Research Article

Majorization Properties for Certain Subclasses of Meromorphic Function of Complex Order

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1.Introduction and Definitions

Let $\mathcal{M}$ represent the class of meromorphic functions $f$ in the form of

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the punctured disc $\mathcal{U} = \{z: 0 < |z| < 1\} = \mathcal{U} \setminus \{0\}$, where $\mathcal{U} = \mathbb{C} \setminus \{0\}$. For the two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{M}$, there exists a Schwartz function $w$, which is analytic in $\mathcal{U}$ with $|w(z)| \leq |z|$ and $w(0) = 0$, such that $f(z) = g(w(z))$, and the function $f$ is subordinate to $g$, written as $f \prec g$. The following relationship holds if $g$ is univalent:

$$f \prec g \iff f(0) = g(0), \text{ and } f(\mathcal{U}) \subseteq g(\mathcal{U}).$$

(2)

Because of its use in a variety of mathematical sciences, the study of $q$-calculus (quantum calculus) has fascinated and motivated many scholars. One of the primary contributors among all the mathematicians who introduced the concept of $q$-calculus theory was Jackson [1, 2]. The formulation of this concept is widely used to investigate the nature of different structures of function theory, such as $q$-calculus was used in other branches of mathematics.

Though the authors of the first article [3] discussed the geometric nature of $q$-starlike functions, Srivastava [4] laid a solid foundation for the use of $q$-calculus in the context of function theory. Also, in [5], Srivastava provided a brief overview of basic or $q$-calculus operators and fractional $q$-calculus operators, as well as their applications in the geometric function theory of complex analysis. Later, the authors [6–8] investigated a number of useful properties for the newly defined $q$-linear differential operator, and Mehmood and Sokół [9] discussed the Ruscheweyh $q$-differential operator, while Srivastava et al. [10] introduced a generalized operator for meromorphic harmonic functions by using the idea of convolution.

Let $0 < q < 1$. For any nonnegative integer $n$, the $q$-integer number $n$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [0]_q = 0.$$

(3)

In general, we will denote

$$[\delta]_q = \frac{1 - q^\delta}{1 - q},$$

(4)

for a noninteger number $\delta$. Also, the $q$-number shifted factorial is defined by
\[ [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \quad [0]_q! = 1. \]

Clearly,
\[
\lim_{q \to 1^{-}} [n]_q = n,
\lim_{q \to 1^{-}} [n]_q = n!.
\]

Let \( a, q \in \mathbb{C} \) \((|q| < 1)\) and \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Then, the \( q \)-shifted factorial \((a; q)_n\) is defined by
\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=1}^{n} (1 - aq^{j-1}), \quad n \in \mathbb{N}.
\]

Let \( x \in \mathbb{C} \) \(- \{n; n \in \mathbb{N}_0\}\). Then, \( q \)-gamma function is as follows:
\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^{1-q}; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1.
\]

In a subset of \( \mathbb{C} \), the \( q \)-derivative (or \( q \)-difference) operator \( \mathcal{D}_q f \) of function \( f \) is defined by
\[
(\mathcal{D}_q f)(z) = \begin{cases} 
\frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\
\frac{f'(0)}{q}, & z = 0,
\end{cases}
\]

provided that \( f'(0) \) exists. We can easily observe from the definition of (9) that \((\mathcal{D}_q f)(z)\big|_{z=0} = f'(z)\).

Suppose that \( q \in [0, 1) \), then \( q \)-analog derivative of \( f \) as
\[
\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \in \mathcal{U}),
\]
or
\[
(\mathcal{D}_q f)(z) = -\frac{1}{qz} + \sum_{n=1}^{\infty} [n]_q a_n z^n.
\]

In 1967, Mac Gregor [11] introduced the notion of majorization as follows.

**Definition 1.** Let complex-valued functions \( f \) and \( g \) be analytic in \( \mathcal{U} \). We say that \( f \) is majorized by \( g \) in \( \mathcal{U} \) and write
\[
f(z) \ll g(z) \quad (z \in \mathcal{U}),
\]
if there exists a function \( \phi(z) \) (complex-valued function in \( \mathcal{U} \)), satisfying
\[
|\phi(z)| \leq 1 \text{ and } f(z) = \phi(z) g(z) \quad (z \in \mathcal{U}).
\]

Majorization (12) is closely related to the concept of quasi-subordination between analytic functions in \( \mathcal{U} \). Several researchers have published articles on this topic; for example, Tang et al. [12] gave the concept of majorization for subclasses of starlike functions based on the sine and cosine functions, Arif et al. [13] discussed majorization for various new defined classes, Cho et al. [14] obtained coefficient estimates for majorization, and Tang and Deng [15] defined the majorization problem connected with Liu-Owa integral operator and exponential function. This concept is also defined for \( p \)-valent function by Altintas and Srivastava [16] and for complex order by Altintas et al. [17].

The basic goal of this article is to examine and explain the idea of majorization in the context of the meromorphic function. Many researchers have shown their interest in this site. Goyal and Goswami [18, 19] studied this concept for majorization for meromorphic function with the integral operator, Tang et al. [12] discussed it for meromorphic sin and cosine functions, Bulut et al., Tang et al., and Janani [20–22] explained this concept for meromorphic multivalent functions, Rasheed et al. [23] investigated a majorization problem for the class of meromorphic spiral-like functions related with a convolution operator, and Panigrahi and El-Ashwah [24] defined majorization for subclasses of multivalent meromorphic functions through iterations and combinations of the Liu–Srivastava operator and Cho–Kwon–Srivastava operator and much more. In addition, there are several other articles on this topic [18].

Here is the definition of our main function.

**Definition 2.** A function \( f(z) \in \mathcal{M} \) is said to be in the class \( \mathcal{M} \mathcal{S}^q(\gamma) \) of meromorphic functions of complex order \( \gamma \neq 0 \) in \( \mathcal{U} \), if
\[
1 - \frac{1}{\gamma} \left[ \frac{z q \mathcal{D}_q f(z)}{f(z)} + 1 \right] < \Psi(z).
\]

Now, we are going to choose some particular functions instead of \( \Psi(z) \). These choices are
\[
\Psi(z) = 1 + \sin z,
\]
or \( \Psi(z) = \cos z \),
\[
\Psi(z) = \sqrt{1 + z},
\]
or \( \Psi(z) = \frac{1 + z}{1 - z} \),

and by applying the above-mentioned concepts, we now consider the following cases:

\[
\mathcal{M} \mathcal{S}^q_{\sin}(\gamma) = \left\{ f(z) \in \mathcal{M} \mathcal{S}^q : 1 - \frac{1}{\gamma} \left[ \frac{z \mathcal{D}_q f(z)}{f(z)} + 1 \right] < 1 + \sin z \right\},
\]
\[
\mathcal{M} \mathcal{S}^q_{\cos}(\gamma) = \left\{ f(z) \in \mathcal{M} \mathcal{S}^q : 1 - \frac{1}{\gamma} \left[ \frac{z \mathcal{D}_q f(z)}{f(z)} + 1 \right] < \cos z \right\},
\]
\[
\mathcal{M} \mathcal{S}^q_{\sqrt{1+z}}(\gamma) = \left\{ f(z) \in \mathcal{M} \mathcal{S}^q : 1 - \frac{1}{\gamma} \left[ \frac{z \mathcal{D}_q f(z)}{f(z)} + 1 \right] < \sqrt{1 + z} \right\},
\]
\[
\mathcal{M} \mathcal{S}^q_{\frac{1+z}{1-z}}(\gamma) = \left\{ f(z) \in \mathcal{M} \mathcal{S}^q : 1 - \frac{1}{\gamma} \left[ \frac{z \mathcal{D}_q f(z)}{f(z)} + 1 \right] < \frac{1 + z}{1 - z} \right\}.
\]

In the present article, we discussed majorization problems for each of the above-defined classes of \( \mathcal{M} \mathcal{S}^q(\gamma) \).
2. Majorization Problem for the Classes $\mathcal{M}^{q, g}(Y)$

We state the following $q$–analogue of the result given by Nehari [25] and Salvakumaran et al. [26].

**Lemma 1** (see [27]). If the function $\varphi(z)$ is analytic and $|\varphi(z)|<1$ in $\mathcal{U}$, then

$$|D_q \varphi(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}.$$  \hspace{1cm} (17)

**Theorem 1.** Let the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{M}^{\varphi, g}(y)$ if $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, i.e.,

$$f(z) \ll g(z).$$  \hspace{1cm} (18)

Then, for $|z| \leq r_1$,

$$|qzD_q f(z)| \leq |qzD_q g(z)|,$$  \hspace{1cm} (19)

where $r_1$ is the smallest positive root of the following equation:

$$(1-r^2)(1-\gamma \sin/hr) - 2qr = 0.$$  \hspace{1cm} (20)

**Proof.** Since $g \in \mathcal{M}^{\varphi, g}(y)$, by using (19), we can find if

$$1 - \frac{1}{\gamma} \left[ \frac{zqD_q g(z)}{g(z)} + 1 \right] \ll \Psi(z),$$  \hspace{1cm} (21)

$z \in \mathcal{U}$ and

$$\Psi(z) = 1 + \sin z.$$  \hspace{1cm} (22)

By Lemma 1, there exists a bounded analytic function $w$ in $\mathcal{U}$ and

$$1 - \frac{1}{\gamma} \left[ \frac{zqD_q g(z)}{g(z)} + 1 \right] = 1 + |w(z)|,$$  \hspace{1cm} (23)

with $w(\infty) = \infty$. From (24), we obtain

$$\frac{zqD_q g(z)}{g(z)} = -(1 + \gamma |w(z)|).$$  \hspace{1cm} (24)

Let $w(z) = \text{Re} i \theta$ with $R \leq |z| < r$ and $-\pi \leq t \leq \pi$. By simple calculation, we show that

$$|\sin w(z)|^2 = \left| \sin \left( \text{Re} i \theta \right) \right|^2 = \cos^2(R \cos t) \sin^2(R \sin t) + \sin^2(R \cos t) \cos^2(R \sin t) = \delta(t).$$  \hspace{1cm} (25)

We easily see that the equation,

$$\delta'(t) = \sin h(2R \sin t) - \sin(2R \cos t) = 0,$$  \hspace{1cm} (26)

has five roots in $[-\pi, \pi]$, that is, $0, \pm \pi/2$ and $\pm \pi$. Because $\delta(-t) = \delta(t)$, we just need to consider $t \in [0, \pi]$. Also, noticing that $\delta(0) = \delta(\pi) = \sin^2 R$, $\delta(\pi/2) = \sin h^2 R$ and

$$\text{max} \left\{ \delta(0), \delta(\pi), \delta(\pi/2) \right\} = \delta(\pi/2) = \sin h^2 R.$$  \hspace{1cm} (27)

Thus, we have

$$|\sin w(z)| = |\sin(\text{Re} i \theta)| \leq \sin hR \leq \sinh \gamma.$$  \hspace{1cm} (28)

From (42) and (28), we find that

$$qzD_q f(z) \leq qzD_q g(z) \ll g(z) \ll \Psi(z),$$  \hspace{1cm} (29)

since $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, from (13), we have

$$f(z) = \varphi(z)g(z).$$  \hspace{1cm} (30)

By applying $q$–derivative on the previous equation w.r.t $z$ as in [27] and then multiplying by $qz$, we have

$$qzD_q f(z) = qzD_q \varphi(z)g(z) + qz\varphi(z)D_q g(z)$$

$$= qzD_q g(z) \left[ \varphi(z) + \frac{D_q \varphi(z)g(z)}{g(z)} \right].$$  \hspace{1cm} (31)

Noting that $\varphi(z)$ is the Schwartz function, so $\Re(\varphi(z)) > 0$ in $\mathcal{U}$, $\varphi(z) \neq 0$ for all $z \in \mathcal{U}$, satisfies the $q$–analogue result given by [25] proved in Lemma 1.

$$|D_q \varphi(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}.$$  \hspace{1cm} (32)

Now, using (29) and (32) in (31), we have

$$|qzD_q f(z)| \leq \left| qzD_q g(z) \left[ \varphi(z) + \frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{rq}{1-\gamma \sinh r} \right] \right|.$$  \hspace{1cm} (33)

By setting

$$|z| = r < 1 \text{ and } |\varphi(z)| = \zeta, \hspace{1cm} 0 \leq \zeta \leq 1,$$  \hspace{1cm} (34)

we get the inequality

$$|qzD_q f(z)| \leq \left| qzD_q g(z) \left[ \varphi(z) + \frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{rq}{1-\gamma \sinh r} \right] \right|.$$  \hspace{1cm} (35)

We define

$$Y(r, \zeta) = \zeta + \frac{rq(1-\zeta^2)}{(1-r^2)(1-\gamma \sinh r)} \hspace{1cm} (0 \leq \zeta \leq 1, 0 < r < 1).$$  \hspace{1cm} (36)

To determine $r_1$, it is sufficient to choose

$$r_1 = \max\{r \in [0, 1]: Y(r, \zeta) \leq 1, \forall \zeta \in [0, 1]\},$$  \hspace{1cm} (37)

equivalently,

$$r_1 = \max\{r \in [0, 1]: Y^*(r, \zeta) \geq 0, \forall \zeta \in [0, 1]\},$$  \hspace{1cm} (38)

where

$$Y^*(r, \zeta) = \left( 1-r^2 \right)(1-\gamma \sinh r) - rq(1+\zeta).$$  \hspace{1cm} (39)

Clearly, when $\zeta = 1$, the above function $Y^*(r, \zeta)$ assumes its minimum value, namely,
\[
\min \{ Y^* (r, \zeta) : \zeta \in [0, 1] \} = Y^* (r, 1) = \psi^* (r), \tag{40}
\]

where
\[
\psi^* (r) = \left( 1 - r^2 \right) (1 - \psi \sinh r) - 2qr. \tag{41}
\]

Next, we obtain the following inequalities:
\[
\psi^* (0) = 1 > 0 \quad \text{and} \quad \psi^* (1) = -2q < 0. \tag{42}
\]

There exists \( r_1 \) such that \( \psi^* (r) \geq 0 \) for all \( r \in [0, r_1] \), where \( r_1 \) is the smallest positive root of (20). The proof of Theorem 1 is completed. \( \square \)

**Theorem 2.** Let the function \( f (z) \in \mathcal{M} \) and suppose \( g \in \mathcal{M} \mathcal{S}_\text{cof} (\gamma) \) if \( f (z) \) is majorized by \( g(z) \) in \( \mathcal{U} \), i.e.,
\[
f (z) \ll g(z). \tag{43}
\]

Then, for \( |z| \leq r_2 \),
\[
|qz\mathcal{D}_q f (z)| \leq |qz\mathcal{D}_q g (z)|, \tag{44}
\]

where \( r_2 \) is the smallest positive root of the following equation:
\[
\left( 1 - r^2 \right) (\gamma(1 + \cos hr) + 1) - 2qr = 0. \tag{45}
\]

**Proof.** Since \( g \in \mathcal{M} \mathcal{S}_\text{cof} (\gamma) \), from (11) and the subordination relationship, we see that
\[
\frac{qz\mathcal{D}_q g (z)}{g(z)} = \gamma(1 - \cos (w(z))) - 1, \tag{46a}
\]

where \( w(z) \) is as same as in (24). Similar to (28), we can verify that
\[
|\cos (w(z))| = |\cos (Re^{i\theta})| \leq \cos hr \leq \cos hr, \tag{47}
\]

where \( w(z) = \Re \, i\theta \) with \( R_\infty \leq |z| = r \in \mathbb{C} \) and \( -\pi \leq \theta \leq \pi \).

Combining (46a) and (47), it is easy to see that
\[
\left| \frac{g(z)}{qz\mathcal{D}_q g(z)} \right| \leq \frac{rq}{\gamma(1 - \cos hr) + 1}, \tag{48}
\]

By virtue of (32) as well as (48) in (31), we immediately obtain
\[
|qz\mathcal{D}_q f (z)| \leq |qz\mathcal{D}_q g (z)| \left| \frac{\psi(z) + \frac{1 - |\psi(z)|^2}{1 - |z|^2} \frac{rq}{\gamma(1 - \cos hr) + 1}}{\gamma(1 - \sqrt{1 - r}) - 1} \right|. \tag{49}
\]

In succession, according to (34) and just as the proof of Theorem 1, we can deduce the required result (45). Hence, we have completed the proof of Theorem 2. \( \square \)

**Theorem 3.** Let the function \( f (z) \in \mathcal{M} \) and suppose \( g \in \mathcal{M} \mathcal{S}_\text{cof} (\gamma) \) if \( f (z) \) is majorized by \( g(z) \) in \( \mathcal{U} \), i.e.,
\[
f (z) \ll g(z). \tag{50}
\]

Then, for \( |z| \leq r_3 \),
\[
|qz\mathcal{D}_q f (z)| \leq |qz\mathcal{D}_q g (z)|, \tag{51}
\]

where \( r_3 \) is the smallest positive root of the following equation:
\[
\left( 1 - r^2 \right) (\gamma(1 - \sqrt{1 - r}) - 1) - 2qr = 0. \tag{52}
\]

**Proof.** Let \( g(z) \in \mathcal{M} \mathcal{S}_\text{cof}^* (\gamma) \). Then, from definition (16) in terms of the Schwartz function, we have
\[
1 - \frac{1}{\gamma} \left[ \frac{qz\mathcal{D}_q g (z)}{g(z)} + 1 \right] = \sqrt{1 + w(z)}, \tag{53}
\]

which implies
\[
\left[ 1 - \frac{1}{\gamma} \left( \frac{qz\mathcal{D}_q g (z)}{g(z)} + 1 \right) \right]^2 = (1 + w(z)), \tag{54}
\]

\[
\left[ 1 - \frac{1}{\gamma} \left( \frac{qz\mathcal{D}_q g (z)}{g(z)} + 1 \right) \right]^2 = |1 + w(z)| \leq 1 - |w(z)|. \tag{55}
\]

Now, as \( w(z) = \Re \, i\theta \) with \( |w(z)| \leq R_\infty \leq |z| = r \leq 1 \) and \( -\pi \leq \theta \leq \pi \), we have
\[
\left[ 1 - \frac{1}{\gamma} \left( \frac{qz\mathcal{D}_q g (z)}{g(z)} + 1 \right) \right] \leq \sqrt{1 - r}, \tag{56}
\]

which implies
\[
\left| \frac{qz\mathcal{D}_q g (z)}{g(z)} \right| \leq \gamma(1 - \sqrt{1 - r}) - 1. \tag{57}
\]

Now, as in Theorem 2, we use (32), as well as (56) in (31), and we obtain
\[
|qz\mathcal{D}_q f (z)| \leq |qz\mathcal{D}_q g (z)| \left| \frac{\psi(z) + \frac{1 - |\psi(z)|^2}{1 - |z|^2} \frac{rq}{\gamma(1 - \sqrt{1 - r}) - 1}}{\gamma(1 - \sqrt{1 - r}) - 1} \right|. \tag{58}
\]

We define
\[
Y (r, \zeta) = \zeta + \frac{rq(1 - \zeta)}{(1 - r^2)(\gamma(1 - \sqrt{1 - r}) - 1)} \left( 0 \leq \zeta \leq 1, 0 < r < 1 \right). \tag{59}
\]

To determine \( r_3 \), it is sufficient to choose
\[
r_3 = \max \{ r \in [0, 1] : Y (r, \zeta) \leq 1, \quad \forall \zeta \in [0, 1] \}, \tag{60}
\]

equivalently,
\[
r_3 = \max \{ r \in [0, 1] : Y^* (r, \zeta) \leq 0, \quad \forall \zeta \in [0, 1] \}, \tag{61}
\]

where
\[
Y^* (r, \zeta) = \left( 1 - r^2 \right) (\gamma(1 - \sqrt{1 - r}) - 1) - rq(1 + \zeta). \tag{62}
\]

This clearly shows the result that, when \( \zeta = 1 \), the above function \( Y^* (r, \zeta) \) assumes its minimum value, namely,
where

$$\psi^*(r) = \left(1 - r^2\right)(\gamma(1 - \sqrt{1-r}) - 1) - 2qr.$$  

Next, we obtain the following inequalities:

$$\psi^*(0) = -1 < 0,$$

$$\psi^*(1) = -2q < 0,$$  

there exists $r_3$ such that $\psi^*(r) \geq 0$ for all $r \in [0,r_3]$, where $r_3$ is the smallest positive root of (52). The proof of Theorem 3 is completed. \hfill \Box

**Theorem 4.** Let the function $f(z) \in \mathcal{M}$ and suppose $g \in \mathcal{M} \, \mathcal{D}_q^*(\gamma)$ if $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, i.e.,

$$f(z) \ll g(z).$$  

Then, for $|z| \leq r_4$, 

$$|qz \mathcal{D}_q f(z)| \leq |qz \mathcal{D}_q g(z)|,$$  

where $r_4$ is the smallest positive root of the following equation:

$$(1 + r)((1 - 2\gamma)r - 1) - 2qr = 0.$$  

**Proof.** Since $g(z) \in \mathcal{M} \, \mathcal{D}_q^*(\gamma)$, we readily obtained from definition (17) that

$$1 - \frac{1}{\gamma} \left[ \frac{zq \mathcal{D}_q g(z)}{g(z)} + 1 \right] = \Psi(z),$$

$$\Psi(z) = \frac{1 + w(z)}{1 - w(z)},$$  

where $w(z)$ is the well-known class of bounded analytic functions in $\mathcal{U}$ such that

$$|w(z)| \leq |z| \quad (z \in \mathcal{U}).$$  

From (69) and (70) and making use of (71), we obtain

$$|zq \mathcal{D}_q g(z)| \leq \frac{(1 - 2\gamma)|z| - 1}{1 - |z|}.$$  

Now, just like the above theorems, we use (32) as well as (72) in (31), and we obtain

$$|qz \mathcal{D}_q f(z)| \leq \left[ |\varphi(z)| \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \frac{rq(1 - |z|)}{(1 - 2\gamma)|z| - 1} \right].$$  

Let us take $|z| = r < 1$ and $|\varphi(z)| = \zeta$, $0 \leq \zeta \leq 1$; we obtain

$$|qz \mathcal{D}_q f(z)| \leq Y(r, \zeta) |qz \mathcal{D}_q g(z)|.$$  

We define

$$Y(r, \zeta) = \zeta + \frac{rq(1 - \zeta^2)}{(1 + r)(1 - 2\gamma)r - 1) - 2qr}.$$  

To determine $r_3$, it is sufficient to choose

$$r_3 = \max\{r \in [0,1] : \min\{Y^*(r, \zeta) : \zeta \in [0,1]\} = Y^*(r, 1) = \psi^*(r),$$  

equivalently,

$$r_3 = \max\{r \in [0,1] : Y^*(r, \zeta) \geq 0, \quad \forall \zeta \in [0,1]\}.$$  

Clearly, when $\zeta = 1$, the above function $Y^*(r, \zeta)$ assumes its minimum value, namely,

$$\min\{Y^*(r, \zeta) : \zeta \in [0,1]\} = Y^*(r, 1) = \psi^*(r),$$  

where

$$\psi^*(r) = (1 + r)((1 - 2\gamma)r - 1) - 2qr.$$  

Next, we obtained the following inequalities:

$$\psi^*(0) = -1 < 0 \quad \text{and} \quad \psi^*(1) = -2(\gamma + 2\gamma) < 0,$$  

there exists $r_4$ such that $\psi^*(r) \geq 0$ for all $r \in [0,r_4]$, where $r_4$ is the smallest positive root of (68). The proof of Theorem 4 is completed. \hfill \Box

**3. Conclusion**

In this article, we investigated majorization and other results for such subclasses of meromorphic functions, such as the meromorphic univalent function of complex order associated with the $q$-differential operator. We also highlighted some special cases and new consequences of our main results. In order to conclude our current study, we attract the attention of interested readers to the potential of examining the fundamental or quantum (or $q$-) extensions of the results obtained in this work. Applications of the $q$-th majorization in the real world will be an interesting and encouraging future study for researchers.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors participated in every stage of the research, and all authors read and approved the final manuscript.
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