

Research Article

Rationalizing Denominators Using Gröbner Bases

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Received 29 May 2021; Revised 22 November 2021; Accepted 29 November 2021; Published 6 January 2022

Academic Editor: Zhiwei Gao

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The problem of rationalizing denominators for two types of fractions is discussed in the paper. By using the theory and algorithms of Gröbner bases, we first introduce a method to rationalize the denominators of fractions with square root and cube root, and then, for the denominators with higher radical of the general form, the problem of rationalizing denominators is converted into the related problem of finding the minimal polynomials. Some interesting results and an executable algorithm for rationalizing the denominator of these type fractions are presented. Furthermore, an example is also established to illustrate the effectiveness of the algorithm.

1. Introduction

A typical topic in algebra is rationalizing denominators [1]. Rationalizing the denominator avoids the problem known as “subtrative cancellation,” deals with the problem of recognizing equivalent expressions, and is commonly used in many of computer algebra systems. Also, rationalizing expression has applications in calculus [2]. More importantly, by rationalizing the denominator, mathematical operations and practical problems can be approximated more accurately [3]. Thus, there are circumstances in which it is advantageous to rationalize an expression.

The general method of denominator rationalization is to seek the rationalized factor of the denominator first and then multiply the numerator and denominator by this factor at the same time [4]. Using this method, the denominator rationalization of quadratic radical fraction has been solved. However, it is difficult to find the rationalized factor for an expression which contains a root higher than a square root [5]. Therefore, a lot of literatures tend to deal with some special case of this problem. In 1929, Paradiso [6] showed that theoretically, in all cases, and practically, in many cases, a rationalizing factor may be found by the method of undetermined coefficients. In 1970, Fateman presented an algorithm named RADCAN that is implemented in MACSYMA for the simplification of expressions containing

radicals [7]. Zhou rationalized the denominator for a class of fractions $1/g(u)$ by theory of minimal polynomials in 1986, where $g(u)$ is a polynomial whose coefficient are rational and u is a complex root of a nonzero rational polynomial [8]. In 1989, Ma showed the possibility of denominator’s rationalization of the irrational expressions: $A/(a_1 \sqrt[n_1]{b_1} + a_2 \sqrt[n_2]{b_2} + \dots + a_m \sqrt[n_m]{b_m} + c)$, where A is the combination of radical and rational addition, subtraction, and multiplication and $a_1 \sqrt[n_1]{b_1} + a_2 \sqrt[n_2]{b_2} + \dots + a_m \sqrt[n_m]{b_m} + c$ is an algebraic element in the field of rational numbers [9]. In 2000, Liu rationalized the denominator for a class of algebraic fractions as follows $1/f(\sqrt[n]{c}) = 1/(a_0 + a_1 \sqrt[n]{c} + \dots + a_n \sqrt[n]{c^n})$ (where $f(x) = a_0 + a_1 x + \dots + a_n x^n$, $c, a_1, a_2, \dots, a_n \in \mathbb{Q}$, $c > 0$) using the knowledge of determinants [10]. In 2002, using polynomials, Tang also discussed the algebraic fractions whose denominator is the same with the above fraction [11]. And in 2015, Berele and Catoiu produced an exact formula for rationalizing any fraction whose denominator is a linear combination with rational coefficients of square roots of rational numbers [1].

Existing results mentioned above mainly deal with several kinds of denominators rationalization of radical fractions by using related theory of polynomials. And many other kinds of denominators of irrational fraction are unsolved, such as $1/(x + 5\sqrt[3]{8} + 6\sqrt[4]{9})$. This kind of fraction is a very common form in mathematical calculation, so it is of

high application value to study a general method to rationalize the denominator. In this paper, we will investigate this by the theory of Gröbner bases as it is an important tool to solve many problems in polynomial ideal [12, 13]. And it has been implemented in many computational softwares including Singular, Maple, CoCoA, Mathematica, Macaulay 2, etc. Many fundamental problems in commutative algebra, computational algebraic number theory, algebraic geometry, graph theory, image processing, cryptography and encoding, and science and engineering can be solved by it algorithmically [13–23]. And the minimal polynomial can be obtained by the reduced Gröbner basis algorithm easily. Based on this and results mentioned above, we consider using the theory of Gröbner bases to explore the relationship between minimal polynomials and denominator rationalization and discuss mainly denominator rationalization of the fraction with the form as

$$\frac{1}{x + c_1 \cdot \sqrt[n]{a} + c_2 \cdot \sqrt[n]{b}}, \quad (1)$$

where $m, n \in \mathbb{N}^+$, $m, n \geq 2$, $a, b, c_1, c_2 \in \mathbb{Q}$, $a, b > 0$. We hope to establish a simplified method for rationalizing the denominator of this type fraction.

The rest of the paper is organized as follows. We present some preliminary knowledge, basic concepts, and a special method to rationalize the denominators with square root and cube root in Section 2. Main results on rationalizing the denominator of a type fraction are shown in Section 3. A simplified algorithm and an example established to illustrate the algorithm are given in Section 4. Section 5 concludes the paper.

2. Denominators with Square Root and Cube Root

In what follows, $K[x_1, x_2, \dots, x_n]$ will denote the polynomial ring in n variables x_1, x_2, \dots, x_n with coefficients in a field K , L/K will be an algebraic extension of the field K , and $[L: K]$ will denote the extension times of the extension field L/K . \mathbb{Q} will denote the rational number field, \mathbb{N} will be the set of integers, and $\text{lcm}(m, n)$ will denote the least common multiple of m and n . For a nonzero polynomial $f(x) \in K[x_1, x_2, \dots, x_n]$, we use $\text{lt}(f)$, $\text{lc}(f)$, and $\text{lp}(f)$ to denote the leading term, the leading coefficient, and the leading monomial of $f(x)$, respectively. For a set $F \subset K[x_1, x_2, \dots, x_n]$, we denote $\text{lt}(F) = \{\text{lt}(f) | f \in F\}$. Then, we introduce several related definitions and algorithm.

Definition 1. Let I be an ideal in $K[x_1, x_2, \dots, x_n]$. A finite subset G of I is called a Gröbner basis of I if

$$\langle \text{lt}(G) \rangle = \langle \text{lt}(I) \rangle. \quad (2)$$

Definition 2 (see [13]). Let $f, g \in K[x_1, x_2, \dots, x_n]$, $f, g \neq 0$, and $L = \text{lcm}(\text{lp}(f), \text{lp}(g))$, and the S-polynomial of f and g is defined as (Algorithm 1)

$$S(f, g) = \frac{L}{\text{lt}(f)}f - \frac{L}{\text{lt}(g)}g. \quad (3)$$

This section focuses on the problem of rationalizing the denominator with square root and cube root. For the sake of convenience in researching the problem, we put the coefficient in the radical sign, and then, the fraction can be reduced to one of the following forms:

$$(1) \frac{1}{x + \sqrt[n]{a} + \sqrt[n]{b}} \quad (a > 0, \text{ and } a, b \in \mathbb{Q}),$$

$$(2) \frac{1}{x - \sqrt[n]{a} + \sqrt[n]{b}} \quad (a > 0, \text{ and } a, b \in \mathbb{Q}).$$

We first consider the denominators rationalization of form (1).

Lemma 1 (see [13]). Let $G = \{g_1, g_2, \dots, g_s\}$ be a subset of an ideal $I \in K[x_1, x_2, \dots, x_n]$, and then, G is a Gröbner bases for I if and only if $S(g_i, g_j) \xrightarrow{G} 0$, for all $i \neq j, 1 \leq i, j \leq s$.

Theorem 1. Let I be an ideal in $K[x_1, x_2, \dots, x_n]$, and then, $fg - 1 \in I \iff 1 \in \langle I, f \rangle$.

Proof. If $fg - 1 \in I$, then there exists $h \in I$, such that $fg - 1 = h$, that is, $1 = fg - h \in \langle I, f \rangle$.

In turn, if $1 \in \langle I, f \rangle$, then there exists $g \in K[x_1, x_2, \dots, x_n]$ and $h \in I$, such that $1 = fg + h$, that is, $fg - 1 = -h \in I$.

Next, we introduce how to rationalize the denominator of fraction as form (1). First, consider the following ideal:

$$I = \langle y_1^2 - a, y_2^3 - b \rangle \subseteq \mathbb{Q}(x)[y_1, y_2], \quad (5)$$

where $\mathbb{Q}(x)[y_1, y_2]$ denotes the polynomial ring in variables y_1 and y_2 on $\mathbb{Q}(x)$ and $\mathbb{Q}(x)$ is an extension of the field \mathbb{Q} with x as a variable. Let

$$f_1(y_1, y_2) = y_1^2 - a, f_2(y_1, y_2) = y_2^3 - b. \quad (6)$$

In the following, multivariate polynomials $h(x, y)$ are denoted as h for convenience. And the term order of the multivariate polynomial ring is the lexicographical ordering, which is defined by $y_1 > y_2$.

Note that

$$S(f_1, f_2) = by_1^2 - ay_2^3 \xrightarrow{f_2} by_1^2 + ab \xrightarrow{f_1} 0. \quad (7)$$

By Lemma 1, we have that $\{f_1, f_2\}$ is a Gröbner basis of I . Let

Input: $F = f_1, f_2, \dots, f_s$.
 Output: $G = g_1, g_2, \dots, g_s$, G is a Gröbner basis of $\langle f_1, f_2, \dots, f_s \rangle$.
 Initialization: $G := F, \Omega := \{\{f_i, f_j\} | f_i \neq f_j \in G\}$.
 When $\Omega \neq \emptyset$
 select $f, g \in \Omega$
 $\Omega := \Omega / \{f, g\}$
 $S(f, g) \xrightarrow{G} h$, h is reductive with respect to G .
 If $h \neq 0$, then
 $\Omega := \Omega \cup \{u, h\} | \forall u \in G$
 $G := G \cup \{h\}$

ALGORITHM 1: Buchberger algorithm [14].

$$f_3(y_1, y_2) = x + y_1 + y_2, \quad (8) \quad \text{where}$$

$y_1 = \sqrt{a}, y_2 = \sqrt[3]{b}$, and then, $f_3(\sqrt{a}, \sqrt[3]{b}) = x + \sqrt{a} + \sqrt[3]{b}$, which is the denominator we want to deal with. Replace f by f_3 in Theorem 1, and then, the condition $f_3 \cdot g - 1 \in I$ is equivalent to $f_3(\sqrt{a}, \sqrt[3]{b}) \cdot g(\sqrt{a}, \sqrt[3]{b}) = 1$, where $g(\sqrt{a}, \sqrt[3]{b}) \in Q[\sqrt{a}, \sqrt[3]{b}]$ is the polynomial we need.

So, the key issue is whether $1 \in \langle I, f_3 \rangle$, that is, whether $1 \in \langle f_1, f_2, f_3 \rangle$. If true, we can rationalize the denominator of $1/(x + \sqrt{a} + \sqrt[3]{b})$ by using this method.

Now, we use Buchberger algorithm to obtain the Gröbner basis of $J = \langle f_1, f_2, f_3 \rangle$ and then determine whether $1 \in \langle f_1, f_2, f_3 \rangle$.

Initialize

$f_1 = y_1^2 - a, f_2 = y_2^3 - b$, and $f_3 = x + y_1 + y_2$, and using the algorithm, we get the Gröbner basis of J is

$$\{f_1, f_2, f_3, f_4, f_5, f_6\}, \quad (9)$$

$$f_4 = y_2^2 - xy_1 + xy_2 - a, \quad (10)$$

$$f_5 = (a + 3x^2)y_2 + 2x^3 - 2ax - b,$$

$$f_6 = \frac{-(2x^3 - 2ax - b)^3}{(3x^2 + a)^3} - b. \quad (11)$$

Observe f_6 , and it is easy to find that f_6 is independent of y_1 and y_2 , and thus, $f_6 \in Q(x)$. Note that $Q(x)$ is a field, and thus, $f_6^{-1} = 1/f_6 \in Q(x)$. Consequently, $1 = f_6 \cdot 1/f_6 \in J$. Therefore, we can calculate the rationalizing result of $1/(x + \sqrt{a} + \sqrt[3]{b})$.

Then, express f_6 as the combination of f_1, f_2, f_3, f_4, f_5 :

$$f_6 = f_2 - \left(\frac{y_2^2}{3x^2 + a} + \frac{-2x^3 + 2ax + b}{(3x^2 + a)^2} + \frac{(2x^3 - 2ax - b)^2}{(3x^2 + a)^3} \right) \cdot f_5, \quad (12)$$

that is,

$$\frac{f_2 - \left(\frac{y_2^2}{3x^2 + a} + \frac{-2x^3 + 2ax + b}{(3x^2 + a)^2} + \frac{(2x^3 - 2ax - b)^2}{(3x^2 + a)^3} \right) \cdot f_5}{f_6} = 1. \quad (13)$$

Hence, divide both sides of equation (13) by $x + \sqrt{a} + \sqrt[3]{b}$, and substitute the value of f_2, f_5 and $y_1 = \sqrt{a}, y_2 = \sqrt[3]{b}$ into the equation, and then, we obtain

$$\begin{aligned}
\frac{1}{x + \sqrt{a} + \sqrt[3]{b}} &= \frac{(2x^3 - 2ax - b)^2 \cdot 2x^2 + bx \cdot (3x^2 + a)^2 + b(2x^3 + 2ax + b) \cdot (3x^2 + a)}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \\
&+ \frac{b(3x^2 + a)^2 - 2x \cdot (2x^3 - 2ax - b)^2}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \cdot \sqrt{a} \\
&+ \frac{-b(3x^2 + a)^2 + 2x^2 \cdot (-2x^3 + 2ax + b) \cdot (3x^2 + a) + x(2x^3 - 2ax - b)^2}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \cdot \sqrt[3]{b} \\
&+ \frac{2x^2 \cdot (3x^2 + a)^2 + x \cdot (3x^2 + a) \cdot (-2x^3 + 2ax + b) - (2x^3 - 2ax - b)^2}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \cdot \sqrt[3]{b^2} \\
&+ \frac{2x \cdot (2x^3 - 2ax - b) \cdot (3x^2 + a) + (2x^3 - 2ax - b)^2}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \cdot \sqrt{a} \cdot \sqrt[3]{b} \\
&+ \frac{-2x \cdot (3x^2 + a)^2 + (-2x^3 + 2ax + b) \cdot (3x^2 + a)}{(2x^3 - 2ax - b)^3 + b(3x^2 + a)^3} \cdot \sqrt{a} \cdot \sqrt[3]{b^2}.
\end{aligned} \tag{14}$$

Using Matlab to simplify the equation above, we obtain

$$\begin{aligned}
\frac{1}{x + \sqrt{a} + \sqrt[3]{b}} &= \frac{1}{x^6 - 3ax^4 + 2bx^3 + 3a^2x^2 + 6abx + b^2 - a^3} \left(x^5 - 2ax^3 + bx^2 \right. \\
&+ a^2x + ab + (-x^4 + 2ax^2 + 2bx - a^2) \cdot \sqrt{a} + (-x^4 - bx + a^2) \\
&\cdot \sqrt[3]{b} + (x^3 + 3ax + b) \cdot \sqrt[3]{b^2} + (2x^3 - 2ax - b) \cdot \sqrt{a} \cdot \sqrt[3]{b} \\
&\left. + (-3x^2 - a) \cdot \sqrt{a} \cdot \sqrt[3]{b^2} \right).
\end{aligned} \tag{15}$$

For form (2), we just need to change $f_3(y_1, y_2) = x + y_1 + y_2$ to $f_3(y_1, y_2) = x - y_1 + y_2$, and the other steps are the same. \square

3. Denominators with Higher Radical

In this section, we focus on rationalizing the denominator of general form as

$$\frac{1}{x + c_1 \cdot \sqrt[m]{a} + c_2 \cdot \sqrt[n]{b}} \tag{16}$$

where $m, n \in \mathbb{N}^+$, $m, n \geq 2$, $a, b, c_1, c_2 \in \mathbb{Q}$, $a, b > 0$.

By the method in Section 2, we first construct the ideal $J = \langle f_1, f_2, f_3 \rangle \in \mathbb{Q}(x)[y_1, y_2]$, where $f_1 = y_1^m - a$, $f_2 = y_2^n - b$, and $f_3 = x + c_1 y_1 + c_2 y_2$. Then, compute the Gröbner basis G and the reduced Gröbner basis G_0 of J . If $G_0 = \{1\}$, then calculate $f_s = G \cap \mathbb{Q}(x)$. Finally, express f_s as a combination of $G/\{f_s\}$.

However, there are two important uncertainties, one is whether $G_0 = \{1\}$ and the other is whether f_s is a combination of $G/\{f_s\}$. So, this method can only solve the case when

c_1, c_2, m, n are specific values. Because of this limitation, we hope to find another method for the general form.

First, we introduce several related definitions and lemmas.

Definition 3 (see [24]). Let L/K be an algebraic extension of the field K , $[L: K] = n$. Suppose $\sigma_i: L \rightarrow C$ ($1 \leq i \leq n$) is n K -insertion of L , where C is the complex field. For $\alpha \in L$, define

$$T_{L/K}(\alpha) = \sum_{i=1}^n \sigma_i(\alpha), \tag{17}$$

as the trace of α to the expansion L/K .

Definition 4 (see [24]). Let L/K be an algebraic extension of the field K . Suppose $\sigma_1, \dots, \sigma_n$ is the K -insertion of L and $\alpha_1, \dots, \alpha_n \in L$, and we define

$$d_{L/K}(\alpha_1, \dots, \alpha_n) = \det^2(\sigma_i(\alpha_j)), 1 \leq i \leq n, 1 \leq j \leq n, \tag{18}$$

as the discriminant of $\{\alpha_1, \dots, \alpha_n\}$ for the expansion L/K .

Definition 5. Let K be a field, $u \in K$, and $f(x) \in K[x]$ be a polynomial. We call $f(x)$ is the minimal polynomial of u in K if $f(x)$ satisfies the following:

- (1) $f(x)$ is monic and $f(u) = 0$
- (2) If $g(x) \in K[x]$ and $g(u) = 0$, then $f(x)|g(x)$

Lemma 2 (see [24]). Let L/K be the expansion of number field and $[L:K] = n$. Suppose $\alpha \in L$, $f(x) = x^n - c_1x^{n-1} + \dots + (-1)^n c_n$ is the minimal polynomial of α in K , where $m = [K(\alpha):K]$, and then, $T_{L/K}(\alpha) = nc_1/m$.

Lemma 3 (see [23]). Let $d_{L/K}(\alpha_1, \dots, \alpha_n)$ be the discriminant of element $\{\alpha_1, \dots, \alpha_n\}$ for the expansion L/K , and then, $d_{L/K}(\alpha_1, \dots, \alpha_n) \neq 0$ if and only if $\alpha_1, \dots, \alpha_n$ is K -linearly independent.

Lemma 4 (see [25], Eisenstein criterion). Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be an integral coefficient univariate polynomial. If there is a prime number p such that

- (1) $p \nmid a_n$
- (2) $p \mid a_{n-1}, a_{n-2}, \dots, a_0$
- (3) $p \nmid a_0^2$

Then, $f(x)$ is irreducible over the rational number field \mathbb{Q} .

Using the lemmas above, we can prove the following results.

Theorem 2. If p is prime number, then $f(x) = x^m - p$ is the minimal polynomial for $\sqrt[m]{p}$ in rational number field \mathbb{Q} , where $m \in \mathbb{N}^+$ and $m \geq 2$.

Proof. It is straightforward that $\sqrt[m]{p}$ is a root of $f(x) = x^m - p = 0$. In the following, we prove that $f(x)$ is irreducible.

Let $g(x) = f(x+p) = x^m + mp_x^{m-1} + \dots + C_m^i p^i x^{m-i} + \dots + p^m - p$, where $C_m^i = m!/(i!(m-i)!)$. It is obvious that $p \nmid 1$, $p \mid C_m^i p^i$ ($1 \leq i \leq m-1$), and $p \mid (p^m - p)$. In fact, we can prove that $p^2 \nmid (p^m - p)$. Suppose $p^2 \mid (p^m - p)$, and then, $p \mid (p^{m-1} - 1)$. Combining $p \mid p^{m-1}$ ($m \in \mathbb{N}^+, m \geq 2$), we have that $p \mid (p^{m-1} - (p^{m-1} - 1))$, that is, $p \mid 1$, and this is a contradiction. So, $p^2 \nmid (p^m - p)$. By the Eisenstein criterion, $g(x)$ is irreducible, so $f(x)$ is irreducible, and it is the minimal polynomial for $\sqrt[m]{p}$ in \mathbb{Q} . \square

Theorem 3. If p_1, p_2, \dots, p_s are different prime numbers, then

$$\sqrt[m_1]{p_1^{t_1}} \cdot \sqrt[m_2]{p_2^{t_2}} \cdot \dots \cdot \sqrt[m_s]{p_s^{t_s}}, \quad (t_i < m_i, t_i, m_i \in \mathbb{N}^+, i = 1, 2, \dots, s), \quad (19)$$

is an irrational number.

Proof. It is straightforward that $\sqrt[m_1]{p_1^{t_1}} \cdot \sqrt[m_2]{p_2^{t_2}} \cdot \dots \cdot \sqrt[m_s]{p_s^{t_s}}$ is a real number. In the following, we prove that it is an irrational number. Suppose it is a rational number, and set $f(x) = x^{m_1 m_2 \dots m_s} - p_1^{t_1 m_2 \dots m_s} p_2^{t_2 m_1 \dots m_s} \dots p_s^{t_s m_1 m_2 \dots t_s}$. Then, $\sqrt[m_1]{p_1^{t_1}} \cdot \sqrt[m_2]{p_2^{t_2}} \cdot \dots \cdot \sqrt[m_s]{p_s^{t_s}}$ is a positive rational root of $f(x) = 0$. Note that $f(x)$ is an integral coefficient polynomial and $\text{lc}(f) = 1$, and then, the positive rational root of $f(x)$ must be a factor of its constant term. So, it has the form as $p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$, where $0 \leq n_i \leq m_1 \dots m_{i-1} t_i m_{i+1} \dots m_s$, $n_i \in \mathbb{N}$, $i = 1, 2, \dots, s$. Then, we have that

$$\sqrt[m_1]{p_1^{t_1}} \cdot \sqrt[m_2]{p_2^{t_2}} \cdot \dots \cdot \sqrt[m_s]{p_s^{t_s}} = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}. \quad (20)$$

From the equation above, we see that some of n_i must be 0. Without loss of generality, we denote the elements whose power exponents are nonzero as p_1, p_2, \dots, p_q , and equation (20) turns into the following:

$$\sqrt[m_1]{p_1^{t_1}} \cdot \sqrt[m_2]{p_2^{t_2}} \cdot \dots \cdot \sqrt[m_s]{p_s^{t_s}} = p_1^{n_1} p_2^{n_2} \dots p_q^{n_q}, \quad (21)$$

that is,

$$p_1^{n_1 - t_1/m_1} p_2^{n_2 - t_2/m_2} \dots p_q^{n_q - t_q/m_q} = p_{q+1}^{t_{q+1}/m_{q+1}} p_{q+2}^{t_{q+2}/m_{q+2}} \dots p_s^{t_s/m_s}. \quad (22)$$

Multiply both sides of equation (22) by the $m_1 m_2 \dots m_s$ power, and then,

$$p_1^{l_1} p_2^{l_2} \dots p_s^{l_s} = p_{q+1}^{l_{q+1}} p_{q+2}^{l_{q+2}} \dots p_s^{l_s}, \quad (23)$$

where $l_i \in \mathbb{N}^+, i = 1, 2, \dots, s$. Hence, $p_1 \mid p_{q+1}^{l_{q+1}} p_{q+2}^{l_{q+2}} \dots p_s^{l_s}$, and this contradicts p_1, p_2, \dots, p_s are different prime numbers. So, the conclusion is correct. \square

Theorem 4. If p_1, \dots, p_s are different prime numbers and m_1, \dots, m_s are positive integers and no less than 2, then the finite extension times of $\mathbb{Q}(\sqrt[m_1]{p_1}, \dots, \sqrt[m_s]{p_s})/\mathbb{Q}$ is no more than $m_1 m_2 \dots m_s$.

Proof. Let $F_i = \mathbb{Q}(\sqrt[m_1]{p_1}, \sqrt[m_2]{p_2}, \dots, \sqrt[m_i]{p_i})$, $1 \leq i \leq s$ and $i \in \mathbb{N}^+$, and then, $F_{i+1} = F_i(\sqrt[m_{i+1}]{p_{i+1}})$, that is, F_{i+1} can be viewed as the single extension of F_i . So, $[F_{i+1}:F_i] = \partial(g_i)$, where $\partial(g_i)$ denotes the degree of the polynomial $g_i(x)$, and $g_i(x)$ is the minimal polynomial of $\sqrt[m_i]{p_i}$ in the field F_i . Obviously, $\sqrt[m_i]{p_i}$ is a root for $f_i(x) = x^{m_i} - p_i$, $1 \leq i \leq s$. By the definition of minimal polynomial, we see that $\partial(g_i) \leq \partial(f_i)$, $2 \leq i \leq s, i \in \mathbb{N}^+$. From Theorem 3, we see that $f_1(x)$ is an irreducible polynomial in \mathbb{Q} . By the property of domain extension, we have $[F_1:\mathbb{Q}] = m_1$. Hence,

$$\begin{aligned}
[Q(\sqrt[m_1]{p_1}, \sqrt[m_2]{p_2}, \dots, \sqrt[m_s]{p_s}): Q] &= [F_s: F_{s-1}][F_{s-1}: F_{s-2}] \cdots [F_2: F_1][F_1: Q] \\
&= \partial(g_s) \cdot \partial(g_{s-1}) \cdots \partial(g_2) \cdot m_1 \\
&\leq m_1 m_2 \cdots m_s.
\end{aligned} \tag{24}$$

Theorem 5. If p_1, p_2, \dots, p_s are different prime numbers and m_1, m_2, \dots, m_s are positive integers and no less than 2, then $x^{m_s} - p_s$ is the minimal polynomial of $\sqrt[m_s]{p_s}$ in the field $Q(\sqrt[m_1]{p_1}, \sqrt[m_2]{p_2}, \dots, \sqrt[m_{s-1}]{p_{s-1}})$.

Proof. Let $b_i = \sqrt[m_i]{p_i}, 1 \leq i \leq s, M = Q(\sqrt[m_1]{p_1}, \sqrt[m_2]{p_2}, \dots, \sqrt[m_s]{p_s})$, and in the following, we prove that all the $m_1 \cdot m_2 \cdots m_s$ elements in set

$$\begin{aligned}
A = \{ & b_1^{n_1} b_2^{n_2} \cdots b_s^{n_s} \mid 0 \leq n_i \\
& \leq m_i - 1, n_i \in N, i = 1, 2, \dots, s\} \subseteq M,
\end{aligned} \tag{25}$$

are linearly independent. First, sort the elements in A . It is easy to observe that each element in A corresponds to such an array (n_1, n_2, \dots, n_s) . So, we can turn the problem into sorting the exponents (n_1, n_2, \dots, n_s) , where the term order is lexicographic order. We denote the element that

$$l = \text{lcm} \left(\frac{m_1}{\gcd(m_1, n_{i_1} + n_{j_1})}, \frac{m_2}{\gcd(m_2, n_{i_2} + n_{j_2})}, \dots, \frac{m_s}{\gcd(m_s, n_{i_s} + n_{j_s})} \right). \tag{26}$$

Obviously, $l > 1$. Let $q = (c_i c_j)^l$ and $B = \{m \in N^+ \mid (c_i c_j)^m \in Q\}$, and by Theorem 3, we have that $m_k \mid (n_{i_k} + n_{j_k})m$, for any $m \in B$, where $k = 1, 2, \dots, s$. Hence, p is the smallest element in B .

In the following, we prove that $f(x) = x^l - q$ is the minimal polynomial of $c_i c_j$ in the rational number field Q .

It is straightforward that $f(c_i c_j) = (c_i c_j)^l - q = 0$. We factorize $f(x)$ in the complex field C as

$$f(x) = (x - \sqrt[l]{q})(x - \varepsilon \sqrt[l]{q}) \cdots (x - \varepsilon^{l-1} \sqrt[l]{q}), \tag{27}$$

where $\varepsilon = e^{2\pi i/l} = \cos(2\pi/l) + i \sin(2\pi/l)$.

Suppose $f(x)$ is reducible in Q , and then, some constant terms of the linear factor in the decomposition above are rational numbers, that is, there is a $t \in N^+$ and $k \in N$ such that $(-1)^t \varepsilon^k q^{t/l} \in Q \subseteq R$, i.e., $e^{2k\pi i/l} = \cos(2k\pi/l) + i \sin(2k\pi/l) \in R$. So, $\sin(2k\pi/l) = 0$, that is, $k = \ln/2, n \in Z$. Hence, $\varepsilon^k = \cos(2k\pi/l) = \cos(n\pi) = \pm 1, n \in Z$, and there is an $r \in Q$ such that $q^{t/l} = r$. Note that $q = (c_i c_j)^l$, and then, $(c_i c_j)^t = r \in Q$ ($t < l$), and this contradicts the selection of l . Thus, $f(x) = x^l - q$ is irreducible in Q , and then, $f(x)$ is the minimal polynomial of $c_i c_j$ in rational number field Q . Combined with Lemma 2, we see that $T_{M/Q}(c_i c_j) = 0$.

Based on the discussion above, we obtain that $d_{M/Q}(c_1, c_2, \dots, c_{m_1 m_2 \cdots m_s}) = \det(T_{M/Q}(c_i c_j)) \neq 0$. Then, all elements in set A are linearly independent in Q by Lemma 3. Combined with Theorem 4, we see that

corresponds to the largest exponential as c_1 , the element that corresponds to the second largest exponential as c_2, \dots , and the element that corresponds to the smallest exponential as $c_{m_1 m_2 \cdots m_s}$, and then, we have sorted out all the elements in A .

Now, we calculate $\det(T_{M/Q}(c_i c_j))$. Suppose $c_i = b_1^{n_{i_1}} b_2^{n_{i_2}} \cdots b_s^{n_{i_s}}$ and $c_j = b_1^{n_{j_1}} b_2^{n_{j_2}} \cdots b_s^{n_{j_s}}$. If one of the following two situations is true,

- (1) $\forall k = 1, 2, \dots, s, n_{i_k} + n_{j_k} = m_k$
- (2) $\forall k = 1, 2, \dots, s, n_{i_k} = n_{j_k} = 0$

We obtain that $c_i c_j \in Q$ by Theorem 3. Hence, $T_{M/Q}(c_i c_j) = n c_i c_j \neq 0$, where $n = [M: Q]$. Otherwise, there exists $k \in \{1, 2, \dots, s\}$ does not satisfy either of the above two cases, and we can calculate $T_{M/Q}(c_i c_j)$ by Lemma 2. We first want to obtain the minimal polynomial of $c_i c_j$ in rational number field Q . Set

$$[Q(\sqrt[m_1]{p_1}, \sqrt[m_2]{p_2}, \dots, \sqrt[m_s]{p_s}): Q] = m_1 m_2 \cdots m_s. \tag{28}$$

So, $x^{m_s} - p_s$ is the minimal polynomial of $\sqrt[m_s]{p_s}$ in the field $Q(\sqrt[m_1]{p_1}, \dots, \sqrt[m_{s-1}]{p_{s-1}})$.

Based on the results above, we can present the steps for rationalizing denominators of the following form:

$$\frac{1}{x + c_1 \sqrt[m]{a} + c_2 \sqrt[n]{b}}, \tag{29}$$

where m and n are integers and $a, b \in Q^+$ and $c_1, c_2 \in Q$.

Step 1: write a and b as fractions in the lowest term, $a = d_1/e_1$ and $b = d_2/e_2$. Then, decompose d_1, e_1, d_2, e_2 into power product of different prime factors, respectively. We use p_1, p_2, \dots, p_t to denote the common prime factors of d_1 and d_2 , q_1, q_2, \dots, q_k to denote the different prime factors in d_1, e_1 , and v_1, v_2, \dots, v_s to denote the different prime factors in d_2, e_2 .

Step 2: let $l = \text{lcm}(m, n)$, and

$$\begin{aligned}
\alpha_1 &= \sqrt[l]{p_1}, \alpha_2 = \sqrt[l]{p_2}, \dots, \alpha_t = \sqrt[l]{p_t}, \\
\alpha_{t+1} &= \sqrt[l]{q_1}, \alpha_{t+2} = \sqrt[l]{q_2}, \dots, \alpha_{t+w} = \sqrt[l]{q_w},
\end{aligned} \tag{30}$$

$$\alpha_{t+w+1} = \sqrt[l]{r_1}, \alpha_{t+w+2} = \sqrt[l]{r_2}, \dots, \alpha_{t+w+s} = \sqrt[l]{r_s}.$$

We construct the rational extension field:

$$M = Q(\alpha_1, \dots, \alpha_t, \alpha_{t+1}, \dots, \alpha_{t+w}, \dots, \alpha_{t+w+s}). \tag{31}$$

Let $\alpha = c_1 \sqrt[m]{a} + c_2 \sqrt[n]{b}$, and there are $f(x_1, \dots, x_{t+w+s}), g(x_1, \dots, x_{t+w+s}) \in Q[x_1, \dots, x_{t+w+s}]$ such that

$$\alpha = f(\alpha_1, \dots, \alpha_{t+w+s})/g(\alpha_1, \dots, \alpha_{t+w+s}). \quad (32)$$

According to Theorem 2, we have that $h_1 = x_1^l - p_1 \in Q[x_1]$ is the minimal polynomial of α_1 in the rational number field Q . It is also known, by Theorem 5, $h_i = x^l - p_i \in Q(\alpha_1, \dots, \alpha_{i-1})[x_i]$ is the minimal polynomial of α_i in $Q(\alpha_1, \dots, \alpha_{i-1})$ for $2 \leq i \leq t$, $h_i = x^m - q_{i-t} \in Q(\alpha_1, \dots, \alpha_{i-1})[x_i]$ is the minimal polynomial of α_i in $Q(\alpha_1, \dots, \alpha_{i-1})$ for $t+1 \leq i \leq t+s$, and $h_i = x^n - r_{i-t-w} \in Q(\alpha_1, \dots, \alpha_{i-1})[x_i]$ is the minimal polynomial of α_i in $Q(\alpha_1, \dots, \alpha_{i-1})$ for $t+w+1 \leq i \leq t+w+s$.

Next, we construct the following homomorphic maps:

$$\varphi_i: Q[x_1, x_2, \dots, x_i] \longrightarrow Q(\alpha_1, \alpha_2, \dots, \alpha_{i-1})[x_i], 2 \leq i \leq t+w+s. \quad (33)$$

For any $f = \sum \lambda_\beta x_1^{\beta_1} x_2^{\beta_2} \dots x_i^{\beta_i} \in Q[x_1, x_2, \dots, x_{i-1}]$, where $\lambda_\beta \in Q, \beta = (\beta_1^i, \beta_2^i, \dots, \beta_i^i) \in N^i$. Define

$$\varphi_i(f) = \sum \lambda_\beta \alpha_1^{\beta_1} \alpha_2^{\beta_2} \dots \alpha_{i-1}^{\beta_{i-1}} x_i^{\beta_i}. \quad (34)$$

Under this map, the corresponding preimage of $h_i \in Q(\alpha_1, \alpha_2, \dots, \alpha_{i-1})[x_i]$ is itself, where $2 \leq i \leq t+w+s$.

Step 3: we find the minimal polynomial of α in Q according to [26].

The term order $<$ in $Q[x_1, \dots, x_t, x_{t+1}, \dots, x_{t+w+s}, y]$ is the lexicographical ordering defined by $y < x_{t+w+s} < \dots < x_{t+1} < x_t < \dots < x_1$. Under this term order, we calculate the reduced Gröbner basis G of the ideal

$$I = \langle h_1, h_2, \dots, h_{t+w+s}, g \cdot y - f \rangle, \quad (35)$$

where

$$h_i = \begin{cases} x^l - p_i, & 1 \leq i \leq t, \\ x^m - q_{i-t}, & t+1 \leq i \leq t+w, \\ x^n - r_{i-t-w}, & t+w+1 \leq i \leq t+w+s. \end{cases} \quad (36)$$

Then, we compute $h(y) = G \cap Q[y]$ and take

$$m(y) = \frac{h(y)}{\lambda}, \lambda = \text{lc}(h(y)). \quad (37)$$

Hence, $m(y)$ is the minimal polynomial of α in Q . And then we can rationalize the denominator by applying the minimal polynomial.

Set $n(y) = x + y$ (x is a parameter) and divide $m(y)$ by $n(y)$ such that

$$m(y) = q(y)n(y) + r(y), \partial(r(y)) < \partial(n(y)). \quad (38)$$

Then, $\partial(r(y)) = 0$ and $r(y) \neq 0$, otherwise, it contradicts that $n(y)$ contains x . Correspondingly, equation (38) can be rewritten as

$$m(y) = q(y)n(y) + r(x), r(x) \in Q[x]. \quad (39)$$

Substitute $y = \alpha$ into the equation above, and then, $0 = (x + \alpha)q(\alpha) + r(x)$, that is,

$$\frac{1}{x + \alpha} = \frac{q(\alpha)}{r(x)}, \quad (40)$$

which is the desired result of rationalizing denominators. \square

4. Algorithm and Example

According to the theorems and discussion in Section 3, we obtain an algorithm for rationalizing the denominators of fractions with the form as $1/(x + c_1 \cdot \sqrt[m]{a} + c_2 \cdot \sqrt[n]{b})$. We describe this algorithm in more detail in Figure 1.

In the following, we construct an example to show the effectiveness of the algorithm.

Example 1. Rationalizing the denominator of

$$\frac{1}{x + \sqrt{6} + \sqrt[3]{4}}. \quad (41)$$

Step 1: we know $a=6$ and $b=4$, and the command “format rat” can be omitted here. Carry out the prime factorization of 6 and 4 by using the function “factorization,” and we obtain

$$6 = 2 \times 3, 4 = 2 \times 2. \quad (42)$$

It is easy to see that $p_1 = 2$ is a common prime factor of 6 and 4. In addition, 6 also contains the factor $q_1 = 3$.

Step 2: note that $m = 2$ and $n = 3$, and then, using the function “min_GBS,” we obtain

$$l = \text{lcm}(m, n) = 6. \quad (43)$$

Let

$$\alpha_1 = \sqrt[6]{2}, \alpha_2 = \sqrt{3}. \quad (44)$$

We construct the extension field $Q(\alpha_1, \alpha_2)$ of the rational number field Q . Set $\alpha = \sqrt{6} + \sqrt[3]{4}$, and then, $\alpha = f(\alpha_1, \alpha_2) = \alpha_1^3 \alpha_2 + \alpha_1^4$.

Step 3: hence, $h_1 = x_1^6 - 2 \in Q[x_1]$ is a minimal polynomial of α_1 on Q and $h_2 = x_2^2 - 3 \in Q(\alpha_1)[x_2]$ is a minimal polynomial of α_2 on $Q(\alpha_1)$.

Set

$$I = \langle x_1^6 - 2, x_2^2 - 3, y - x_1^3 x_2 - x_1^4 \rangle, \quad (45)$$

take the lexicographical ordering $y < x_2 < x_1$ as the term order in $Q[x_1, x_2, y]$, and calculate the reduced Gröbner basis G of ideal I in the software singular using the built-in function “groebner.” Then, the calculation results are as follows:

$$G = \{f_1, f_2, f_3\}, \quad (46)$$

where

An algorithm for rationalizing denominator	
Input:	$m, n \in \mathbb{N}^+, a, b \in \mathbb{Q}^+, c_1, c_2 \in \mathbb{Q}$, the term orders over \mathbb{Q}
Output:	$\alpha, q(x), r(x)$ such that $\frac{1}{x+\alpha} = -\frac{q(\alpha)}{r(x)}$
Step 1.	Write a, b as fractions in lowest term $a = d_1/e_1, b = d_2/e_2$ using the command "format rat". Then decompose $d_i, e_i (i = 1, 2)$ into power product of different prime factors respectively using the function "factorization", denote the common prime factors of d_1, d_2 by p_i , use q_j and r_k to denote the different prime factors in d_1, e_1 and d_2, e_2 respectively, $i, j, k = 1, 2, \dots$
Step 2.	Compute $l = \text{lcm}(m, n)$, set $\alpha_i = \sqrt[l]{p_i}, i = 1, 2, \dots, t; \alpha_{t+j} = \sqrt[l]{q_j}, j = 1, \dots, w$, and $\alpha_{t+w+k} = \sqrt[l]{r_k}, k = 1, \dots, s$. Construct the extension field $M = \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_{t+w+s})$. Find $f(x_1, \dots, x_{t+w+s}), g(x_1, \dots, x_{t+w+s}) \in \mathbb{Q}(x_1, \dots, x_{t+w+s})$ such that $\alpha = \frac{f(\alpha_1, \dots, \alpha_{t+w+s})}{g(\alpha_1, \dots, \alpha_{t+w+s})}$.
Step 3.	Let $h_i = x^l - p_i$ for $i = 1, 2, \dots, t, h_{t+j} = x^m - q_j$ for $j = 1, \dots, w$, and $h_{t+w+k} = x^n - r_k$ for $k = 1, \dots, s$, set $I = \langle h_1, \dots, h_{t+w+s}, g \cdot y - f \rangle$, calculate the reduce Gröbner basis G of I using the function "groebner". Then compute $h(y) = G \cap \mathbb{Q}[y]$. Set $m(y) = h(y)/\text{lc}(h(y)), n(y) = x + y$ divide $m(y)$ by $n(y)$ using the function "polynomialReduce" such that $m(y) = q(y)n(y) + r(y), \partial(r(y)) < \partial(g(y))$. Write $m(y) = q(y)n(y) + r(x)$, and substitute $y = \alpha$ in the equation.
Return:	$\alpha, q(x), r(x)$.

FIGURE 1: Denominator rationalized algorithm.

$$f_1 = y^6 - 18y^4 + 108y^2 - 144y - 200,$$

$$f_2 = x_2^2 - 3,$$

$$f_3 = 7128x_1 + 18x_2^3y^4 - 144x_2^3y^3 - 294x_2^3y^2 + 4756x_2^3y + 64x_2^3 - 3x_2y^5 - 30x_2y^2 - 14904x_2y + 4536x_2.$$

(47)

$$m(y) = q(y) \cdot (x + y) + r(x), \quad (48)$$

Therefore, $G \cap \mathbb{Q}[y] = \{y^6 - 18y^4 - 8y^3 + 108y^2 - 144y - 200\}$. Hence, the minimal polynomial of α in the rational number field \mathbb{Q} is $m(y) = y^6 - 18y^4 - 8y^3 + 108y^2 - 144y - 200$.

Next, divide $m(y)$ by $x + y$ in the software Matlab using the built-in function "polynomialReduce." Then,

where

$$\begin{aligned} q(y) &= y^5 - xy^4 + y^3(x^2 - 18) + y^2(-x^3 + 18x - 8) + y(x^4 - 18x^2 + 8x + 108) + (-x^5 + 18x^3 - 8x^2 - 108x - 144), \\ r(x) &= x^6 - 18x^4 + 8x^3 + 108x^2 + 144x - 200. \end{aligned} \quad (49)$$

Substitute $\alpha = \sqrt{6} + \sqrt[3]{4}$ into

then

$$\frac{1}{x + \alpha} = -\frac{q(\alpha)}{r(x)}, \quad (50)$$

$$\begin{aligned} \frac{1}{x + \sqrt{6} + \sqrt[3]{4}} &= \frac{x^5 - 12x^3 + 4x^2 + 36x + 24}{r(x)} + \frac{-x^4 + 12x^2 + 8x - 36}{r(x)} \sqrt{6} + \frac{-x^4 - 4x + 36}{r(x)} \sqrt[3]{4} \\ &+ \frac{2x^3 + 36x + 8}{r(x)} \sqrt[3]{2} + \frac{2x^3 - 12x - 4}{r(x)} \sqrt{6} \sqrt[3]{4} + \frac{-6x^2 - 12}{r(x)} \sqrt[3]{2} \sqrt{6}. \end{aligned} \quad (51)$$

Remark 1. Note that $m = 2$ and $n = 3$, and we can use the method in Section 2 to do this example. Replace

$a = 6$ and $b = 4$ and do all the steps as in Section 2, and we obtain

$$\frac{1}{x + \sqrt{6} + \sqrt[3]{4}} = \frac{1}{x^6 - 18x^4 + 8x^3 + 108x^2 + 144x - 200} \left(x^5 - 12x^3 + 4x^2 + 36x + 24 + (-x^4 + 12x^2 + 8x - 36) \right. \\ \cdot \sqrt{6} + (-x^4 - 4x + 36) \cdot \sqrt[3]{4} + (2x^3 + 36x + 8) \cdot \sqrt[3]{2} \\ \left. + (2x^3 - 12x - 4) \cdot \sqrt{6} \cdot \sqrt[3]{4} + (-6x^2 - 12) \cdot \sqrt{6} \cdot \sqrt[3]{2} \right). \quad (52)$$

Through the two examples above, it is easy to find that the results obtained by both methods are the same, which proves that the algorithm we designed is correct and the application scope is wider.

Remark 2. Using the method in Section 2 or the method of undetermined coefficients, the problem of denominator rationalization of fractions with the form as $1/(x + c_1 \cdot \sqrt[n]{a} + c_2 \cdot \sqrt[n]{b})$ may not be solved or can be solved but will take a long time. However, from Example 1, it is straightforward that this problem can be solved efficiently and simply according to the three steps of Denominator rationalized algorithm by using the software Matlab and Singular.

5. Conclusion

In this paper, using theory of Gröbner bases, we have achieved in rationalizing denominators for two types of fractions, especially rationalizing the denominator of the fraction with the form as

$$\frac{1}{x + c_1 \cdot \sqrt[n]{a} + c_2 \cdot \sqrt[n]{b}}, \quad (53)$$

where $m, n \in \mathbb{N}^+$, $m, n \geq 2$, $a, b, c_1, c_2 \in \mathbb{Q}$, $a, b > 0$. We have presented some interesting results and an executable algorithm on rationalizing the denominators for this type fractions. Furthermore, we have established an example to illustrate the effectiveness of the algorithm.

The method that we proposed on rationalizing denominators can be realized in computer system such as Maple and Singular, which makes the related computation more quick. Furthermore, it can improve the performance of the related algorithms. For example, we find that, for some new swarm intelligence algorithms proposed in recent years, such as monarch butterfly optimization (MBO) [27], earthworm optimization algorithm (EWA) [28], elephant herding optimization (EHO) [29], etc. the fraction of the form $1/(x + c_1 \cdot \sqrt[n]{a} + c_2 \cdot \sqrt[n]{b})$ occurs during the implementation of these algorithms. The denominator rationalized algorithm we proposed can rationalize the denominator of this type fraction, which can improve the accuracy of these algorithms on calculation.

Future work will investigate in rationalizing denominators of more general forms of radical fraction. The type of fractions discussed in the paper is unnested radical

expressions, and the problem of the nested radical fractions is not involved here, which is also what we will do.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11871207 and 11971161).

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