$

In addition, it follows from Lemma 4 that there exist $r_1 = r_1(\tau, \varepsilon, \omega) > 0$ and $\widehat{M}_2 = \widehat{M}_2(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geq \widehat{M}_2$,

$$\left\|\varphi^{(\beta)}\left(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)}\right)\right\|_{\mathscr{H}\left(\mathbb{R}^n \setminus \mathbb{H}_{r_1}\right)}^2 \leq \varepsilon. \tag{99}$$

Next, by using Lemma 5, there are $N = N(\tau, \varepsilon, \omega) > 0$, $r_2 = r_2(\tau, \varepsilon, \omega) \geq r_1$, and $\widehat{M}_3 = \widehat{M}_3(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geq \widehat{M}_3$,

$$\left\|(I - P_N)\widehat{\varphi}^{(\beta)}\left(\tau, \tau - t_m, \theta_{-\tau}\omega, \widehat{\varphi}_{0,m}^{(\beta)}\right)\right\|_{\mathscr{H}\left(\mathbb{H}_{2r_2}\right)}^2 \leq \varepsilon. \tag{100}$$

Using (73) and (98), we find that $\{P_N\widehat{\varphi}^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau}\omega, \widehat{\varphi}_{0,m}^{(\beta)})\}$ is bounded in the finite-dimensional space $P_N\mathscr{H}(\mathbb{H}_{2r_2})$, which together with (100) implies that $\{\widehat{\varphi}^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau}\omega, \widehat{\varphi}_{0,m}^{(\beta)})\}$ is precompact in $H^2(\mathbb{H}_{2r_2}) \times L^2(\mathbb{H}_{2r_2})$.

Note that $\widehat{\rho}_r(x) = 1$ for $|x| \leq r_2$. Recalling (73), we find that $\{\varphi^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\}$ is precompact in $\mathscr{H}(\mathbb{H}_{r_2})$, which along with (99) shows the precompactness of this sequence in $\mathscr{H}(\mathbb{R}^n)$. This completes the proof.

The main result of this section is given below. □

Theorem 2. *Assume that (2)–(7) and (16) hold. Then, the continuous cocycle* Φ_β *associated with problem (23) has a unique* $\mathscr{D}$*-pullback attractor* $\mathscr{A}_\beta = \{\mathscr{A}_\beta(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\}$ *in* $\mathscr{H}(\mathbb{R}^n)$.

Proof. This is an immediate consequence of Proposition 1 and Lemmas 2 and 6. □

6. Upper Semicontinuity of Pullback Attractors

In this section, we will use Theorem 1 to consider an upper semicontinuity of random attractors $\mathscr{A}_\beta(\omega)$ when $\beta \longrightarrow 0$. To indicate the dependence of solutions on β, we, respectively, write the solutions of problem (23) as $u^{(\beta)}$ and $v^{(\beta)}$, that is, $(u^{(\beta)}, v^{(\beta)})$ satisfies

$$\begin{cases} \dfrac{du^{(\beta)}}{dt} = v^{(\beta)} - \delta u^{(\beta)} + \beta z(\theta_t\omega), \\ \dfrac{dv^{(\beta)}}{dt} = (\delta - \alpha - A)v^{(\beta)} + [\delta(-\delta + \alpha + A) - \lambda - A]u^{(\beta)} \\ +\beta[1 - (\alpha + A - \delta)]z(\theta_t\omega) \\ +M\left(\left\|\nabla u^{(\beta)}\right\|^2\right)\Delta u^{(\beta)} - f\left(x, u^{(\beta)}\right) + g(x, t), \\ u^{(\beta)}(\tau, \tau, x) = u_0^{(\beta)}(x), v^{(\beta)}(\tau, \tau, x) = v_0^{(\beta)}(x). \end{cases} \tag{101}$$

When $\beta = 0$, the random problem (23) reduces to a deterministic one:

$$\begin{cases} \dfrac{du^{(0)}}{dt} = v^{(0)} - \delta u^{(0)}, \\ \dfrac{dv^{(0)}}{dt} = (\delta - \alpha - A)v^{(0)} + [\delta(-\delta + \alpha + A) - \lambda - A]u^{(0)} \\ +M\left(\left\|\nabla u^{(0)}\right\|^2\right)\Delta u^{(0)} \\ -f\left(x, u^{(0)}\right) + g(x, t), \\ u^{(0)}(\tau, \tau, x) = u_0^{(0)}(x), v^{(0)}(\tau, \tau, x) = v_0^{(0)}(x). \end{cases} \tag{102}$$

By Theorem 2, the deterministic and autonomous system Φ_0 generated by (102) is readily verified to admit a global attractor $\mathscr{A}_0$ in $\mathscr{H}(\mathbb{R}^n)$.

Theorem 3. *Assume that (2)–(7) and (16) hold. Then, the random dynamical system* Φ_β *generated by (23) has a unique* $\mathscr{D}$*-pullback attractor* $\{\mathscr{A}_\beta(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ *in* $\mathscr{H}(\mathbb{R}^n)$*. Moreover, the family* $\{\mathscr{A}_\beta\}_{\beta > 0}$ *of random attractors is upper semicontinuous.*

Proof. By Lemma 3 and Theorem 2, Φ_β has a closed measurable random absorbing set $E_\beta(\tau, \omega)$ and a unique random attractor $\mathscr{A}_\beta$.

(i) In Lemma 2, we have proved that the system Φ_β possesses a closed random absorbing set $E_\beta = \{E_\beta(\tau, \omega)_{\tau \in \mathbb{R}, \omega \in \Omega}\}$ in $\mathscr{D}$, which is given by

$$E_\beta(\tau, \omega) = \left\{(u, v) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n): \|u\|_{H^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2 \leq R_1(\tau, \omega)\right\}. \tag{103}$$

Then, we get

$$\limsup_{\beta\longrightarrow 0}\left\|\mathrm{E}_{\beta}(\tau,\omega)\right\|_{\mathscr{H}(\mathbb{R}^n)}\le \mathrm{c}, \tag{104}$$

which deduces condition (i) in Theorem 1 immediately.

(ii) Given $\beta\in(0,1]$, let $E_1(\tau,\omega)=\{(u,v)\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\colon \|u\|^2_{H^2(\mathbb{R}^n)}+\|v\|^2_{L^2(\mathbb{R}^n)}\le R_*(\tau,\omega)\}$, where

$$\begin{aligned}R_*(\tau,\omega)&=c+c\int_{-\infty}^{0}e^{\sigma s}\Big(1+\left\|\Delta z(\theta_s\omega)\right\|^2\\&\quad+\left\|\nabla z(\theta_s\omega)\right\|^2+\left\|z(\theta_s\omega)\right\|^2+\left\|z(\theta_s\omega)\right\|_{H^2}^{k+1}\Big)ds.\end{aligned} \tag{105}$$

Then,

$$\bigcup_{0<\beta\le 1}\mathscr{A}_\beta(\tau,\omega)\subseteq\bigcup_{0<\beta\le 1}E_\beta(\omega)\subseteq E_1(\tau,\omega). \tag{106}$$

First, by (106), Lemma 4, and the invariance of $\mathscr{A}_\beta(\tau,\omega)$, we obtain that for every $\beta>0$ and $P-a.e.\tau\in\mathbb{R}$, $\omega\in\Omega$, there exists $r_0=r_0(\omega,\varepsilon)\ge 1$ such that

$$\int_{|x|\ge r_0}\left(\|u(x)\|^2+\|\Delta u(x)\|^2+\|v(x)\|^2\right)dx\le\varepsilon,\ \text{for all }(u,v)\in\bigcup_{0<\beta\le 1}\mathscr{A}_\beta(\tau,\omega). \tag{107}$$

Second, by (106), the proof of Lemma 5, Lemma 6, and the invariance of $\mathscr{A}_\beta(\tau,\omega)$, we know that there exists $r_1=r_1(\omega,\varepsilon)\ge r_0$ such that for all $r\ge r_1$, the set $\cup_{0<\beta\le 1}\mathscr{A}_\beta(\tau,\omega)$ is precompact in $\mathscr{H}(\mathbb{H}_r)$, which together with (107) implies that $\cup_{0<\beta\le 1}\mathscr{A}_\beta(\tau,\omega)$ is precompact in $\mathscr{H}(\mathbb{R}^n)$.

(iii) Let $\varphi^{(0)}=(u^{(0)},v^{(0)})$ be a mild solution of (102) with initial data $\varphi^{(0)}=(u^{(0)},v^{(0)})$, and $U=u^{(\beta)}-u^{(0)}$, $V=v^{(\beta)}-v^{(0)}$. By (101) and (102), we get

$$\begin{cases}\dfrac{dU}{dt}=V-\delta U+\beta z(\theta_t\omega),\\[2mm] \dfrac{dV}{dt}=(\delta-\alpha-A)V+[\delta(-\delta+\alpha+A-\lambda)-A]U\\[2mm] +\left(M\left(\left\|\nabla u^{(\beta)}\right\|^2\right)-M\left(\left\|\nabla u^{(0)}\right\|^2\right)\Delta U\right)-f\left(x,u^{(\beta)}\right)\\[2mm] +f\left(x,u^{(0)}\right)+\beta[1-(\alpha+A-\delta)]z(\theta_t\omega),\\[2mm] U(\tau,\tau,x)=U_0(x),V(\tau,\tau,x)=V_0(x).\end{cases} \tag{108}$$

Taking the inner product of $(108)_2$ with V in $L^2(\mathbb{R}^n)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|V\|^2+\left(\delta^2+\lambda-\delta\alpha\right)\|U\|^2+(1-\delta)\|\Delta U\|^2\right)\\&\le-\frac{3}{4}(\alpha-\delta)\|V\|^2-\frac{3}{4}\delta\left(\delta^2+\lambda-\delta\alpha\right)\|U\|^2\\&\quad-\frac{3}{4}\delta(1-\delta)\|\Delta U\|^2\\&\quad+\left(M\left(\left\|\nabla u^{(\beta)}\right\|^2\right)-M\left(\left\|\nabla u^{(0)}\right\|^2\right)\Delta U,V\right)\\&\quad+\left(-f\left(x,u^{(\beta)}\right)+f\left(x,u^{(0)}\right),V\right)\\&\quad+c\beta^2\left(1+\left\|\Delta z(\theta_t\omega)\right\|^2+\left\|z(\theta_t\omega)\right\|^2\right).\end{aligned} \tag{109}$$

Taking advantage of $M(a^2)-M(b^2)\le M'(\sup\{a^2,b^2\})|a+b|\cdot|a-b|$ and Lemma 2, we get

$$\begin{aligned}&\left(M\left(\left\|\nabla u^{(\beta)}\right\|^2\right)-M\left(\left\|u^{(0)}\right\|^2\right)\Delta U,V\right)\\&\le\left(M'\left(\sup\left\{\left\|\nabla u^{(\beta)}\right\|^2,\left\|\nabla u^{(0)}\right\|^2\right\}\right)\left\|\nabla u^{(\beta)}\right\|\right.\\&\quad\left.+\left\|\nabla u^{(0)}\right\|\mid\cdot\mid\left\|\nabla u^{(\beta)}\right\|-\left\|\nabla u^{(0)}\right\|\mid\Delta U,V\right)\\&\le c\|\Delta U\|^2+c\|V\|^2.\end{aligned} \tag{110}$$

By (7), we get

$$\left|\left(f\left(x,u^{(0)}\right)-f\left(x,u^{(\beta)}\right),V\right)\right|\leq c\|U\|^2+c\|V\|^2, \tag{111}$$

which along with (107)-(108) implies

$$\begin{aligned}&\frac{d}{dt}\left(\|V\|^2+\left(\delta^2+\lambda-\delta\alpha\right)\|U\|^2+(1-\delta)\|\Delta U\|^2\right)\\&\leq c\left(\|V\|^2+\left(\delta^2+\lambda-\delta\alpha\right)\|U\|^2+(\|1-\delta\|)\|\Delta U\|^2\right)\\&\quad+c\beta^2\left(1+\left\|\Delta z\left(\theta_t\omega\right)\right\|^2+\left\|z\left(\theta_t\omega\right)\right\|^2\right).\end{aligned} \tag{112}$$

Applying Gronwall inequality to (112) over (τ,t), we have

$$\begin{aligned}&\left\|u^{(\beta)}\left(t,\tau,\omega,u_0^{(\beta)}\right)-u^{(0)}\left(t,\tau,u_0^{(0)}\right)\right\|_{H^2(\mathbb{R}^n)}^2+\left\|v^{(\beta)}\left(t,\tau,\omega,v_0^{(\beta)}\right)-v^{(0)}\left(t,\tau,v_0^{(0)}\right)\right\|_{L^2(\mathbb{R}^n)}^2\\&\leq ce^{c(t-\tau)}\left(\left\|u_0^{(\beta)}-u_0^{(0)}\right\|_{H^2(\mathbb{R}^n)}^2+\left\|v_0^{(\beta)}-v_0^{(0)}\right\|_{L^2(\mathbb{R}^n)}^2\right)+c\beta^2\int_\tau^t e^{c(t-s)}1+\left\|\Delta z\left(\theta_s\omega\right)\right\|^2\\&\quad+\left\|z\left(\theta_s\omega\right)\right\|^2ds,\end{aligned} \tag{113}$$

which along with (i), (ii), and Theorem 1 completes the proof [29]. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

XY and ZZ completed the main study together. XY wrote the manuscript, and ZZ checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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