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Research Article

Stochastic Hybrid Hepatitis B Epidemic Model with Markovian Switching

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In this paper, we consider a stochastic epidemic model under regime-switching. We show that our model is well-posed by proving the existence of solution, positivity, and uniqueness. Furthermore, we investigate sufficient conditions for the existence of ergodic stationary distribution to the model, which shows that the hepatitis B epidemic will persist. In addition, we establish sufficient conditions for the extinction of the hepatitis B epidemic. We perform numerical simulations to confirm and illustrate the theoretical results.

1. Introduction

The research of new approaches and techniques in epidemiology has taken the attention of many scholars [1–4] in order to give a theoretical base that can be used to support the government to put programs to control the propagation of epidemics in the population. Likewise, the research in areas of mathematic epidemiology has perceived an immense development and proved their efficiency in solving some real medical problems. Generally, there are two approaches to formulating mathematical epidemic models: (i) deterministic based on ordinary differential equations, fractional, or partial [5–7]; (ii) stochastic founded on stochastic differential equations and their generalization [8–11]. The deterministic epidemic model has some limitations in predicting the future dynamics perfectly, and the stochastic one can show it [12]. In the literature review, many authors are involved in the formulation of the mathematical model of transmission of hepatitis B epidemic (see [13–17]) and provided some significant analysis results in order to understand the mechanism of the translation of this type of epidemic and proposing appropriate solutions to controlling their propagation.

For a long time, the problem of infectious diseases presents a tall threat to the people in the world. In addition, a significant number of people in the world die due to infectious diseases. Among these dangerous diseases, we find hepatitis B (which causes liver inflammation from the hepatitis B virus infection [18]). Additionally, the propagation of hepatitis B disease is compatible with the stochastic propriety since the contact between the individuals is not predictable. Generally, the studied populations can be divided into four classes (see [19]), namely, Susceptible $S(t)$, acute Infected $I_1(t)$, chronically Infected $I_2(t)$, and Recovered $R(t)$. In this paper, we are interested in the following mathematic hepatitis B epidemic model given by the stochastic system:

$$\begin{cases} dS(t) = (A - \beta S(t)I_1(t) - (\nu + \mu_0)S(t))dt + \sigma_1 S(t)dW_1(t), \\ dI_1(t) = (\beta S(t)I_1(t) - (\gamma_1 + \mu_0 + \alpha)I_1(t))dt + \sigma_2 I_1(t)dW_2(t), \\ dI_2(t) = (\alpha I_1(t) - (\gamma_2 + \mu_0 + \mu_1)I_2(t))\mathrm{d}t + \sigma_3 I_2(t)dW_3(t), \\ dR(t) = (\nu S(t) + \gamma_2 I_2(t) - \mu_0 R(t) + \gamma_1 I_1(t))dt + \sigma_4 R(t)dW_4(t), \end{cases} \tag{1}$$

where all parameters of the system are supposed to be nonnegative. A is the birth recruitment rate to the susceptible population, ν is the vaccination rate of hepatitis B virus, μ_0 is the natural death rate, and μ_1 is the death rate due to hepatitis B virus. The parameter β is the disease transmission rate between the infected and susceptible individuals. Therefore, the acutely infected individuals are recovered with the rate γ_1 and chronically infected individuals are recovered with γ_2. However, α represents the rate of the acute class going to the chronic class. $W_i(t)$ $(i = 1, 2, 3, 4)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{ttt\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and σ_i $(i = 1, 2, 3, 4)$ is the intensity of noise. The above model describes the fact that the epidemic model is often subject to random noise [9, 20], which conduit to represent the effect of environmental fluctuation on the parameters of the system, which are not absolute constants but fluctuate randomly around some average values due to continuous fluctuation in the environment. Furthermore, many approaches to formulate the stochastic epidemic model existed [21–24]. Therefore, in model (1), we have assumed that the white noise is directly proportional to the variable. Many works examined the effect of white noise in the epidemic model. Liu et al. in [25] proposed a stochastic model for the heroin epidemic and showed the existence of a unique global positive solution to the system. Also, they showed a sufficient condition for the extinction of the drug users. In addition, they prove the existence of ergodic stationary distribution that implied the persistence of the drug users in the population. In [26], Liu et al. used the parameters perturbation method to express the randomness in an SIRI epidemic model and employed the stochastic Lyapunov technics to show conditions for the extinction and persistence of disease in the system. Wang et al. [27] supposed that the stochastic perturbation is around the positive equilibrium states. They showed that the endemic equilibrium is stochastically asymptotically stable in the large.

Frequently, in the natural population, the system switches between two or more environmental regimes that change in terms of factors [28]. Thus, the hepatitis B epidemic models need perturbing by colored noise to allow the system to switch between two or more environmental regimes [23, 29]. The switching between the environmental regimes is memoryless, and the waiting time for the next switch follows the exponential distribution [30, 31]. Hence, the regime-switching can be modeled by a continuous-time Markov chain $(r(t))_{t\geq 0}$ with values in a finite-state space $\mathbb{L} = \{1, 2, \ldots, m\}$. Many researchers have added the Markov chain process in their models to express the change of regimes. For example, Liu and Wang (see Refs. [32]), have studied a stochastic single-species model with Markovian switching. Other interesting papers on the epidemic of Markovian switches models are listed in references [23, 29, 33–37]. Then, in this paper, we consider the hepatitis B epidemic model with both white and color noises disturbances presented by the following hybrid stochastic differential equation system:

$$\begin{cases} dS(t) = \left(A_{r(t)} - \beta_{r(t)} S I_1 - \left(\nu_{r(t)} + \mu_{0,r(t)}\right) S\right) dt + \sigma_{1,r(t)} S \, dW_1(t), \\ dI_1(t) = \left(\beta_{r(t)} S I_1 - \left(\gamma_{1,r(t)} + \mu_{0,r(t)} + \alpha_{r(t)}\right)\right) I_1 \Big) dt + \sigma_{2,r(t)} I_1 dW_2(t), \\ dI_2(t) = \left(\alpha(t) I_1 - \left(\gamma_{2,r(t)} + \mu_{0,r(t)} + \mu_{1,r(t)}\right) I_2\right) dt + \sigma_{3,r(t)} I_2 dW_3(t), \\ dR(t) = \left(\nu_{r(t)} S + \gamma_{2,r(t)} I_2 - \mu_{0,r(t)} R + \gamma_{1,r(t)} I_1\right) dt + \sigma_{4,r(t)} R \, dW_4(t), \end{cases} \tag{2}$$

where $\{r(t)\}_{t>0}$ is a right-continuous Markov chain defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and supposed independent of Brownian motion $W(.)$. The Markov chain takes values in a finite-state space $\mathbb{L} = \{1, 2, \ldots, m\}$ with the generator $\Phi = (\phi_{uv})_{1\leq u,v\leq m}$ given, for $\Delta t > 0$, by

$$\mathbb{P}(r(t + \Delta t) = v | r(t) = u) = \begin{cases} \phi_{uv}\Delta t + o(\Delta t), & \text{if } u \neq v, \\ 1 + \phi_{uu}\Delta t + o(\Delta t) & \text{if } u = v, \end{cases} \tag{3}$$

with ϕ_{uv} is the transition rate from u to v and $\phi_{uv} \geq 0$ if $u \neq v$, while

$$\phi_{uu} = -\sum_{u\neq v} \phi_{uv}. \tag{4}$$

Assume that the Markov chain $r(.)$ is irreducible, independent of the Brownian motion $W(.)$. Therefore, there exists a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_m)$ of $r(t)$ such that $\pi\Phi = 0$ and $\sum_{i=1}^{m} \pi_i = 1, \pi_i > 0$, $i \in \mathbb{L}$. For any sequence $U = (U(1), \ldots, U(m))^T$, let $\widehat{U} = \min_{i\in\mathbb{L}}\{U(i)\}$ and $\check{U} = \max_{i\in\mathbb{L}}\{U(i)\}$. Now, we are interested in the following stochastic equation:

$$d\xi(t) = H(t, \xi(t), r(t))dt + G(t, \xi(t), r(t))dW(t), \tag{5}$$

(i) $W(t)$ is a d-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$

(ii) $C^2(\mathbb{R}^d \times \mathbb{L}; \mathbb{R}_+)$ represents the family of all nonnegative functions $\mathcal{Y}$ defined on $\mathbb{R}^d \times \mathbb{L}$ such that they are continuously twice differentiable in ξ

We define the operator $\mathcal{L}\mathcal{Y}$ (see [38]) from $\mathbb{R}^d \times \mathbb{L}$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}\mathcal{Y}(\xi, k) = {} & \mathcal{Y}_t(t, \xi, k) + \mathcal{Y}_\xi(t, \xi, k) H(t, \xi, k) \\ & + \frac{1}{2} G(t, \xi, k) \mathcal{Y}_{\xi,\xi}(t, \xi, k) G(t, \xi, k), \\ & + \sum_{i\in\mathbb{L}} \phi_{ki} \mathcal{Y}(t, \xi, i), \end{aligned} \tag{6}$$

where

(i) $\mathcal{Y}_\xi = (\partial\mathcal{Y}/\partial\xi_1, \ldots, \partial\mathcal{Y}/\partial\xi_d)$ and $\mathcal{Y}_{\xi,\xi} = (\partial^2\mathcal{Y}/\partial\xi_i\partial\xi_j)_{d\times d}$

(ii) G^T is the transposed matrix of G

Employing Itô's formula [38], we obtain for $\mathcal{Y}$ defined on $\mathbb{R}^d \times \mathbb{L}$:

$$d\mathcal{Y}(\xi(t),k) = \mathcal{L}\mathcal{Y}(\xi(t),k)dt + \mathcal{Y}_\xi(\xi(t),k)G(\xi(t),k)dW(t). \tag{7}$$

We put the following lemma existing in [28] which we used to prove the existence of ergodic stationary distribution of the solution to equation (17).

Lemma 1 (see [28]). *If the following conditions are satisfied,*

(i) $\phi_{ij} > 0$ *for any* $i \neq j$,

(ii) For each $k \in \mathbb{L}$, $D(x,k) = (d_{ij}(x,k))$ *is symmetric and satisfies*

$$\lambda|\zeta|^2 \leq \langle D(x,k)\zeta,\zeta\rangle \leq \lambda^{-1}|\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^n, \tag{8}$$

with some constant $\lambda \in ((0,1]$ *for all* $x \in \mathbb{R}^n$,

(iii) There exists a nonempty open set $\mathcal{D}$ *with compact closure, satisfying that, for each* $k \in \mathbb{L}$, *there is a nonnegative function* $V(.,k)$: $\mathcal{D}^c \longrightarrow \mathbb{R}$ *such that* $V(.,k)$ *is twice continuously differential and that for some* $\alpha > 0$,

$$LV \leq -\alpha \text{ for all } (x,k) \in \mathcal{D}^c \times \mathbb{L}, \tag{9}$$

then $(x(t),r(t))$ *of system (4) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution* $\mu(.,.)$ *such that for any Borel measurable function* f: $\mathbb{R}^n \times \mathbb{L} \longrightarrow \mathbb{R}$ *satisfying*

$$\sum_{k=1}^{N} \int_{\mathbb{R}^n} |f(x,k)|\mu(dx,k) < \infty, \tag{10}$$

and we have

$$\mathbb{P}\left(\lim_{t\longrightarrow\infty} \frac{1}{t}\int_0^t f(X(s),r(s))ds = \sum_{k=1}^{N}\int_{\mathbb{R}^n}|f(x,k)|\mu(dx,k)\right) = 1. \tag{11}$$

Lemma 2 (see [39]). *Define an* $m \times m$ *matrix*

$$B = \text{diag}\{\chi(1),\ldots,\chi(m)\} - \Phi, \tag{12}$$

then B is a nonsingular M-matrix [40]. Therefore, for any vector $z \in \mathbb{R}^n_+$, *the linear equation* $Bx = z$ *has a unique solution* $x \in \mathbb{R}^n_+$.

Now, we consider the following system:

$$dX(t) = \left(A_{r(t)} - \left(\nu_{r(t)} + \mu_{0,r(t)}X\right)dt + \sigma_{1,r(t)}XdW_1(t). \tag{13}$$

Then, we have the following lemma used in the extinction section.

Lemma 3 (see [39]). *The solution* $(X(t),r(t))$ *of system (13) satisfies the following equality:*

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t \beta_{r(s)}X(s)ds = \sum_{k\in\mathbb{L}}\pi_k A_k u_k, \tag{14}$$

where u_k *verifies* $\beta_k = u_k(\nu_k + \mu_{0,k}) - \sum_{l=1}^m \phi_{kl}u_l$.

2. Existence of Unique Positive Solution

The investigation in this work is to study a stochastic hepatitis B epidemic model with Markovian commutation. Our models represent a general case of many works existing in the literature and give an additional degree of realism in modeling the spread of the hepatitis B epidemic. We establish the existence and uniqueness of the positive solution to system (2). We give a threshold value that permits us to determine the extinction and the persistence of hepatitis B disease in (2). We present numerical simulation by using the Milstein Higher Order Method to confirm the theoretical results.

Our system (2) describes the dynamics of a biological population, thus investigating the problem of global existence, uniqueness, and positivity of the solution is so necessary. Hence, the following theorem guaranteed this statement for initial value $(S(0),I_1(0),I_2(0),R(0),r(0))$ in $\mathbb{R}^4_+ \times \mathbb{L}$.

Theorem 1. *For any initial value* $(S(0),I_1(0),I_2(0),R(0),r(0)) \in \mathbb{R}^4_+ \times \mathbb{L}$, *there is a unique solution* $(S(t),I_1(t),I_2(t),R(t),r(t))$ *to equation (2) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one, namely,* $(S(t),I_1(t),I_2(t),R(t),r(t)) \in \mathbb{R}^4_+ \times \mathbb{L}$ *for all* $t \geq 0$ *almost surely (briefly a.s.).*

Proof. Since the coefficients of the stochastic system (2) are locally Lipschitz continuous, for any given initial value $(S(0),I_1(0),I_2(0),R(0),r(0)) \in \mathbb{R}^4_+ \times \mathbb{L}$, there is a unique local solution $(S(t),I_1(t),I_2(t),R(t),r(t))$ for all $t \in [0,\tau_e)$, where τ_e denotes the explosion time [38]. If we prove that $\tau_e = \infty$ a.s., then this local solution is global. Let $n_0 > 0$ be sufficiently large such that all component of $(S(0),I_1(0),I_2(0),R(0))$ remains in interval $[1/n_0,n_0]$. For each $n \geq n_0$, define the stopping time as

$$\tau_n = \inf\left\{t \in [0,\tau_e): \min\{(S(t),I_1(t),I_2(t),R(t)\} \leq \frac{1}{n} \text{ or } \max\{(S(t),I_1(t),I_2(t),R(t)\} \geq n\right\}. \tag{15}$$

Obviously, τ_n is increasing as n tends to ∞. Set $\lim_{n,\infty}\tau_n = \tau_\infty$, then $\tau_\infty \leq \tau_e$ a.s. To prove that the solution is global, we need to confirm that $\tau_\infty = \infty$ a.s. If $\tau_\infty < \infty$ a.s., then there is a pair of constants $\theta \geq 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbb{P}\{\tau_\infty \le \theta\} \ge \varepsilon. \tag{16}$$

Therefore, there is an integer $n_1 \ge n_0$ such that

$$\mathbb{P}\{\tau_n \le \theta\} \ge \varepsilon, \text{ for all } n \ge n_1. \tag{17}$$

We consider a C^2-function J: $\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ defined by

$$\begin{aligned} J(S, I_1, I_2, R) = &\left(S - c - c\ln\frac{S}{c}\right) + (I_1 - 1 - \ln I_1) \\ &+ (I_2 - 1 - \ln I_2) + (R - 1 - \ln R), \end{aligned} \tag{18}$$

where $c := (\widehat{\mu}_0 + \widehat{\alpha})/\breve{\beta}$. Applying Itô's formula to equation (2), we obtain

$$\begin{aligned} dJ(S, I_1, I_2, R) = &\mathscr{L}J(S, I_1, I_2, R)dt + \sigma_{1,r(t)}(S-1)dW_1(t) \\ &+ \sigma_{2,r(t)}(I_1 - 1)dW_2(t) \\ &+ \sigma_{3,r(t)}(I_2 - 1)dW_3(t) \\ &+ \sigma_{4,r(t)}(R-1)dW_4(t), \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathscr{L}J(S, I_1, I_2, R) = &\left(1 - \frac{c}{S}\right)\left[A_k - \beta_k S I_1 - \left(\nu_k + \mu_{0,k}\right)S\right], \\ &+ \left(1 - \frac{1}{I_1}\right)\left[\beta_k S I_1 - \left(\gamma_{1,k} + \mu_{0,k} + \alpha_k\right)I_1\right], \\ &+ \left(1 - \frac{1}{I_2}\right)\left[\alpha_k I_1 - \left(\gamma_{2,k} + \mu_{0,k} + \mu_{1,k}\right)I_2\right], \\ &+ \left(1 - \frac{1}{R}\right)\left[\nu_k S(t) + \gamma_{2,k} I_2(t)\right. \\ &\left. - \mu_{0,k} R(t) + \gamma_{1,k} I_1(t)\right], \\ &+ \frac{\sigma_{1,k} + \sigma_{2,k} + \sigma_{3,k} + \sigma_{4,k}}{2}, \\ \le &A_k + \left[c\beta_k - \left(\mu_{0,k} + \alpha_k\right)\right]I_1 \\ &+ c\left(\nu_k + \mu_{0,k}\right) + \gamma_{1,k} + \mu_{0,k}, \\ &+ \alpha_k + \gamma_{2,k} + \mu_{0,k} + \mu_{1,k} + \mu_{0,k} \\ &+ \frac{\sigma_{1,k} + \sigma_{2,k} + \sigma_{3,k} + \sigma_{4,k}}{2}, \\ \le &\breve{A} + c\left(\breve{\nu} + t\breve{\mu}_0\right) + \breve{\gamma}_1 + \breve{\mu}_0 \\ &+ \breve{\alpha} + \breve{\gamma}_2 + \breve{\mu}_0 + \breve{\mu}_1 + \breve{\mu}_0 + \frac{\breve{\sigma}_1 + \breve{\sigma}_2 + \breve{\sigma}_3 + \breve{\sigma}_4}{2}, \\ \triangleq &\widetilde{C}. \end{aligned} \tag{20}$$

The rest of the proof is similar to [31] therefore is omitted. □

3. The Existence of Stationary Distribution

This section is sacred to provide sufficient conditions for the existence of a unique ergodic stationary distribution. To establish this statement, we define the following threshold value:

$$\mathscr{R}_{sw} = \frac{\sum_{k=1}^m \pi_k A_k u_k}{\sum_{k=1}^m \pi_k\left(\gamma_{1,k} + \mu_{0,k} + \alpha_k + \sigma_{2,k}/2\right)}, \tag{21}$$

where u_k is the solution of the following equation:

$$\beta_k = u_k\left(\nu_k + \mu_{0,k}\right) - \sum_{l=1}^m \phi_{kl} u_l. \tag{22}$$

To prove that system (2) has a unique ergodic stationary distribution, we must show the conditions (i), (ii), and (iii) in Lemma 1.

(i) Using the assumption $\phi_{ij} > 0$ for $i \ne j$ in Section 2, we obtain that condition (i) in Lemma 3.1 is satisfied.

(ii) It is easy to show that the diffusion matrix

$$D(S, I_1, I_2, R) = \operatorname{diag}\left\{\sigma_1^2(k)S^2, \sigma_2^2(k)I_1^2, \sigma_3^2(k)I_2^2, \sigma_4^2(k)R^2\right\}, \tag{23}$$

of system (2) is positive definitely, and then condition (ii) is verified.

It remains to verify the condition (iii) in Lemma 1. For this, we propose the following theorem.

Theorem 2. *Let $\mathscr{R}_{sw} > 1$, then for any initial value $(S(0), I_1(0), I_2(0), R(0), r(0)) \in \mathbb{R}_+^4 \times \mathbb{L}$, the stochastic switched system (2) has a unique ergodic stationary distribution.*

Proof. Define a C^2-function $\widehat{F}$ by

$$\begin{aligned} \widehat{F}(S, I_1, I_2, R, k) = &-M\left(\ln I_1 + u_k\left(S + I_1\right) - \eta_k\right) \\ &+ \frac{1}{\kappa + 1}\left(S + I_1 + I_2 + R\right)^{\kappa+1} \\ &+ \left(S + I_1 + I_2 + R\right) - \ln R - \ln I_2 - \ln S \\ = &-MU_1 + U_2 + U_3 + U_4 + U_5 + U_6, \end{aligned} \tag{24}$$

where

$$\begin{aligned} U_1 &= -\ln I_1 - u_k(S + I_1) + \eta_k, \\ U_2 &= \frac{1}{\kappa + 1}(S + I_1 + I_2 + R)^{\kappa+1}, \\ U_3 &= (S + I_1 + I_2 + R), \\ U_4 &= -\ln R, \\ U_5 &= -\ln I_2, \end{aligned} \tag{25}$$

M satisfying the following condition:

$$K_2 - M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \leq -2, \tag{26}$$

and $\kappa \in (0,1)$, K_2, u_k, η_k will be determined later.

Clearly,

$$\liminf_{k\longrightarrow\infty,\,(S,I_1,I_2,R)\in\mathbb{R}_+^4/U_k} \widehat{F}(S, I_1, I_2, R, k) = +\infty, \tag{27}$$

where $U_k = (1, 1/k) \times (1, 1/k) \times (1, 1/k) \times (1, 1/k)$. Furthermore, $\widehat{F}(S, I_1, I_2, R, k)$ is a continuous function on $\overline{U}_k$, and then $\widehat{F}(S, I_1, I_2, R, k)$ has a minimum value point $(\widehat{S}, \widehat{I}_1, \widehat{I}_2, \widehat{R}, \widehat{k})$ in the interior of $\mathbb{R}_+^4$.

We define a C^2-function U: $\mathbb{R}_+^4 \times \mathbb{L} \longrightarrow \mathbb{R}$ as follows:

$$U(S, I_1, I_2, R, k) = \widehat{F}(S, I_1, I_2, R, k) - \widehat{F}\left(\widehat{S}, \widehat{I}_1, \widehat{I}_2, \widehat{R}, \widehat{k}\right). \tag{28}$$

By using Itô's formula and system (2), we obtain

$$\begin{aligned} \mathscr{L}U_1 &= -\beta_k S + \gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2} - u_k A_k \\ &\quad + u_k(\nu_k + \mu_{0,k})S + u_k(\gamma_{1,k} + \mu_{0,k} + \alpha_k)I_1, \\ &\quad - \sum_{l=1}^{m}\phi_{kl}u_l S - \sum_{l=1}^{m}\phi_{kl}u_l I_1 - \sum_{l=1}^{m}\phi_{kl}\eta_l, \\ &= \gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2} - u_k A_k \\ &\quad - \sum_{l=1}^{m}\phi_{kl}\eta_l - \left[\beta_k - u_k(\nu_k + \mu_{0,k}) + \sum_{l=1}^{m}\phi_{kl}u_l\right]S, \\ &\quad + \left[u_k(\gamma_{1,k} + \mu_{0,k} + \alpha_k) - \sum_{l=1}^{m}\phi_{kl}u_l\right]I_1. \end{aligned} \tag{29}$$

Let u_k be the solution of the following equation:

$$\beta_k = u_k(\nu_k + \mu_{0,k}) - \sum_{l=1}^{m}\phi_{kl}u_l, \tag{30}$$

then

$$\begin{aligned} \mathscr{L}U_1 &= \gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2} - u_k A_k - \sum_{l=1}^{m}\phi_{kl}\eta_l \\ &\quad + \left[\beta_k + u_k(\gamma_{1,k} + \mu_{0,k} - \nu_k)\right]I_1 \\ &\triangleq -\overline{R}_k - \sum_{l=1}^{m}\phi_{kl}\eta_l + \left[\beta_k + u_k(\gamma_{1,k} + \mu_{0,k} - \nu_k)\right]I_1. \end{aligned} \tag{31}$$

Since the generator matrix Φ is irreducible, then for $\overline{R} = (\overline{R}_1, \ldots, \overline{R}_m)^T$, there exists $\eta = (\eta_1, \ldots, \eta_m)^T$ solution of the following Poisson system (see [41]):

$$\Phi\eta = -\overline{R} + \left(\sum_{k=1}^{m}\pi_k\overline{R}_k\right)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{32}$$

Then, we have

$$\begin{aligned} \mathscr{L}U_1 &\leq -\sum_{k=1}^{m}\pi_k\overline{R}_k + \left[\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ &= -\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &\quad + \left[\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1. \end{aligned} \tag{33}$$

In addition, we can obtain

$$\begin{aligned} \mathscr{L}U_2 &= (S + I_1 + I_2 + R)^{\kappa}\left[A_k - \mu_{0,k}(S + I_1 + I_2 + R) - \mu_{1,k}I_2\right] \\ &\quad + \frac{\kappa}{2}(S + I_1 + I_2 + R)^{\kappa-1}\left[\sigma_{1,k}^2 S^2 + \sigma_{2,k}^2 I_1^2 + \sigma_{3,k}^2 I_2^2 + \sigma_{4,k}^2 R^2\right] \\ &\leq (S + I_1 + I_2 + R)^{\kappa}\left[\check{A} - \widehat{\mu}_0(S + I_1 + I_2 + R)\right] \\ &\quad + \frac{\kappa}{2}(S + I_1 + I_2 + R)^{\kappa+1}\left[\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right] \\ &= \check{A}(S + I_1 + I_2 + R)^{\kappa} \\ &\quad - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right](S + I_1 + I_2 + R)^{\kappa+1} \\ &\leq K_1 - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right](S + I_1 + I_2 + R)^{\kappa+1} \\ &\leq K_1 - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\left(S^{\kappa+1} + I_1^{\kappa+1} + I_2^{\kappa+1} + R^{\kappa+1}\right), \end{aligned} \tag{34}$$

with

$$K_1 \triangleq \sup_{(S+I_1+I_2+R)\in\mathbb{R}_+^4}\left\{\check{A}(S + I_1 + I_2 + R)^{\kappa} - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right](S + I_1 + I_2 + R)^{\kappa+1}\right\}. \tag{35}$$

We have

$$\mathscr{L}U_3 = A_k - \mu_{0,k}(S + I_1 + I_2 + R) - \mu_{1,k}I_2 \le \check{A}, \tag{36}$$

$$\mathscr{L}U_4 = -\nu_k\frac{S}{R} - \gamma_{2,k}\frac{I_2}{R} - \gamma_{1,k}\frac{I_1}{R} + \mu_{0,k} + \frac{\sigma_{4,k}^2}{2} \le -\widehat{\nu}\frac{S}{R} - \widehat{\gamma}_2\frac{I_2}{R} - \widehat{\gamma}_1\frac{I_1}{R} + \check{\mu}_0 + \frac{\check{\sigma}_4^2}{2}, \tag{37}$$

$$\mathscr{L}U_5 = -\alpha_k\frac{I_1}{I_2} + \gamma_{2,k} + \mu_{0,k} + \mu_{1,k} + \frac{\sigma_{3,k}^2}{2} \le -\widehat{\alpha}\frac{I_1}{I_2} + \check{\gamma}_2 + \check{\mu}_0 + \check{\mu}_1 + \frac{\check{\sigma}_3^2}{2}, \tag{38}$$

$$\mathscr{L}U_6 = -\frac{A_k}{S} + \beta_k I_1 + \nu_k + \mu_{0,k} + \frac{\sigma_{1,k}^2}{2} \le -\frac{\widehat{A}}{S} + \check{\beta}I_1 + \check{\nu} + \check{\mu}_0 + \frac{\check{\sigma}_1^2}{2}. \tag{39}$$

Combining (33) and (34) with (36)–(39) yields

$$\begin{aligned}\mathscr{L}U \le & -M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ & + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ & - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\left(S^{\kappa+1} + I_1^{\kappa+1} + I_2^{\kappa+1} + R^{\kappa+1}\right) \\ & - \frac{\widehat{A}}{S} - \widehat{\nu}\frac{S}{R} - \widehat{\gamma}_2\frac{I_2}{R} - \widehat{\gamma}_1\frac{I_1}{R} - \widehat{\alpha}\frac{I_1}{I_2} + K_2,\end{aligned} \tag{40}$$

where

$$K_2 = \check{A} + \check{\gamma}_2 + 3\check{\mu}_0 + \check{\mu}_1 + \check{\nu} + K_1 + \frac{\check{\sigma}_1^2}{2} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_4^2}{2}. \tag{41}$$

Next, we define a compact set

$$\mathscr{D} = \left\{\varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I_1 \le \frac{1}{\varepsilon}, \varepsilon' \le I_2 \le \frac{1}{\varepsilon'}, \varepsilon' \le R \le \frac{1}{\varepsilon'}\right\}, \tag{42}$$

where ε and ε' are a sufficiently small positive constant satisfying the following conditions:

$$K_2 + h^* - \frac{\widehat{A}}{\varepsilon} < -1, \tag{43}$$

$$M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]\varepsilon \le 1, \tag{44}$$

$$K_2 + h^* - \frac{\widehat{\alpha}}{\varepsilon} \le -1, \tag{45}$$

$$K_2 + h^* - \frac{\widehat{\gamma}_1}{\varepsilon} \le -1, \tag{46}$$

$$K_2 + h^* - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\kappa+1}} \le -1, \tag{47}$$

$$K_2 - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\kappa+1}} + C \le -1, \tag{48}$$

$$-\frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\kappa+1}} + E \le -1, \tag{49}$$

$$-\frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon'^{\kappa+1}} + S \le -1, \tag{50}$$

where K_2, C, E, and $\mathscr{S}$ are positive constants to be determined. We divide $\mathbb{R}_+^4, \mathscr{D}$ into the following eight domains:

$$\begin{aligned}\mathscr{D}_1 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, 0 < S < \varepsilon\right\}, \\ \mathscr{D}_2 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, 0 < I_1 < \varepsilon\right\}, \\ \mathscr{D}_3 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, S \ge \varepsilon, I_1 \ge \varepsilon, 0 < I_2 < \varepsilon'\right\}, \\ \mathscr{D}_4 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, I_1 \ge \varepsilon, 0 < R < \varepsilon'\right\}, \\ \mathscr{D}_5 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, S > \frac{1}{\varepsilon}\right\}, \\ \mathscr{D}_6 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, I_1 > \frac{1}{\varepsilon}\right\}, \\ \mathscr{D}_7 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, I_2 > \frac{1}{\varepsilon'}\right\}, \\ \mathscr{D}_8 &= \left\{(S, I_1, I_2, R) \in \mathbb{R}_+^4, R > \frac{1}{\varepsilon'}\right\},\end{aligned} \tag{51}$$

with

$$\mathscr{D}^c = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3 \cup \mathscr{D}_4 \cup \mathscr{D}_5 \cup \mathscr{D}_6 \cup \mathscr{D}_7 \cup \mathscr{D}_8. \tag{52}$$

In the next step, we must prove that $\mathscr{L}U < -1$ for any $(S, I_1, I_2, R, k) \in \mathscr{D}^c \times \mathbb{L}$. For this, we discuss the following eight cases:

Case 1. If $(S, I_1, I_2, R, k) \in \mathscr{D}_1 \times \mathbb{L}$, for (40) and (43), we can obtain that

$$\mathscr{L}U \le K_2 + h(I_1) - \frac{\widehat{A}}{S} \le K_2 + h^* - \frac{\widehat{A}}{\varepsilon} \le -1, \tag{53}$$

where

$$\begin{aligned}h(I_1) = & -M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ & + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ & - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_1^{\kappa+1},\end{aligned} \tag{54}$$

and $h^* = \sup_{I_1 \in (0,\infty)}\{h(I_1)\}$.

Case 2. If $(S, I_1, I_2, R, k) \in \mathscr{D}_2 \times \mathbb{L}$, from (7), (40), and (44), we obtain

$$\begin{aligned}\mathscr{L}U &\le K_2 - M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &\quad + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ &\le K_2 - M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &\quad + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]\varepsilon \le -1.\end{aligned} \tag{55}$$

Case 3. If $(S, I_1, I_2, R, k) \in \mathscr{D}_3 \times \mathbb{L}$, choosing $\varepsilon' = \varepsilon^2$, we have

$$\mathscr{L}U \le K_2 + h(I_1) - \widehat{\alpha}\frac{I_1}{I_2} \le K_2 + h^* - \widehat{\alpha}\frac{\varepsilon}{\varepsilon'} \le K_2 + h^* - \frac{\widehat{\alpha}}{\varepsilon}. \tag{56}$$

According to (46), we obtain

$$\mathscr{L}U \le K_2 + h^* - \frac{\widehat{\alpha}}{\varepsilon} \le -1. \tag{57}$$

Case 4. If $(S, I_1, I_2, R, k) \in \mathscr{D}_4 \times \mathbb{L}$, from (40) and (46), we have

$$\begin{aligned}\mathscr{L}U &\le K_2 + h(I_1) - \widehat{\gamma}_1\frac{I_1}{R} \\ &\le K_2 + h^* - \widehat{\gamma}_1\frac{\varepsilon}{\varepsilon'} \\ &\le K_2 + h^* - \frac{\widehat{\gamma}_1}{\varepsilon} \\ &\le -1.\end{aligned} \tag{58}$$

Case 5. If $(S, I_1, I_2, R, k) \in \mathscr{D}_5 \times \mathbb{L}$, from (40) and (47), we have

$$\begin{aligned}\mathscr{L}U &\le K_2 + h(I_1) - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]S^{\kappa+1} \\ &\le K_2 + h^* - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\kappa+1}} \le -1.\end{aligned} \tag{59}$$

Case 6. If $(S, I_1, I_2, R, k) \in \mathscr{D}_5 \times \mathbb{L}$, from (40) and (48), we get

$$\begin{aligned}\mathscr{L}U &\le K_2 - M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &\quad + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ &\quad - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_1^{\kappa+1} \\ &= K_2 - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_1^{\kappa+1} \\ &\quad - M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &\quad + M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ &\quad - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_1^{\kappa+1} \\ &\le K_2 - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\kappa+1}} + C \le -1,\end{aligned} \tag{60}$$

where

$$\begin{aligned}C = \sup_{I_1 \in (0,\infty)}\Bigg\{&-M\sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1) \\ &+ M\left[2\check{\beta} + \check{u}(\check{\gamma}_1 + \check{\mu}_0)\right]I_1 \\ &- \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_1^{\kappa+1}\Bigg\}.\end{aligned} \tag{61}$$

Case 7. If $(S, I_1, I_2, R, k) \in \mathscr{D}_7 \times \mathbb{L}$, from (40) and (49), we have

$$\begin{aligned}\mathscr{L}U &\le K_2 + h(I_1) - \frac{1}{2}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_2^{\kappa+1} \\ &\le -\frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_2^{\kappa+1} + E \\ &\le -\frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon'^{\kappa+1}} + E \le -1,\end{aligned} \tag{62}$$

where

$$E = \sup_{I_2 \in (0,\infty)}\left\{K_2 + h^* - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]I_2^{\kappa+1}\right\}. \tag{63}$$

Case 8. If $(S, I_1, I_2, R, k) \in \mathscr{D}_7 \times \mathbb{L}$, from (40) and (50) and using the same method in Case 7, we obtain

$$\begin{aligned}\mathscr{L}U &\leq -\frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]\frac{1}{\varepsilon^{\prime\kappa+1}} + \mathcal{S} \\ &\leq -1,\end{aligned} \tag{64}$$

where

$$\mathcal{S} = \sup_{R\in(0,\infty)}\left\{K_2 + h^* - \frac{1}{4}\left[\widehat{\mu}_0 - \frac{\kappa}{2}\left(\check{\sigma}_1^2 \vee \check{\sigma}_2^2 \vee \check{\sigma}_3^2 \vee \check{\sigma}_4^2\right)\right]R^{\kappa+1}\right\}. \tag{65}$$

From the above eight cases, one can conclude that

$$\mathscr{L}U \leq -1, \text{for all } (S, I_1, I_2, R, k) \in \mathscr{D}^c \times \mathbb{L}. \tag{66}$$

Thus, condition (iii) in Lemma 1 holds. Therefore, we conclude that system (2) is ergodic and has a unique stationary distribution. □

Remark 1. Theorem 2 confirms the existence of a unique ergodic stationary distribution to system (2) that represents significant propriety across which we obtain from that the hepatitis B disease will persist in the population.

4. Extinction

The principal question of investigating the dynamics of the epidemic model is to give a process to reduce and control the spread of the epidemic in a population. Then, in this section, we investigate the extinction of I_1 of the stochastic system (2).

Theorem 3. *Let $(S(t), I_1(t), I_2(t), R(t), r(t))$ be the solution of system (2) with any initial value $(S(0), I_1(0), I_2(0), R(0), r(0)) \in \mathbb{R}_+^4 \times \mathbb{L}$. If $\mathscr{R}_{sw} \leq 1$, then the acute infected I_1 will die out exponentially with probability one, i.e.,*

$$\limsup_{t\longrightarrow\infty}\frac{\ln I_1(t)}{t} \leq \sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1)\, a.s. \tag{67}$$

Proof. Making use of Itô's formula to the second equation of system (2) leads to

$$\begin{aligned}d\ln I_1(t) = &\left[\beta_{r(t)}S(t) - \left(\gamma_{1,r(t)} + \mu_{0,r(t)} + \alpha_{r(t)} + \frac{\sigma_{2,r(t)}}{2}\right)\right]dt \\ &+ \sigma_{2,r(t)}dW_2(t).\end{aligned} \tag{68}$$

Integrating both sides of (68) from 0 to t and dividing by t, we get

$$\begin{aligned}\frac{\ln I_1(t)}{t} &= \frac{1}{t}\int_0^t\left[\beta_{r(s)}S(s) - \left(\gamma_{1,r(s)} + \mu_{0,r(s)} + \alpha_{r(s)} + \frac{\sigma_{2,r(s)}}{2}\right]ds \right. \\ &\quad + \frac{1}{t}\int_0^t \sigma_{2,r(s)}dW_2(s) - \frac{\ln I_1(0)}{t} \\ &\leq \frac{1}{t}\int_0^t\left[\beta_{r(s)}X(s) - \left(\gamma_{1,r(s)} + \mu_{0,r(s)} + \alpha_{r(s)} + \frac{\sigma_{2,r(s)}}{2}\right)\right]ds \\ &\quad + \frac{M(t)}{t} - \frac{\ln I_1(0)}{t},\end{aligned} \tag{69}$$

where

$$M(t) = \int_0^t \sigma_{2,r(s)}dW_2(s), \tag{70}$$

is a real-valued continuous local martingale, and its quadratic variation is presented by

$$\langle M(t), M(t)\rangle_t = \int_0^t \sigma_{2,r(s)}^2 ds \leq \check{\sigma}^2 t. \tag{71}$$

Using the large number theorem for local martingales (see Ref. [38]), we obtain

$$\limsup_{t\longrightarrow\infty}\frac{M(t)}{t} = 0\, a.s. \tag{72}$$

Taking the superior limit on the both sides of (69), combining with (72) and the ergodic property of Markov chain [38], we obtain

$$\limsup_{t\longrightarrow\infty}\frac{\ln I_1(t)}{t} \leq \sum_{k=1}^{m}\pi_k\left[\gamma_{1,k} + \mu_{0,k} + \alpha_k + \frac{\sigma_{2,k}}{2}\right](\mathscr{R}_{sw} - 1)\, a.s. \tag{73}$$

Since $\mathscr{R}_{sw} \leq 1$, we obtain

$$\lim_{t\longrightarrow\infty} I_1(t) = 0\, a.s. \tag{74}$$

□

Remark 2. According to Theorem 3, we conclude that the large noises can show the extinction of disease in the system.

Remark 3. By Theorems 2 and 3, the value of $\mathscr{R}_{sw}$ can represent the stochastic threshold and determine the persistence and extinction of disease, namely, if $\mathscr{R}_{sw} < 1$, then the infected extinct. On the other hand, the disease persist when $\mathscr{R}_{sw} > 1$.

5. Numerical Example

In this section, we simulate our model (2) using Milstein Higher Order Method explained in [42] to validate the analysis results concerning the dynamical of the stochastic hepatitis B model in a switching environment. Firstly, we can write the model as the following discretization systems:

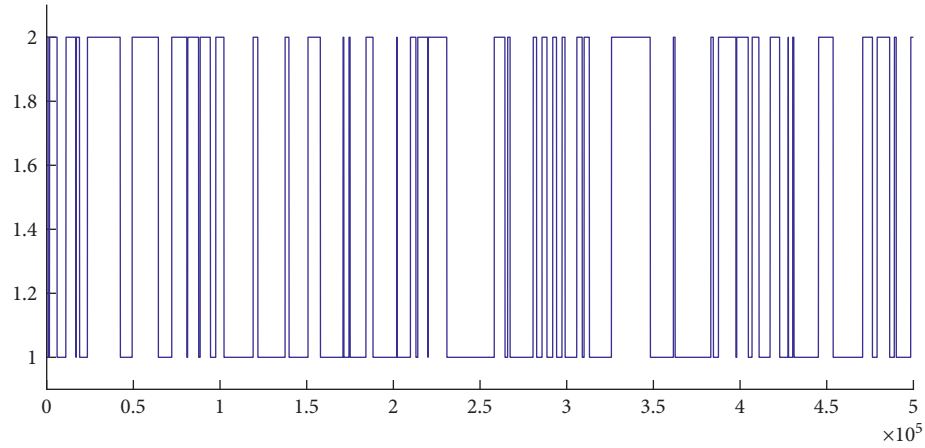

Figure 1: Trajectories of Markov chain $r(t)$.

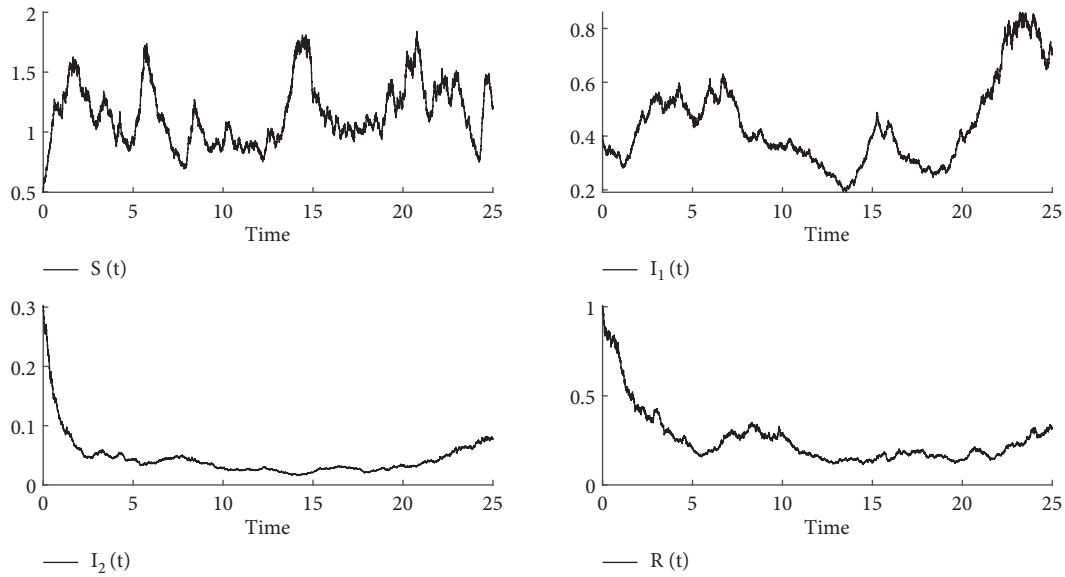

Figure 2: Numerical simulations of the path $S(t)$, $I_1(t)$, $I_2(t)$, and $R(t)$ for the stochastic systems (2).

$$\begin{cases} S_{i+1} = S_i + \left[A_{r(i\Delta t)} - \beta_{r(i\Delta t)} S_i I_{1,i} - \left(\nu_{r(i\Delta t)} + \mu_{0,r(i\Delta t)}\right) S_i\right]\Delta t + \sigma_{1,r(i\Delta t)} S_i \sqrt{\Delta t}\, \zeta_{1,i}, \\ I_{1,i+1} = I_{1,i} + \left[\beta_{r(i\Delta t)} S_i I_{1,i} - \left(\gamma_{1,r(i\Delta t)} + \mu_{0,r(i\Delta t)} + \alpha_{r(i\Delta t)}\right) I_{1,i}\right]\Delta t + \sigma_{2,r(i\Delta t)} I_{1,i} \sqrt{\Delta t}\, \zeta_{2,i}, \\ I_{2,i+1} = I_{2,i} + \left[\alpha_{r(i\Delta t)} I_{1,i} - \left(\gamma_{2,r(i\Delta t)} + \mu_{0,r(i\Delta t)} + \mu_{1,r(i\Delta t)}\right) I_{2,i}\right]\Delta t + \sigma_{3,r(i\Delta t)} I_{2,i} \sqrt{\Delta t}\, \zeta_{3,i}, \\ R_{i+1} = R_i + \left[\nu_{r(i\Delta t)} S_i + \gamma_{2,r(i\Delta t)} I_{2,i} - \mu_{0,r(i\Delta t)} R_i + \gamma_{1,r(i\Delta t)} I_{1,i}\right]\Delta t + \sigma_{4,r(i\Delta t)} R_i \sqrt{\Delta t}\, \zeta_{4,i}, \end{cases} \tag{75}$$

where $\{r(i\Delta t), i = 0, 1, 2 \ldots\}$ is a discrete-time Markov chain. $\{\zeta_{j,i}, i \geq 0\}$, $j = 1, 2, 3, 4$, are random numbers with standard normal distribution: $\mathcal{N}(0, 1)$ and Δt are arbitrary time step.

Then, we consider the Markov chain $r(t)$ taking value in state space $\mathbb{L} = \{1, 2\}$ with the generator as follows:

$$\Phi = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \tag{76}$$

The Markov chain $r(t)$ has a unique stationary distribution as follows:

$$\pi = (\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right). \tag{77}$$

The path of the Markov chain $r(t)$ is plotted in Figure 1. The parameter values of system (2) are chosen as follows:

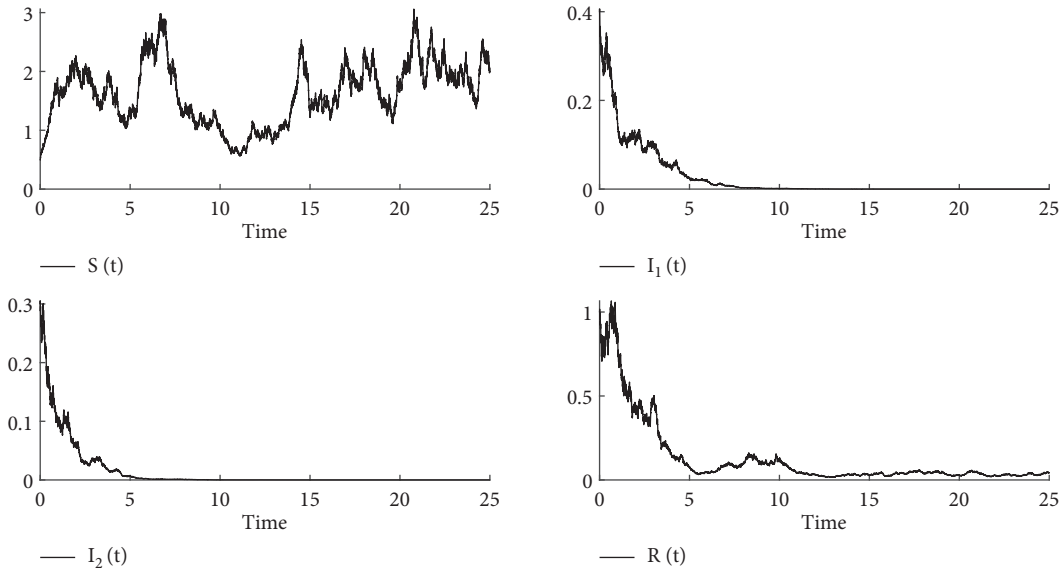

Figure 3: Numerical simulations of the path $S(t)$, $I_1(t)$, $I_2(t)$, and $R(t)$ for the stochastic systems (2).

$$\begin{aligned} &A(1) = 0.78; \beta(1) = 0.8; \nu(1) = 0.01; \mu_0(1) = 0.6, \gamma_1(1) = 0.2, \gamma_2(1) = 0.4, \\ &\alpha(1) = 0.1, \mu_1(1) = 0.2, \sigma_1(1) = 0.15, \sigma_2(1) = 0.12, \sigma_3(1) = 0.15, \sigma_4(1) = 0.2, \\ &A(2) = 0.75; \beta(2) = 0.7, \nu(2) = 0.01, \mu_0(2) = 0.6, \gamma_1(2) = 0.2, \gamma_2(2) = 0.4, \\ &\alpha(2) = 0.1, \mu_1(2) = 0.2, \sigma_1(2) = 0.2, \sigma_2(2) = 0.15, \sigma_3(2) = 0.2, \sigma_4(2) = 0.14. \end{aligned} \tag{78}$$

By computation, we obtain

$$\mathcal{R}_{sw} = \frac{\sum_{k=1}^{m} \pi_k A_k u_k}{\sum_{k=1}^{m} \pi_k \left(\gamma_{1,k} + \mu_{0,k} + \alpha_k + \sigma_{2,k}/2\right)} \approx 1.1130 > 1. \tag{79}$$

Hence, according to Theorem 3, the solution $(S(t), I_1(t), I_2(t), R(t), r(t))$ of system (2) with an initial value $(0.5, 0.4, 0.3, 0.2)$ has a unique stationary distribution with the ergodic property, which means that the disease will be prevailing in the population. Figure 2 confirms this result. In the following, we explore the extinction of the diseases for the stochastic system (2). To proceed, we take the parameters as follows:

$$\begin{aligned} &A(1) = 0.5, \beta(1) = 0.1, \nu(1) = 0.01, \mu_0(1) = 0.5, \gamma_1(1) = 0.2, \gamma_2(1) = 0.4, \\ &\alpha(1) = 0.1, \mu_1(1) = 0.2, \sigma_1(1) = 0.4, \sigma_2(1) = 0.6, \sigma_3(1) = 0.45, \sigma_4(1) = 0.4; \\ &A(2) = 0.5, \beta(2) = 0.1, \nu(2) = 0.01, \mu_0(2) = 0.5, \gamma_1(2) = 0.2, \gamma_2(2) = 0.4, \\ &\alpha(2) = 0.1, \mu_1(2) = 0.2, \sigma_1(2) = 0.4, \sigma_2(2) = 0.5, \sigma_3(2) = 0.44, \sigma_4(2) = 0.4. \end{aligned} \tag{80}$$

We obtain

$$\mathcal{R}_{sw} = \frac{\sum_{k=1}^{m} \pi_k A_k u_k}{\sum_{k=1}^{m} \pi_k \left(\gamma_{1,k} + \mu_{0,k} + \alpha_k + \sigma_{2,k}/2\right)} \approx 0.0530 < 1, \tag{81}$$

then, according to Theorem 3, the infected $I_1(t)$ will tend to zero exponentially with probability one. Therefore, the stochastic hepatitis B infection process (2) switches between states 1 and 2 and also goes extinct (see Figure 3).

6. Conclusion

In this work, we addressed a stochastic epidemic model characterizing the dynamics of hepatitis B by using hybrid environmental perturbations modeled with two types of noise, namely, the white and colored noises. The objective of this investigation is to study the random noise effect in a population divided into four groups, namely, (S) Susceptible, (I_1) acute Infected, (I_2) chronically Infected, and (R) recovered. Firstly, the existence of a unique global positive

solution with any positive initial value is established. Then, we showed a sufficient condition for the existence of ergodic stationary distribution of the solution for system (2) and conditions for the extinction of the infection. Precisely,

(i) If $\mathscr{R}_{sw} < 1$, then the infected $I_1(t)$ will die out exponentially with probability one.

(ii) If $\mathscr{R}_{sw} > 1$, then for any initial value $(S(0), I_1(0), I_2(0), R(0), r(0)) \in \mathbb{R}_+^4 \times \mathbb{L}$, the solution $(S(t), I_1(t), I_2(t), R(t), r(t))$ of system (2) has an ergodic unique stationary distribution.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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