Hindawi
Complexity
Volume 2022, Article ID 2385739, 6 pages
https://doi.org/10.1155/2022/2385739

Research Article

Majorization Properties for Certain Subclasses of Meromorphic Function of Complex Order

Neelam Khan,[1] Muhammad Arif,[1] and Maslina Darus[2]

[1]*Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan*
[2]*Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia*

Correspondence should be addressed to Maslina Darus; maslina@pkrisc.cc.ukm.my

Received 4 June 2022; Revised 14 September 2022; Accepted 26 October 2022; Published 21 November 2022

Academic Editor: Abdellatif Ben Makhlouf

By making use of $q-$differential operators, many distinct subclasses of analytic and meromorphic functions have already been defined and investigated from numerous perspectives. In this article, we investigated several majorization results for the class of meromorphic univalent functions of complex order, defined by $q-$differential operator. Moreover, we pointed out some new or known consequences of our results, which is in the form of corollaries.

1. Introduction and Definitions

Let $\mathscr{M}$ represent the class of meromorphic functions f in the form of

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured disc $\mathring{\mathscr{U}} = \{z\colon 0 < |z| < 1\} = \mathscr{U}\backslash\{0\}$, where $\mathscr{U} = \mathring{\mathscr{U}} \cup \{0\}$. For the two functions $f(z)$ and $g(z)$ belonging to the class $\mathscr{M}$, there exists a Schwartz function w, which is analytic in $\mathscr{U}$ with $|w(z)| \le |z|$ and $w(0) = 0$, such that $f(z) = g(w(z))$, and the function f is subordinate to g, written as $f \prec g$. The following relationship holds if g is univalent:

$$f \prec g \Longleftrightarrow f(0) = g(0), \text{ and } f(\mathring{\mathscr{U}}) \subseteq g(\mathring{\mathscr{U}}). \tag{2}$$

Because of its use in a variety of mathematical sciences, the study of $q-$calculus (quantum calculus) has fascinated and motivated many scholars. One of the primary contributors among all the mathematicians who introduced the concept of $q-$calculus theory was Jackson [1, 2]. The formulation of this concept is widely used to investigate the nature of different structures of function theory, such as $q-$calculus was used in other branches of mathematics. Though the authors of the first article [3] discussed the geometric nature of $q-$starlike functions, Srivastava [4] laid a solid foundation for the use of $q-$calculus in the context of function theory. Also, in [5], Srivastava provided a brief overview of basic or $q-$calculus operators and fractional $q-$calculus operators, as well as their applications in the geometric function theory of complex analysis. Later, the authors [6–8] investigated a number of useful properties for the newly defined $q-$linear differential operator, and Mehmood and Sokół [9] discussed the Ruscheweyh $q-$differential operator, while Srivastava et al. [10] introduced a generalized operator for meromorphic harmonic functions by using the idea of convolution.

Let $0 < q < 1$. For any nonnegative integer n, the q-integer number n is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n+1}, \quad [0]_q = 0. \tag{3}$$

In general, we will denote

$$[\delta]_q = \frac{1 - q^\delta}{1 - q}, \tag{4}$$

for a noninteger number δ. Also, the q-number shifted factorial is defined by

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q,\ [0]_q! = 1. \tag{5}$$

Clearly,

$$\begin{aligned} \lim_{q \longrightarrow 1^-} [n]_q &= n, \\ \lim_{q \longrightarrow 1^-} [n]_q &= n!. \end{aligned} \tag{6}$$

Let $a, q \in \mathbb{C}\,(|q| < 1)$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, the $q-$shifted factorial $(a;q)_n$ is defined by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{j=1}^{n} \left(1 - aq^{j-1}\right), \quad n \in \mathbb{N}. \tag{7}$$

Let $x \in \mathbb{C} - \{-n: n \in \mathbb{N}_0\}$. Then, $q-$gamma function is as follows:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1. \tag{8}$$

In a subset of $\mathbb{C}$, the $q-$derivative (or $q-$difference) operator $\mathcal{D}_q f$ of function f is defined by

$$\left(\mathcal{D}_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, z \neq 0, \\ f'(0), \quad z = 0, \end{cases} \tag{9}$$

provided that $f'(0)$ exists. We can easily observe from the definition of (9) that $(\mathcal{D}_q f)(z)_{\lim_{q \longrightarrow 1^-}} = f'(z)$.

Suppose that $q \in [0,1]$, then $q-$ analog derivative of f as

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)},\ (z \in \mathcal{U}), \tag{10}$$

or

$$\left(\mathcal{D}_q f\right)(z) = -\frac{1}{qz^2} + \sum_{n=1}^{\infty} [n]_q a_n z^n. \tag{11}$$

In 1967, Mac Gregor [11] introduced the notion of majorization as follows.

Definition 1. Let complex-valued functions f and g be analytic in $\mathcal{U}$. We say that f is majorized by g in $\mathcal{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}), \tag{12}$$

if there exists a function $\varphi(z)$ (complex−valued function in $\mathcal{U}$), satisfying

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z) g(z) \quad (z \in \mathcal{U}). \tag{13}$$

Majorization (12) is closely related to the concept of quasi-subordination between analytic functions in $\mathcal{U}$. Several researchers have published articles on this topic; for example, Tang et al. [12] gave the concept of majorization for subclasses of starlike functions based on the sine and cosine functions, Arif et al. [13] discussed majorization for various new defined classes, Cho et al. [14] obtained coefficient estimates for majorization, and Tang and Deng [15] defined the majorization problem connected with Liu-Owa integral operator and exponential function. This concept is also defined for $p-$valent function by Altintas and Srivastava [16] and for complex order by Altintas et al. [17].

The basic goal of this article is to examine and explain the idea of majorization in the context of the meromorphic function. Many researchers have shown their interest in this site. Goyal and Goswami [18, 19] studied this concept for majorization for meromorphic function with the integral operator, Tang et al. [12] discussed it for meromorphic sin and cosine functions, Bulut et al., Tang et al., and Janani [20–22] explained this concept for meromorphic multivalent functions, Rasheed et al. [23] investigated a majorization problem for the class of meromorphic spiral-like functions related with a convolution operator, and Panigrahi and El-Ashwah [24] defined majorization for subclasses of multivalent meromorphic functions through iterations and combinations of the Liu–Srivastava operator and Cho–Kwon–Srivastava operator and much more. In addition, there are several other articles on this topic [18].

Here is the definition of our main function.

Definition 2. A function $f(z) \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^q(\gamma)$ of meromorphic functions of complex order $\gamma \neq 0$ in $\mathring{\mathcal{U}}$, if

$$1 - \frac{1}{\gamma}\left[\frac{zq\mathcal{D}_q f(z)}{f(z)} + 1\right] \prec \Psi(z). \tag{14}$$

Now, we are going to choose some particular functions instead of $\Psi(z)$. These choices are

$$\begin{aligned} &\Psi(z) = 1 + \sin z, \\ \text{or } &\Psi(z) = \cos z, \\ \text{or } &\Psi(z) = \sqrt{1+z}, \\ \text{or } &\Psi(z) = \frac{1+z}{1-z}, \end{aligned} \tag{15}$$

and by applying the above-mentioned concepts, we now consider the following cases:

$$\begin{aligned} \mathcal{MS}^q_{\sin}(\gamma) &= \left\{f(z) \in \mathcal{M}_{\mathcal{H}}: 1 - \frac{1}{\gamma}\left[\frac{z\mathcal{D}_q f(z)}{f(z)} + 1\right] \prec 1 + \sin z\right\}, \\ \mathcal{MS}^q_{\cos}(\gamma) &= \left\{f(z) \in \mathcal{M}_{\mathcal{H}}: 1 - \frac{1}{\gamma}\left[\frac{z\mathcal{D}_q f(z)}{f(z)} + 1\right] \prec \cos z\right\}, \\ \mathcal{MS}^q_{\mathfrak{SL}}(\gamma) &= \left\{f(z) \in \mathcal{M}_{\mathcal{H}}: 1 - \frac{1}{\gamma}\left[\frac{z\mathcal{D}_q f(z)}{f(z)} + 1\right] \prec \sqrt{1+z}\right\}, \\ \mathcal{MS}^q_{\mathfrak{S}}(\gamma) &= \left\{f(z) \in \mathcal{M}_{\mathcal{H}}: 1 - \frac{1}{\gamma}\left[\frac{z\mathcal{D}_q f(z)}{f(z)} + 1\right] \prec \frac{1+z}{1-z}\right\}. \end{aligned} \tag{16}$$

In the present article, we discussed majorization problems for each of the above-defined classes of $\mathcal{MS}^q(\gamma)$.

2. Majorization Problem for the Classes $\mathscr{MS}^q(\gamma)$

We state the following q–analogue of the result given by Nehari [25] and Salvakumaran et al. [26].

Lemma 1 (see [27]). *If the function $\varphi(z)$ is analytic and $|\varphi(z)|<1$ in $\mathring{\mathscr{U}}$, then*

$$\left|\mathscr{D}_q\varphi(z)\right| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{17}$$

Theorem 1. *Let the function $f(z)\in\mathscr{M}$ and suppose $g\in\mathscr{MS}_{\sin^q}(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\mathring{\mathscr{U}}$, i.e.,*

$$f(z) \ll g(z). \tag{18}$$

Then, for $|z|\leqq r_1$,

$$\left|qz\mathscr{D}_q f(z)\right| \le \left|qz\mathscr{D}_q g(z)\right|, \tag{19}$$

where r_1 is the smallest positive root of the following equation:

$$\left(1-r^2\right)(1-\gamma\sin \mathrm{hr}) - 2qr = 0. \tag{20}$$

Proof. Since $g\in\mathscr{MS}_{\sin^q}(\gamma)$, by using (19), we can find if

$$1-\frac{1}{\gamma}\left[\frac{zq\mathscr{D}_q g(z)}{g(z)}+1\right] \prec \Psi(z), \tag{21}$$

$z\in\mathring{\mathscr{U}}$ and

$$\Psi(z) = 1+\sin z. \tag{22}$$

By Lemma 1, there exists a bounded analytic function w in $\mathscr{U}$ and

$$1-\frac{1}{\gamma}\left[\frac{zq\mathscr{D}_q g(z)}{g(z)}+1\right] = 1+\sin(w(z)), \tag{23}$$

with $w(\infty)=\infty$. From (24), we obtain

$$\frac{zq\mathscr{D}_q g(z)}{g(z)} = -(1+\gamma\sin(w(z))). \tag{24}$$

Let $w(z)=$ Re $i\theta$ with $R\leqq|z|=r<1$ and $-\pi\leqq t\leqq\pi$. By simple calculation, we show that

$$\begin{aligned}|\sin w(z)|^2 &= \left|\sin\left(\mathrm{Re}^{it}\right)\right|^2\\ &= \cos^2(R\cos t)\sin h^2(R\sin t)\\ &\quad + \sin^2(R\cos t)\cos h^2(R\sin t) =: \delta(t).\end{aligned} \tag{25}$$

We easily see that the equation,

$$\delta'(t) = \sin h(2R\sin t) - \sin(2R\cos t) = 0, \tag{26}$$

has five roots in $[-\pi,\pi]$, that is, 0, $\pm\,\pi/2$ and $\pm\,\pi$. Because $\delta(-t)=\delta(t)$, we just need to consider $t\in[0,\pi]$. Also, noticing that $\delta(0)=\delta(\pi)=\sin^2 R$, $\delta(\pi/2)=\sin h^2 R$ and

$$\max\left\{\delta(0),\delta(\pi),\delta\left(\frac{\pi}{2}\right)\right\} = \delta\left(\frac{\pi}{2}\right) = \sin h^2 R. \tag{27}$$

Thus, we have

$$|\sin w(z)| = \left|\sin\left(\mathrm{Re}^{it}\right)\right| \leqq \sin hR \leqq \sinh r. \tag{28}$$

From (24) and (28), we find that

$$\left|\frac{g(z)}{\mathscr{D}_q g(z)}\right| \le \frac{q|z|}{|1+\gamma\sin(w(z))|} \leqq \frac{q|z|}{1-\gamma|\sin(w(z))|} \leqq \frac{rq}{1-\gamma\sinh r}. \tag{29}$$

Since $f(z)$ is majorized by $g(z)$ in $\mathring{\mathscr{U}}$, from (13), we have

$$f(z) = \varphi(z)g(z). \tag{30}$$

By applying q–derivative on the previous equation w.r.t z as in [27] and then multiplying by qz, we have

$$\begin{aligned}qz\mathscr{D}_q f(z) &= qz\mathscr{D}_q\varphi(z)g(z) + qz\varphi(z)\mathscr{D}_q g(z)\\ &= qz\mathscr{D}_q g(z)\left[\varphi(z)+\frac{\mathscr{D}_q\varphi(z)g(z)}{\mathscr{D}_q g(z)}\right].\end{aligned} \tag{31}$$

Noting that $\varphi(z)$ is the Schwartz function, so $\mathfrak{R}(\varphi(z))>0$ in $\mathring{\mathscr{U}}$, $\varphi(z)\neq 0$ for all $z\in\mathring{\mathscr{U}}$, satisfies the q–analogue result given by [25] proved in Lemma 1.

$$\left|\mathscr{D}_q\varphi(z)\right| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{32}$$

Now, using (29) and (32) in (31), we have

$$\left|qz\mathscr{D}_q f(z)\right| \leqq \left|qz\mathscr{D}_q g(z)\right|\left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{rq}{1-\gamma\sinh r}\right]. \tag{33}$$

By setting

$$|z| = r < 1 \text{ and } |\varphi(z)| = \zeta, \quad (0\le\zeta\le 1), \tag{34}$$

we get the inequality

$$\left|qz\mathscr{D}_q f(z)\right| \le Y(r,\zeta)\left|qz\mathscr{D}_q g(z)\right|. \tag{35}$$

We define

$$\Upsilon(r,\zeta) = \zeta + \frac{rq\left(1-\zeta^2\right)}{\left(1-r^2\right)(1-\gamma\sinh r)} \quad (0\leqq\zeta\leqq 1,\ 0<\mathrm{r}<1). \tag{36}$$

To determine r_1, it is sufficient to choose

$$r_1 = \max\{r\in[0,1)\colon Y(r,\zeta)\leqq 1, \quad \forall\zeta\in[0,1]\}, \tag{37}$$

equivalently,

$$r_1 = \max\{r\in[0,1)\colon Y^*(r,\zeta)\geqq 0, \quad \forall\zeta\in[0,1]\}, \tag{38}$$

where

$$\Upsilon^*(r,\zeta) = \left(1-r^2\right)(1-\gamma\sinh r) - rq(1+\zeta). \tag{39}$$

Clearly, when $\zeta=1$, the above function $Y^*(r,\zeta)$ assumes its minimum value, namely,

$$\min\{Y^*(r,\zeta): \zeta \in [0,1]\} = Y^*(r,1) := \psi^*(r), \tag{40}$$

where

$$\psi^*(r) = \left(1-r^2\right)(1-\gamma \sinh r) - 2qr. \tag{41}$$

Next, we obtain the following inequalities:

$$\psi^*(0) = 1 > 0 \text{ and } \psi^*(1) = -2q < 0. \tag{42}$$

There exists r_1 such that $\psi^*(r) \geqq 0$ for all $r \in [0, r_1]$, where r_1 is the smallest positive root of (20). The proof of Theorem 1 is completed. □

Theorem 2. *Let the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{MS}_{\cos^q}(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\mathring{\mathcal{U}}$, i.e.,*

$$f(z) \ll g(z). \tag{43}$$

Then, for $|z| \leqq r_2$,

$$\left|qz\mathcal{D}_q f(z)\right| \leq \left|qz\mathcal{D}_q g(z)\right|, \tag{44}$$

where r_2 is the smallest positive root of the following equation:

$$\left(1-r^2\right)(\gamma(1+\cos hr)+1) - 2qr = 0. \tag{45}$$

Proof. Since $g \in \mathcal{MS}_{\cos^q}(\gamma)$, from (11) and the subordination relationship, we see that

$$\frac{zq\mathcal{D}_q g(z)}{g(z)} = \gamma(1-\cos(w(z))) - 1, \tag{46a}$$

where $w(z)$ is as same as in (24). Similar to (28), we can verify that

$$|\cos(w(z))| = \left|\cos\left(\mathrm{Re}^{i\theta}\right)\right| \leqq \cos hR \leqq \cos hr, \tag{47}$$

where $w(z) = \mathrm{Re}\ i\theta$ with $R \leqq |z| = r < 1$ and $-\pi \leqq t \leqq \pi$.

Combining (46a) and (47), it is easy to see that

$$\left|\frac{g(z)}{qz\mathcal{D}_q g(z)}\right| \leqq \frac{rq}{\gamma(1-\cos hr)+1}. \tag{48}$$

By virtue of (32) as well as (48) in (31), we immediately obtain

$$\left|qz\mathcal{D}_q f(z)\right| \leqq \left|qz\mathcal{D}_q g(z)\right| \left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{rq}{\gamma(1-\cos hr)+1}\right]. \tag{49}$$

In succession, according to (34) and just as the proof of Theorem 1, we can deduce the required result (45). Hence, we have completed the proof of Theorem 2. □

Theorem 3. *Let the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{MS}^q_{\mathrm{SL}}(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\mathring{\mathcal{U}}$, i.e.,*

$$f(z) \ll g(z). \tag{50}$$

Then, for $|z| \leqq r_3$,

$$\left|qz\mathcal{D}_q f(z)\right| \leq \left|qz\mathcal{D}_q g(z)\right|, \tag{51}$$

where r_3 is the smallest positive root of the following equation:

$$\left(1-r^2\right)\left(\gamma\left(1-\sqrt{1-r}\right)-1\right) - 2qr = 0. \tag{52}$$

Proof. Let $g(z) \in \mathcal{MS}^q_{\mathrm{SL}}(\gamma)$. Then, from definition (16) in terms of the Schwartz function, we have

$$1 - \frac{1}{\gamma}\left[\frac{zq\mathcal{D}_q g(z)}{g(z)} + 1\right] = \sqrt{1+w(z)}, \tag{53}$$

which implies

$$\begin{aligned} &\left[1 - \frac{1}{\gamma}\left\{\frac{zq\mathcal{D}_q g(z)}{g(z)} + 1\right\}\right]^2 = (1+w(z)), \\ &\left|1 - \frac{1}{\gamma}\left\{\frac{zq\mathcal{D}_q g(z)}{g(z)} + 1\right\}\right|^2 = |1+w(z)| \leqq 1 - |w(z)|. \end{aligned} \tag{54}$$

Now, as $w(z) = \mathrm{Re}\ i\theta$ with $|w(z)| \leqq R \leqq |z| = r < 1$ and $-\pi \leqq t \leqq \pi$ we have

$$\left|1 - \frac{1}{\gamma}\left\{\frac{zq\mathcal{D}_q g(z)}{g(z)} + 1\right\}\right| \leqq \sqrt{1-r}, \tag{55}$$

which implies

$$\left|\frac{zq\mathcal{D}_q g(z)}{g(z)}\right| \leqq \gamma\left(1-\sqrt{1-r}\right) - 1. \tag{56}$$

Now, as in Theorem 2, we use (32), as well as (56) in (31), and we obtain

$$\left|qz\mathcal{D}_q f(z)\right| \leqq \left|qz\mathcal{D}_q g(z)\right| \left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{rq}{\gamma\left(1-\sqrt{1-r}\right)-1}\right]. \tag{57}$$

Let us take $|z| = r < 1$ and $|\varphi(z)| = \zeta$, $(0 \leqq \zeta \leqq 1)$; we obtain

$$\left|qz\mathcal{D}_q f(z)\right| \leq Y(r,\zeta)\left|qz\mathcal{D}_q g(z)\right|. \tag{58}$$

We define

$$\Upsilon(r,\zeta) = \zeta + \frac{rq\left(1-\zeta^2\right)}{\left(1-r^2\right)\left(\gamma\left(1-\sqrt{1-r}\right)-1\right)} \quad (0 \leqq \zeta \leqq 1, 0 < r < 1). \tag{59}$$

To determine r_3, it is sufficient to choose

$$r_3 = \max\{r \in [0,1) : Y(r,\zeta) \leqq 1, \quad \forall \zeta \in [0,1]\}, \tag{60}$$

equivalently,

$$r_3 = \max\{r \in [0,1) : Y^*(r,\zeta) \geqq 0, \quad \forall \zeta \in [0,1]\}, \tag{61}$$

where

$$Y^*(r,\zeta) = \left(1-r^2\right)\left(\gamma\left(1-\sqrt{1-r}\right)-1\right) - rq(1+\zeta). \tag{62}$$

This clearly shows the result that, when $\zeta = 1$, the above function $Y^*(r,\zeta)$ assumes its minimum value, namely,

$$\min\{\Upsilon^*(r,\zeta):\zeta\in[0,1]\}=\Upsilon^*(r,1)\coloneqq\psi^*(r), \tag{63}$$

where

$$\psi^*(r)=\left(1-r^2\right)\left(\gamma\left(1-\sqrt{1-r}\right)-1\right)-2qr. \tag{64}$$

Next, we obtain the following inequalities:

$$\begin{aligned}\psi^*(0)&=-1<0,\\ \psi^*(1)&=-2q<0,\end{aligned} \tag{65}$$

there exists r_3 such that $\psi^*(r)\geqq 0$ for all $r\in[0,r_3]$, where r_3 is the smallest positive root of (52). The proof of Theorem 3 is completed. □

Theorem 4. *Let the function $f(z)\in\mathscr{M}$ and suppose $g\in\mathscr{M}\mathscr{S}^q_{\mathfrak{S}}(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\mathring{\mathscr{U}}$, i.e.,*

$$f(z)\ll g(z). \tag{66}$$

Then, for $|z|\leqq r_4$,

$$\left|qz\mathscr{D}_q f(z)\right|\leq\left|qz\mathscr{D}_q g(z)\right|, \tag{67}$$

where r_4 is the smallest positive root of the following equation:

$$(1+r)((1-2\gamma)r-1)-2qr=0. \tag{68}$$

Proof. Since $g(z)\in\mathscr{M}\mathscr{S}^q_{\mathfrak{S}}(\gamma)$, we readily obtained from definition (17) that

$$1-\frac{1}{\gamma}\left[\frac{zq\mathscr{D}_q g(z)}{g(z)}+1\right]=\Psi(z), \tag{69}$$

$$\Psi(z)=\frac{1+w(z)}{1-w(z)}, \tag{70}$$

where $w(z)$ is the well-known class of bounded analytic functions in $\mathscr{U}$ such that

$$|w(z)|\leqq|z|\quad(z\in\mathscr{U}). \tag{71}$$

From (69) and (70) and making use of (71), we obtain

$$\left|\frac{zq\mathscr{D}_q g(z)}{g(z)}\right|\leqq\frac{(1-2\gamma)|z|-1}{1-|z|}. \tag{72}$$

Now, just like the above theorems, we use (32) as well as (72) in (31), and we obtain

$$\left|qz\mathscr{D}_q f(z)\right|\leqq\left|qz\mathscr{D}_q g(z)\right|\left[|\varphi(z)|+\frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{rq(1-|z|)}{(1-2\gamma)|z|-1}\right]. \tag{73a}$$

Let us take $|z|=r<1$ and $|\varphi(z)|=\zeta$, $(0\leqq\zeta\leqq 1)$; we obtain

$$\left|qz\mathscr{D}_q f(z)\right|\leq\Upsilon(r,\zeta)\left|qz\mathscr{D}_q g(z)\right|. \tag{74}$$

We define

$$Y(r,\zeta)=\zeta+\frac{rq\left(1-\zeta^2\right)}{(1+r)((1-2\gamma)r-1)},\quad(0\leqq\zeta\leqq 1,0<r<1). \tag{75}$$

To determine r_3, it is sufficient to choose

$$r_3=\max\{r\in[0,1):Y(r,\zeta)\leqq 1,\quad\forall\zeta\in[0,1]\}, \tag{76}$$

equivalently,

$$r_3=\max\{r\in[0,1):Y^*(r,\zeta)\geqq 0,\quad\forall\zeta\in[0,1]\}, \tag{77}$$

where

$$\Upsilon^*(r,\zeta)=(1+r)((1-2\gamma)r-1)-rq(1+\zeta). \tag{78}$$

Clearly, when $\zeta=1$, the above function $Y^*(r,\zeta)$ assumes its minimum value, namely,

$$\min\{Y^*(r,\zeta):\zeta\in[0,1]\}=Y^*(r,1)\coloneqq\psi^*(r), \tag{79}$$

where

$$\psi^*(r)=(1+r)((1-2\gamma)r-1)-2qr. \tag{80}$$

Next, we obtained the following inequalities:

$$\psi^*(0)=-1<0\text{ and }\psi^*(1)=-2(q+2\gamma)<0, \tag{81}$$

there exists r_4 such that $\psi^*(r)\geqq 0$ for all $r\in[0,r_4]$, where r_4 is the smallest positive root of (68). The proof of Theorem 4 is completed. □

3. Conclusion

In this article, we investigated majorization and other results for such subclasses of meromorphic functions, such as the meromorphic univalent function of complex order associated with the q–differential operator. We also highlighted some special cases and new consequences of our main results. In order to conclude our current study, we attract the attention of interested readers to the potential of examining the fundamental or quantum (or q–) extensions of the results obtained in this work. Applications of the q th majorization in the real world will be an interesting and encouraging future study for researchers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors participated in every stage of the research, and all authors read and approved the final manuscript.

Acknowledgments

This work was supported by Universiti Kebangsaan Malaysia.

References

[1] F. H. Jackson, "On q-functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[2] F. H. Jackson, "On q-definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[3] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions," *Complex Variables, Theory and Application: An International Journal*, vol. 14, no. 1-4, pp. 77–84, 1990.

[4] H. M. Srivastava, "Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis," *Iranian Journal of Science and Technology Transaction A-Science*, vol. 44, no. 1, pp. 327–344, 2020.

[5] H. M. Srivastava, "Univalent functions, fractional calculus, and associated generalized hypergeometric functions," in *Univalent Functions, Fractional Calculus, and Their Applications*, H. M. Srivastava and S. Owa, Eds., pp. 329–354, John Wiley & Sons, New York, NY, USA, 1989.

[6] M. Arif and B. Ahmad, "New subfamily of meromorphic multivalent starlike functions in circular domain involving $_{q\text{-}}$ differential operator," *Mathematica Slovaca*, vol. 68, no. 5, pp. 1049–1056, 2018.

[7] B. Ahmad and M. Arif, "New subfamily of meromorphic convex functions in circular domain involving q-operator," *International Journal of Analysis and Applications*, vol. 16, no. 1, pp. 75–82, 2018.

[8] M. Arif, M. U. Haq, and J. L. Liu, "A subfamily of univalent functions associated with q-analogue of noor integral operator," *Journal of Function Spaces*, vol. 2018, pp. 1–5, 2018.

[9] S. Mahmood and J. Sokół, "New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator," *Results in Mathematics*, vol. 71, no. 3-4, pp. 1345–1357, 2017.

[10] H. M. Srivastava, M. Arif, and M. Raza, "Convolution properties of meromorphically harmonic functions defined by a generalized convolution $ q $-derivative operator," *AIMS Mathematics*, vol. 6, pp. 5869–5885, 2021.

[11] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, no. 1, pp. 95–102, 1967.

[12] H. Tang, H. M. Srivastava, S. H. Li, and G. T. Deng, "Majorization results for subclasses of starlike functions based on the sine and cosine functions," *Bulletin of the Iranian Mathematical Society*, vol. 46, no. 2, pp. 381–388, 2020.

[13] M. Arif, M. Ul-Haq, O. Barukab, S. A. Khan, and S. Abullah, "Majorization results for certain subfamilies of analytic functions," *Journal of Function Spaces*, vol. 2021, pp. 1–6, 2021.

[14] N. E. Cho, Z. Oroujy, E. Analouei Adegani, and A. Ebadian, "Majorization and coefficient problems for a general class of starlike functions," *Symmetry*, vol. 12, no. 3, p. 476, 2020.

[15] H. Tang and G. Deng, "Majorization problems for two subclasses of analytic functions connected with the Liu–Owa integral operator and exponential function," *Journal of Inequalities and Applications*, vol. 2018, no. 1, pp. 277–311, 2018.

[16] O. Altintas and H. M. Srivastava, "Some majorization problems associated with p-valently starlike and convex functions of complex order," *East Asian mathematical journal*, vol. 17, no. 2, pp. 175–183, 2001.

[17] O. Altintaş, Ö. Özkan, and H. M. Srivastava, "Majorization by starlike functions of complex order," *Complex Variables, Theory and Application: An International Journal*, vol. 46, no. 3, pp. 207–218, 2001.

[18] S. P. Goyal and P. Goswami, "Majorization for certain classes of meromorphic functions defined by integral operator," *Annales UMCS, Mathematica*, vol. 66, no. 2, pp. 57–62, 2012.

[19] K. Dhuria and R. Mathur, "Majorization for certain classes of meromorphic functions defined by integral operator," *International Journal of Open Problems in Complex Analysis*, vol. 5, no. 3, pp. 50–56, 2013.

[20] S. Bulut, E. A. Adegani, and T. Bulboaca, "Majorization results for a general subclass of meromorphic multivalent functions," *Politeh. Univ. Buchar. Sci. Bull. Ser. A Appl. Math. Phys*, vol. 83, pp. 121–128, 2021.

[21] H. Tang, M. K. Aouf, and G. Deng, "Majorization problems for certain subclasses of meromorphic multivalent functions associated with the Liu-Srivastava operator," *Filomat*, vol. 29, no. 4, pp. 763–772, 2015.

[22] T. Janani and G. Murugusundaramoorthy, "Majorization problems for p- valently meromorphic functions of complex order involving certain integral operator," *Global Journal of Mathematical Analysis*, vol. 2, no. 3, pp. 146–151, 2014.

[23] A. Rasheed, S. Hussain, S. Ghoos Ali Shah, M. Darus, and S. Lodhi, "Majorization problem for two subclasses of meromorphic functions associated with a convolution operator," *AIMS Mathematics*, vol. 5, no. 5, pp. 5157–5170, 2020.

[24] T. Panigrahi and R. El-Ashwah, "Majorization for subclasses of multivalent meromorphic functions defined through iterations and combinations of the Liu-Srivastava operator and a meromorphic analogue of the Cho-Kwon-Srivastava operator," *Filomat*, vol. 31, no. 20, pp. 6357–6365, 2017.

[25] Z. Nehari, *Conformal Mappings*, MacGrow-Hill Book Company, New york, Torinto and london, 1955.

[26] K. A. Selvakumaran, S. D. Purohit, and A. Secer, "Majorization for a class of analytic functions defined by q-differentiation," *Mathematical Problems in Engineering*, vol. 2014, Article ID 653917, 5 pages, 2014.

[27] K. Vijaya, G. Murugusundaramoorthy, and N. E. Cho, "Majorization Problems for Uniformly Starlike functions based on Ruscheweyh q− differential operator defined with exponential function," *Nonlinear Functional Analysis and Applications*, vol. 26, no. 1, pp. 71–81, 2021.