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Research Article

The Analysis of Fractional-Order Proportional Delay Physical Models via a Novel Transform

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In this paper, we deal with an alternative analytical analysis of fractional-order partial differential equations with proportional delay, achieved by applying Yang decomposition method, where the fractional derivative is taken in Caputo sense. The suggested series results are discovered to quickly converge to an exact solution. The computation of three test problems of fractional-order with proportional delay partial differential equations was presented to confirm the validity and efficiency of suggested method. The system appears to be a very dependable, effective, and powerful method for solving a variety of physical problems that arise in engineering and science.

1. Introduction

Because of its numerous applications in modelling, $\mu(0,\eta)=\beta_0(\eta),\quad \mu(1,\eta)=\beta_1(\eta),\quad \eta\in(0,T]$, the study of fractional calculus has recently attracted a lot of attention of fluid dynamics, electrodynamics, biological mathematics, signal processing, and numerous different fields of science. The researchers have written a number of books on fractional calculus such as Ross and Miller [1], Zhou et al. [2], Kilbas et al. [3], Podlubny [4], and Agarwal et al. [5]. Furthermore, delay differential problems have various uses in transportation networks, biological and chemical, which can be identified in [6–11]. In this paper, we suggest with time delay coefficient variable partial differential equation (PDE), provided by

$$D_\eta^\delta\mu(\zeta,\eta)-\left(\nu(\zeta)\mu_\zeta(\zeta,\eta)\right)_\zeta+\mu^p(\zeta,\eta)\mu_\zeta(\zeta,\eta)=g(\zeta,\eta,\mu(\zeta,\eta-\eta)),\quad (\zeta,\eta)\in(0,1)\times(0,T],\tag{1}$$

the boundaries condition and initial condition

$$\mu(\zeta,\eta)=\varphi(\zeta,\eta),\quad (\zeta,\eta)\in[0,1]\times[-\eta,0],\tag{2}$$

where $\mu(\zeta,\eta)\in C^{2,1}([0,1]\times[-\eta,T])$ is the unknown term with ζ,η space and time variables. When $\nu(\zeta)$ is variable coefficient of space which provides $0<c_0\le\nu(\zeta)\le c_1$, p is any positive integer and $\varphi(\zeta,\eta)$ is sufficiently smooth prehistory function.

For $\delta=1$, (1) becomes classical semilinear with time delay convection reaction diffusion equation (CRDE). Numerous numeric systems have been investigated at $\delta=1$. For example, Zhang and Zhang [12] described a linearize splitting multicompact system to overcome nonlinear PDEs with delay time. Pao [13] suggested a monotone iterative algorithm for the numeric solution of a delay CRDE. For nonlinear CRDEs with time delay, Zhang et al. [14]

introduced explicit implicit multistep finite element technique. For variable coefficient time delay PDEs, Ran and He [15] suggested a linearized Crank–Nicolson method.

In the meantime, fractional with delay differential equations have attracted the interest of investigators due to its wide implementation in dynamics population, finance, automatic control, etc. [16–18]. Furthermore, by using variable coefficients, more complex natural phenomena can be introduced in [19–21]. It is not straightforward to solve fractional delay PDEs effectively and accurately. The evolution of the dependent variable of fractional delay partial differential equations is at time η, but also on all earlier findings, due to the nature of history dependence of a fractional derivative. In a few cases, analytical solutions to fractional differential equations with delay can be found. Ouyang [22], for example, investigated the uniqueness and existence of results for nonlinear fractional-order delay PDEs. To analyze time fractional delay PDEs, Rihan [23] suggest a different approach which is unconditionally implicit stable. For fractional nonlinear delay diffusion equations, Pimenov and Ahmed [24] suggested a numerical solution for a class of time fractional diffusion equations with delay. For semilinear fractional PDEs with time delay, Zang et al. [25] suggested a compact finite difference approach. For the numeric solution of semilinear fractional delay partial differential equations, Mohebbi [26] suggested a spectral collocation and finite difference methods. For the mathematical investigation of variable order delay Burgers equation, Sweilam et al. [27] presented a nonstandard weighted average finite difference approach. To explore the propagation of the population growth model, Jaradat et al. [28] suggested two numerical schemes based on fractional homotopy perturbation and power series methods.

Adomian decomposition method and Yang transform are two well-known techniques that have been applied to have introduced Yang decomposition method. Several physical phenomena which are modeled by partial differential equations and fractional partial differential equations are solved by applying Adomian transform decomposition method, such as the analytical investigation of fractional system partial differential equations are proposed in [29–31], the analysis of nonlinear ordinary differential equations is successfully shown in [32], nonlinear partial differential equations [33–35], fractional unsteady flow of fractional telegraph equations [36], polytropic gas model [37], fractional Schrodinger equation [38], and Fokker–Plank equation [39–41].

2. Basic Definitions

In this portion, we described a few basic concepts of fractional calculus along with properties of Laplace transformation.

2.1. Definition. The Caputo fractional derivative is define as

$$D_{\eta}^{\delta}\nu(\psi,\eta)=\frac{1}{\Gamma(k-\delta)}\int_{0}^{\eta}(\eta-\rho)^{k-\delta-1}\nu^{(k)}(\psi,\rho)d\rho,\quad k-1<\delta\leq k,\quad k\in N. \tag{3}$$

2.2. Definition. Xiao Jun Yang introduced the Yang Laplace transform in 2018. The Yang transformation for a term $\nu(\eta)$ is determined by $Y\{\nu(\eta)\}$ or $M(u)$ and is given as

$$Y\{\nu(\eta)\}=M(u)=\int_{0}^{\infty}e^{-\eta/u}\nu(\eta)d\eta\ ,\quad \eta>0,u\in(-\eta_1,\eta_2). \tag{4}$$

The inverse Yang transformation is expressed as

$$Y^{-1}\{M(u)\}=\nu(\eta), \tag{5}$$

where Y^{-1} is the inverse Yang operator.

2.3. Definition. The nth derivatives of Yang transformation are given as

$$Y\{\nu^{n}(\eta)\}=\frac{M(u)}{u^{n}}-\sum_{k=0}^{n-1}\frac{\nu^{k}(0)}{u^{n-k-1}},\quad \forall n=1,2,3,\ldots. \tag{6}$$

2.4. Definition. The fractional-order derivatives of Yang transformation are define as

$$Y\{\nu^{\delta}(\eta)\}=\frac{M(u)}{u^{\gamma}}-\sum_{k=0}^{n-1}\frac{\nu^{k}(0)}{u^{\delta-(k+1)}},\quad 0<\delta\leq n. \tag{7}$$

3. Idea of Yang Decomposition Technique

In this portion, the YDM to investigate the general solution of fractional delay partial differential equations:

$$D_{\eta}^{\delta}\mu(\zeta,\eta)+L\mu(\zeta,\eta)+N\mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)=q(\zeta,\eta),\quad \zeta,\eta\geq 0,\ell-1<\delta<\ell,$$
$$\mu(\zeta,\eta)=k(\zeta), \tag{8}$$

where $D^{\delta}=\partial^{\delta}/\partial\eta^{\delta}$ is the Caputo operator $\delta,\ell\in\mathbb{N}$, where L is linear and N is the nonlinear term, and q is the source term. Using the Yang transformation to (8), we have

$$\mathbb{Y}\left[D^{\delta}\mu(\zeta,\eta)\right]+\mathbb{Y}\left[L\mu(\zeta,\eta)+N\mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\right]=\mathbb{Y}[q(\zeta,\eta)], \tag{9}$$

and applying the Yang transformation of differentiation property, we get

$$\frac{1}{s^{\delta}}\mathbb{Y}[\mu(\zeta,\eta)] - s\mu(\zeta,0) = \mathbb{Y}[q(\zeta,\eta)] - \mathbb{Y}\left[L\mu(\zeta,\eta) + N\mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\right],$$
$$\mathbb{Y}[\mu(\zeta,\eta)] = sk(\zeta) + s^{\delta}\mathbb{Y}[q(\zeta,\eta)] - s^{\delta}\mathbb{Y}\left[L\mu(\zeta,\eta) + N\mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\right]. \tag{10}$$

The YDM solution $\mu(\zeta,\eta)$ is represented by the following infinite series:

$$\mu(\zeta,\eta) = \sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta), \tag{11}$$

and the nonlinear functions are expressed by the Adomian polynomials,

$$N\mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right) = \sum_{\ell=0}^{\infty}A_{\ell},$$
$$A_{\ell} = \frac{1}{\ell!}\left[\frac{d^{\ell}}{d\lambda^{\ell}}\left[N\sum_{\ell=0}^{\infty}\left(\lambda^{\ell}\mu_{\ell}\right)\right]\right]_{\lambda=0}, \quad \ell = 0,1,2,\ldots, \tag{12}$$

and substituting equations (10) and (11) in (10), we get

$$\mathbb{Y}\left[\sum_{\ell=0}^{\infty}\mu(\zeta,\eta)\right] = sk(\zeta) + s^{\delta}\mathbb{Y}[q(\zeta,\eta)] - s^{\delta}\mathbb{Y}\left[L\sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta) + \sum_{\ell=0}^{\infty}A_{\ell}\right],$$
$$\mathbb{Y}[\mu_0(\zeta,\eta)] = s\mu(\zeta,0) + s^{\delta}\mathbb{Y}[q(\zeta,\eta)],$$
$$\mathbb{Y}[\mu_1(\zeta,\eta)] = -s^{\delta}\mathbb{Y}[L\mu_0(\zeta,\eta) + A_0]. \tag{13}$$

Generally, we can write

$$\mathbb{Y}[\mu_{\ell+1}(\zeta,\eta)] = -s^{\delta}\mathbb{Y}[L\mu_{\ell}(\zeta,\eta) + A_{\ell}], \quad \ell \geq 1. \tag{14}$$

Applying the inverse Yang transform in (14), we get

$$\mu_0(\zeta,\eta) = k(\zeta,\eta),$$
$$\mu_{\ell+1}(\zeta,\eta) = -\mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}[L\mu_{\ell}(\zeta,\eta) + A_{\ell}]\right]. \tag{15}$$

4. Numerical Results

Example 1. Consider the generalized Burgers equation with proportional delay as defined by [42]:

$$\frac{\partial^{\delta}\mu}{\partial\eta^{\delta}} - \frac{\partial^2\mu(\zeta,\eta)}{\partial\zeta^2} - \mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu(\zeta,(\eta/2))}{\partial\zeta} - \frac{1}{2}\mu(\zeta,\eta) = 0, \tag{16}$$

with initial condition

$$\mu(\zeta,0) = \zeta. \tag{17}$$

Taking Yang transform of (16), we get

$$\frac{1}{s^{\delta}}\mathbb{Y}[\mu(\zeta,\eta)] - s\mu(\zeta,0) = \mathbb{Y}\left[\frac{\partial^2\mu(\zeta,\eta)}{\partial\zeta^2} + \mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu(\zeta,(\eta/2))}{\partial\zeta} + \frac{1}{2}\mu(\zeta,\eta)\right]. \tag{18}$$

Applying inverse Yang transform,

$$\mu(\zeta,\eta) = \mathbb{Y}^{-1}\left[s\mu(\zeta,0) - s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu(\zeta,\eta)}{\partial\zeta^2} + \mu\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu(\zeta,(\eta/2))}{\partial\zeta} + \frac{1}{2}\mu(\zeta,\eta)\right\}\right]. \tag{19}$$

Using ADM procedure, we get

$$\begin{aligned}
&\mu_0(\zeta,\eta) = \mathbb{Y}^{-1}[s\mu(\zeta,0)] = \zeta,\\
&\sum_{\ell=0}^{\infty}\mu_{\ell+1}(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\sum_{\ell=0}^{\infty}\left(\mu_{\zeta\zeta}(\zeta,\eta)\right)_{\ell} + \sum_{\ell=0}^{\infty}A_{\ell}\left(\mu\mu_{\zeta}\right) + \frac{1}{2}\sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta)\right\}\right], \quad \ell = 0,1,2,\ldots,\\
&A_0\left(\mu\mu_{\zeta}\right) = \mu_0\frac{\partial\mu_0}{\partial\zeta},\\
&A_1\left(\mu\mu_{\zeta}\right) = \mu_0\frac{\partial\mu_1}{\partial\zeta} + \mu_1\frac{\partial\mu_0}{\partial\zeta},\\
&A_2\left(\mu\mu_{\zeta}\right) = \mu_0\frac{\partial\mu_2}{\partial\zeta} + \mu_1\frac{\partial\mu_1}{\partial\zeta} + \mu_2\frac{\partial\mu_0}{\partial\zeta},
\end{aligned} \tag{20}$$

for $\ell = 0$,

$$\begin{aligned}
&\mu_1(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu_0(\zeta,\eta)}{\partial\zeta^2} + \mu_0\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_0(\zeta,(\eta/2))}{\partial\zeta} + \frac{1}{2}\mu_0(\zeta,\eta)\right\}\right],\\
&\mu_1(\zeta,\eta) = \zeta\frac{\eta^{\delta}}{\Gamma(\delta+1)}.
\end{aligned} \tag{21}$$

The subsequent terms are

$$\begin{aligned}
&\mu_2(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu_1(\zeta,\eta)}{\partial\zeta^2} + \mu_0\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_1(\zeta,(\eta/2))}{\partial\zeta} + \mu_1\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_0(\zeta,(\eta/2))}{\partial\zeta} + \frac{1}{2}\mu_1(\zeta,\eta)\right\}\right],\\
&\mu_2(\zeta,\eta) = \frac{\zeta\left(2+2^{\delta}\right)\eta^{2\delta}}{2^{\delta}2\Gamma(2\delta+1)},\\
&\mu_3(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu_2(\zeta,\eta)}{\partial\zeta^2} + \mu_0\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_2(\zeta,\eta/2)}{\partial\zeta} + \mu_1\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_1(\zeta,\eta/2)}{\partial\zeta} + \mu_2\left(\frac{\zeta}{2},\frac{\eta}{2}\right)\frac{\partial\mu_0(\zeta,\eta/2)}{\partial\zeta} + \frac{1}{2}\mu_2(\zeta,\eta)\right\}\right],\\
&\mu_3(\zeta,\eta) = \frac{\zeta\eta^{3\delta}}{4\Gamma(3\delta+1)}\left(1 + \frac{2}{2^{\delta}} + \frac{2}{2^{2\delta}} + \frac{2^2}{2^{3\delta}} + \frac{2\Gamma(1+2\delta)}{2^{\delta}\Gamma(1+\delta)^2}\right),\\
&\mu_4(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left[\begin{array}{c}\dfrac{\partial^2\mu_3(\zeta,\eta)}{\partial\zeta^2} + \mu_0\left(\dfrac{\zeta}{2},\dfrac{\eta}{2}\right)\dfrac{\partial\mu_3(\zeta,(\eta/2))}{\partial\zeta} + \mu_1\left(\dfrac{\zeta}{2},\dfrac{\eta}{2}\right)\dfrac{\partial\mu_2(\zeta,(\eta/2))}{\partial\zeta}\\ + \mu_2\left(\dfrac{\zeta}{2},\dfrac{\eta}{2}\right)\dfrac{\partial\mu_1(\zeta,(\eta/2))}{\partial\zeta} + \mu_3\left(\dfrac{\zeta}{2},\dfrac{\eta}{2}\right)\dfrac{\partial\mu_0(\zeta,(\eta/2))}{\partial\zeta} + \dfrac{1}{2}\mu_3(\zeta,\eta)\end{array}\right]\right],\\
&\mu_4(\zeta,\eta) = \frac{\zeta\eta^{4\delta}}{8\Gamma(4\delta+1)}\left[\begin{array}{c}1 + \dfrac{2^9}{2^{6\delta}} + \dfrac{2^8}{2^{5\delta}} + \dfrac{3\times 2^7}{2^{3\delta}} + \dfrac{2^7}{2^{2\delta}} + \dfrac{2^7}{2^{\delta}} + \dfrac{2^8}{2^{4\delta}}\\ + \left(\dfrac{2^8}{2^{5\delta}} + \dfrac{2^7}{2^{2\delta}}\right)\dfrac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + \left(\dfrac{2^9}{2^{4\delta}} + \dfrac{2^8}{2^{3\delta}}\right)\dfrac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\end{array}\right].
\end{aligned} \tag{22}$$

The YDM series form solution for Examples 1–3 is

$$\mu(\zeta,\eta) = \mu_0(\zeta,\eta) + \mu_1(\zeta,\eta) + \mu_2(\zeta,\eta) + \mu_3(\zeta,\eta) + \mu_4(\zeta,\eta)\ldots,$$

$$\mu(\zeta,\eta) = \zeta + \zeta\frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\zeta\left(2+2^{\delta}\right)\eta^{2\delta}}{2^{\delta}2\Gamma(2\delta+1)} + \frac{\zeta\eta^{3\delta}}{4\Gamma(3\delta+1)}\left(1+\frac{2}{2^{\delta}}+\frac{2}{2^{2\delta}}+\frac{2^2}{2^{3\delta}}+\frac{2\Gamma(1+2\delta)}{2^{\delta}\Gamma(1+\delta)^2}\right)$$
$$+\frac{\zeta\eta^{4\delta}}{8\Gamma(4\delta+1)}\left[1+\frac{2^9}{2^{6\delta}}+\frac{2^8}{2^{5\delta}}+\frac{3\times 2^7}{2^{3\delta}}+\frac{2^7}{2^{2\delta}}+\frac{2^7}{2^{\delta}}+\frac{2^8}{2^{4\delta}}+\left(\frac{2^8}{2^{5\delta}}+\frac{2^7}{2^{2\delta}}\right)\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}+\left(\frac{2^9}{2^{4\delta}}+\frac{2^8}{2^{3\delta}}\right)\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right]+\ldots,$$

$$\mu(\zeta,\eta) = \zeta + \zeta\frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\zeta\left(2+2^{\delta}\right)\eta^{2\delta}}{2^{\delta}2\Gamma(2\delta+1)} + \frac{\zeta\eta^{3\delta}}{4\Gamma(3\delta+1)}\left(1+\frac{2}{2^{\delta}}+\frac{2}{2^{2\delta}}+\frac{2^2}{2^{3\delta}}+\frac{2\Gamma(1+2\delta)}{2^{\delta}\Gamma(1+\delta)^2}\right)$$
$$+\frac{\zeta\eta^{4\delta}}{8\Gamma(4\delta+1)}\left[1+\frac{2^9}{2^{6\delta}}+\frac{2^8}{2^{5\delta}}+\frac{3\times 2^7}{2^{3\delta}}+\frac{2^7}{2^{2\delta}}+\frac{2^7}{2^{\delta}}+\frac{2^8}{2^{4\delta}}+\left(\frac{2^8}{2^{5\delta}}+\frac{2^7}{2^{2\delta}}\right)\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}+\left(\frac{2^9}{2^{4\delta}}+\frac{2^8}{2^{3\delta}}\right)\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right]+\ldots, \tag{23}$$

when $\delta = 1$, then YDM series form result is

$$\mu(\zeta,\eta) = \zeta\left(1+\eta+\frac{\eta^2}{2!}+\frac{\eta^3}{3!}+\frac{\eta^4}{4!}+\ldots\right). \tag{24}$$

The exact result is

$$\mu(\zeta,\eta) = \zeta e^{\eta}. \tag{25}$$

In Figure 1, the approximate solution graph of problem 4.1, at $\delta = 1$ and 0.8, which shows the close contact with each other. Figure 2 shows the approximate result graph of problem 4.1, at $\delta = 0.6$ and 0.4. Figure 3 shows the analytical solution graph of different fractional-order of δ.

Example 2. Consider the fractional partial differential equation with proportional delay as defined in [42]:

$$\frac{\partial^{\delta}\mu(\zeta,t)}{\partial\eta^{\delta}} - \mu\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu(\zeta,\eta/2)}{\partial\zeta^2} + \mu(\zeta,\eta) = 0, \tag{26}$$

with the initial condition

$$\mu(\zeta,0) = \zeta^2. \tag{27}$$

Taking Yang transform of (26), we get

$$\frac{1}{s^{\delta}}\mathbb{Y}[\mu(\zeta,\eta)] - s\mu(\zeta,0) = \mathbb{Y}\left[\mu\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu(\zeta,\eta/2)}{\partial\zeta^2} - \mu(\zeta,\eta)\right]. \tag{28}$$

Applying inverse Yang transform,

$$\mu(\zeta,\eta) = \mathbb{Y}^{-1}\left[s\mu(\zeta,0) = \mathbb{Y}\left[\mu\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu(\zeta,\eta/2)}{\partial\zeta^2} - \mu(\zeta,\eta)\right]\right]. \tag{29}$$

Using ADM procedure, we get

$$\begin{aligned}
\mu_0(\zeta,\eta) &= \mathbb{Y}^{-1}[s\mu(\zeta,0)] = \zeta^2,\\
\sum_{\ell=0}^{\infty}\mu_{\ell+1}(\zeta,\eta) &= \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\sum_{\ell=0}^{\infty}B_{\ell}\left(\mu\mu_{\zeta\zeta}\right) - \sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta)\right\}\right], \quad \ell = 0,1,2,\\
B_0\left(\mu\mu_{\zeta\zeta}\right) &= \mu_0\frac{\partial^2\mu_0}{\partial\zeta^2},\\
B_1\left(\mu\mu_{\zeta\zeta}\right) &= \mu_0\frac{\partial^2\mu_1}{\partial\zeta^2} + \mu_1\frac{\partial^2\mu_0}{\partial\zeta^2},\\
B_2\left(\mu\mu_{\zeta\zeta}\right) &= \mu_0\frac{\partial^2\mu_2}{\partial\zeta^2} + \mu_1\frac{\partial^2\mu_1}{\partial\zeta^2} + \mu_2\frac{\partial^2\mu_0}{\partial\zeta^2},
\end{aligned} \tag{30}$$

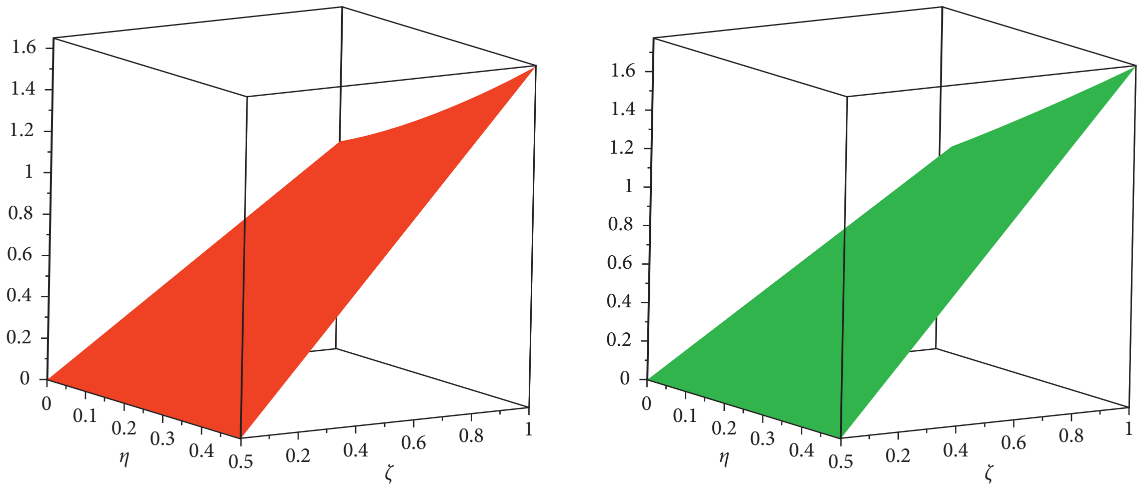

FIGURE 1: The approximate solution graph of problem 4.1, at $\delta = 1$ and 0.8.

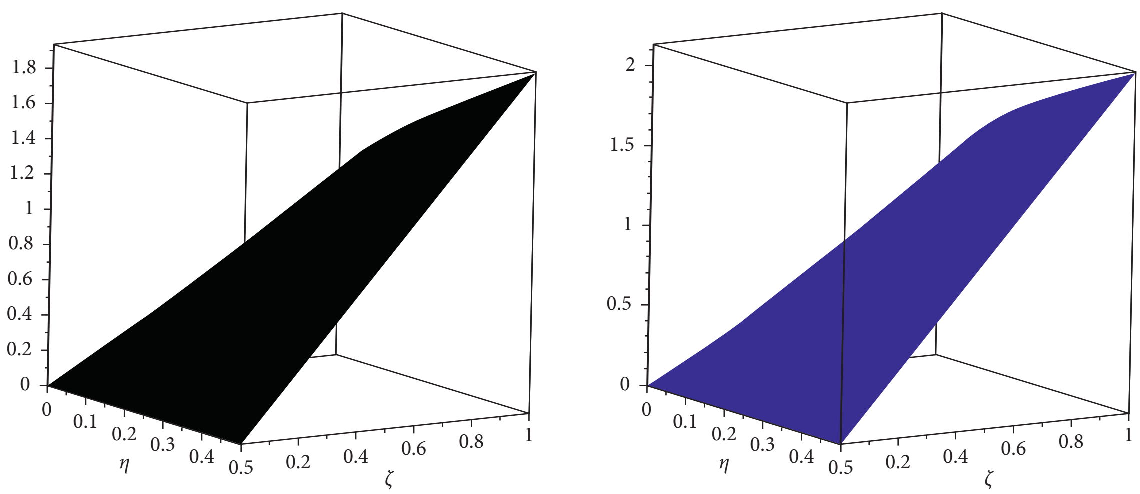

FIGURE 2: The approximate result graph of problem 4.1, at $\delta = 0.6$ and 0.4.

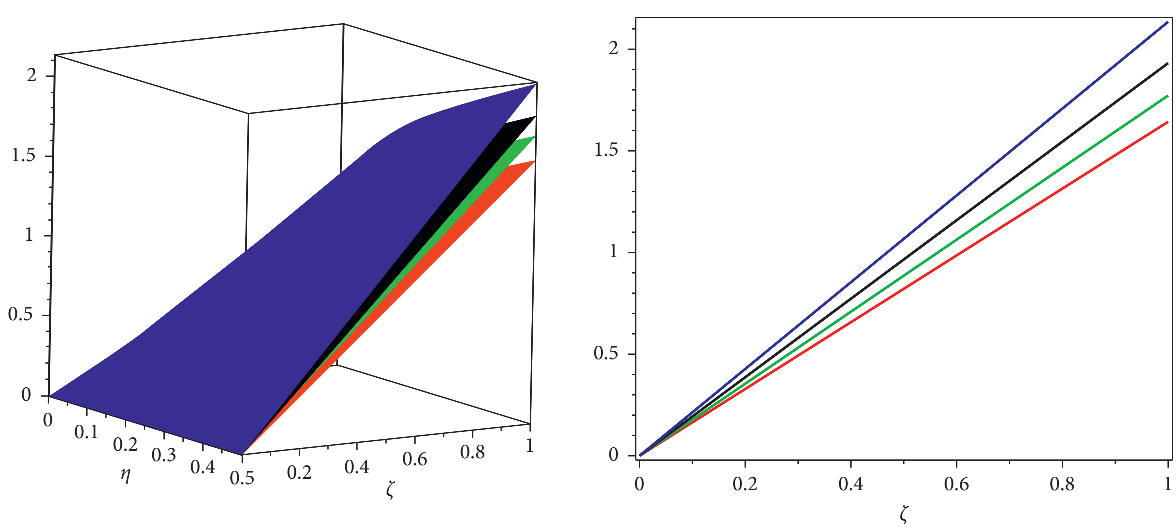

FIGURE 3: The approximate result graph of problem 4.1, at different fractional-order of δ.

for $\ell = 0$,

$$\mu_1(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\mu_0\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu_0(\zeta,\eta/2)}{\partial\zeta^2} + \mu_0(\zeta,\eta)\right\}\right],$$
$$\mu_1(\zeta,\eta) = \zeta^2\frac{\eta^{\delta}}{\Gamma(\delta+1)}. \tag{31}$$

The subsequent terms are

$$\mu_2(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\mu_0\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu_1(\zeta,\eta/2)}{\partial\zeta^2} + \mu_1\left(\zeta,\frac{\eta}{2}\right)\frac{\partial^2\mu_0(\zeta,\eta/2)}{\partial\zeta^2} - \mu_1(\zeta,\eta)\right\}\right],$$
$$\mu_2(\zeta,\eta) = \frac{\zeta^2\left(2-2^{\delta}\right)\eta^{2\delta}}{2^{\delta}\Gamma(2\delta+1)},$$
$$\mu_3(\zeta,\eta) = \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\mu_0\left(\zeta,\frac{\eta}{2}\right)\frac{\partial\mu_2(\zeta,\eta/2)}{\partial\zeta} + \mu_1\left(\zeta,\frac{\eta}{2}\right)\frac{\partial\mu_1(\zeta,\eta/2)}{\partial\zeta} + \mu_2\left(\zeta,\frac{\eta}{2}\right)\frac{\partial\mu_0(\zeta,\eta/2)}{\partial\zeta} - \mu_2(\zeta,\eta)\right\}\right],$$
$$\mu_3(\zeta,\eta) = \frac{\zeta^2\eta^{3\delta}}{\Gamma(3\delta+1)}\left(1 - \frac{2}{2^{\delta}} - \frac{2^2}{2^{2\delta}} + \frac{2^4}{2^{3\delta}} + \frac{2\Gamma(2\delta+1)}{2^{\delta}\Gamma(\delta+1)^2}\right). \tag{32}$$

The YDM result for problem 4.2 is

$$\mu(\zeta,\eta) = \mu_0(\zeta,\eta) + \mu_1(\zeta,\eta) + \mu_2(\zeta,\eta) + \mu_3(\zeta,\eta) + \mu_4(\zeta,\eta)\ldots,$$
$$\mu(\zeta,\eta) = \zeta^2 + \zeta^2\frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\zeta^2\left(2-2^{\delta}\right)\eta^{2\delta}}{2^{\delta}\Gamma(2\delta+1)} + \frac{\zeta^2\eta^{3\delta}}{\Gamma(3\delta+1)}\left(1 - \frac{2}{2^{\delta}} - \frac{2^2}{2^{2\delta}} + \frac{2^4}{2^{3\delta}} + \frac{2\Gamma(2\delta+1)}{2^{\delta}\Gamma(\delta+1)^2}\right)\ldots, \tag{33}$$

when $\delta = 1$, then YDM series form result is

$$\mu(\zeta,\eta) = \zeta^2\left(1 + \eta + \frac{\eta^2}{2!} + \frac{\eta^3}{3!} + \frac{\eta^4}{4!} + \ldots\right). \tag{34}$$

The exact solution

$$\mu(\zeta,\eta) = \zeta^2 e^{\eta}. \tag{35}$$

In Figure 4, approximate and exact result graph of problem 4.2, at $\delta = 1$, which shows the close contact with each other. Figure 5 shows the approximate solution graph of problem 4.1, at different fractional order of $\delta = 1$ with respect to ζ and η. Figure 6 shows the different fractional-order graphs of problem 4.2, with respect to time.

Example 3. Consider the fractional partial differential equation with proportional delay as given by [42]:

$$\frac{\partial^{\delta}\mu(\zeta,\eta)}{\partial\eta^{\delta}} - \frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta) = 0, \tag{36}$$

with initial condition

$$\mu(\zeta,0) = \zeta^2. \tag{37}$$

Using Yang transformation of (36), we get

$$\frac{1}{s^{\delta}}\mathbb{Y}[\mu(\zeta,\eta)] - s\mu(\zeta,0) = \mathbb{Y}\left[\frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta)\right]. \tag{38}$$

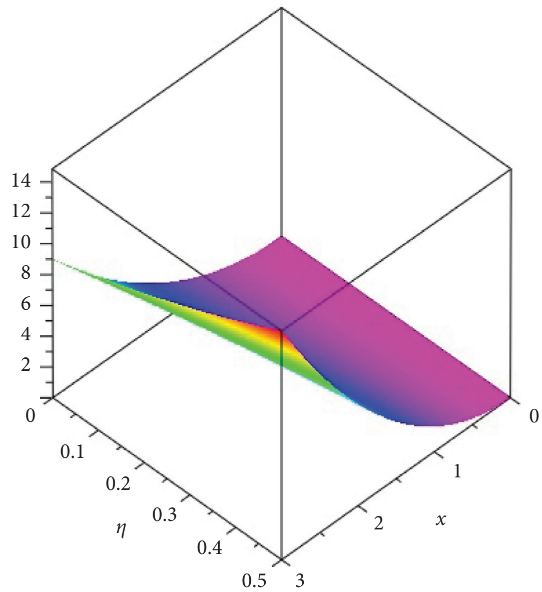

Figure 4: The approximate and exact result graph of problem 4.2, at $\delta = 1$.

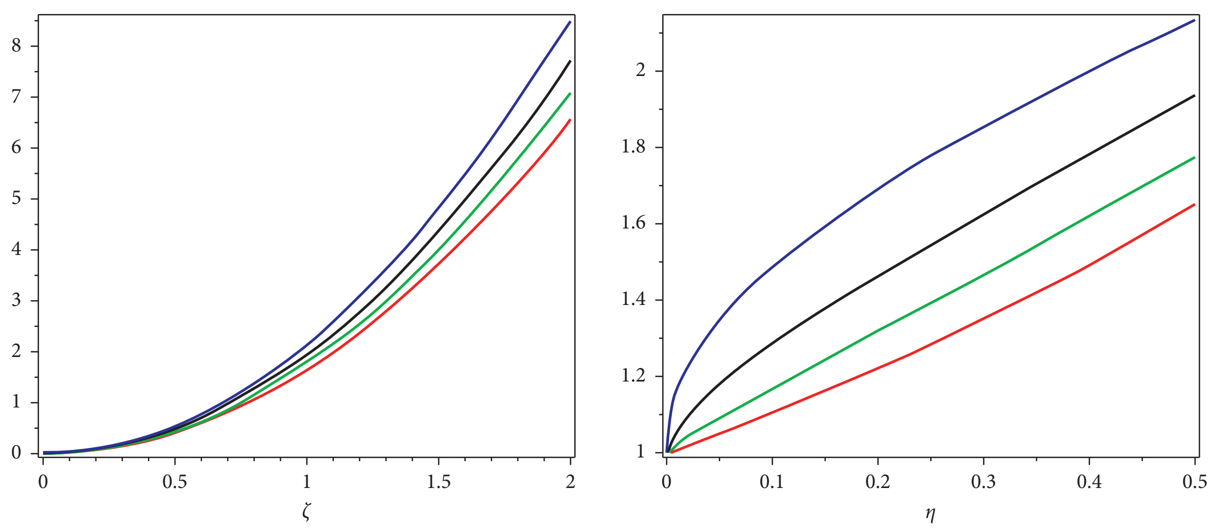

Figure 5: The approximate solution graph of problem 4.1, at different fractional order of $\delta = 1$ with respect to ζ and η.

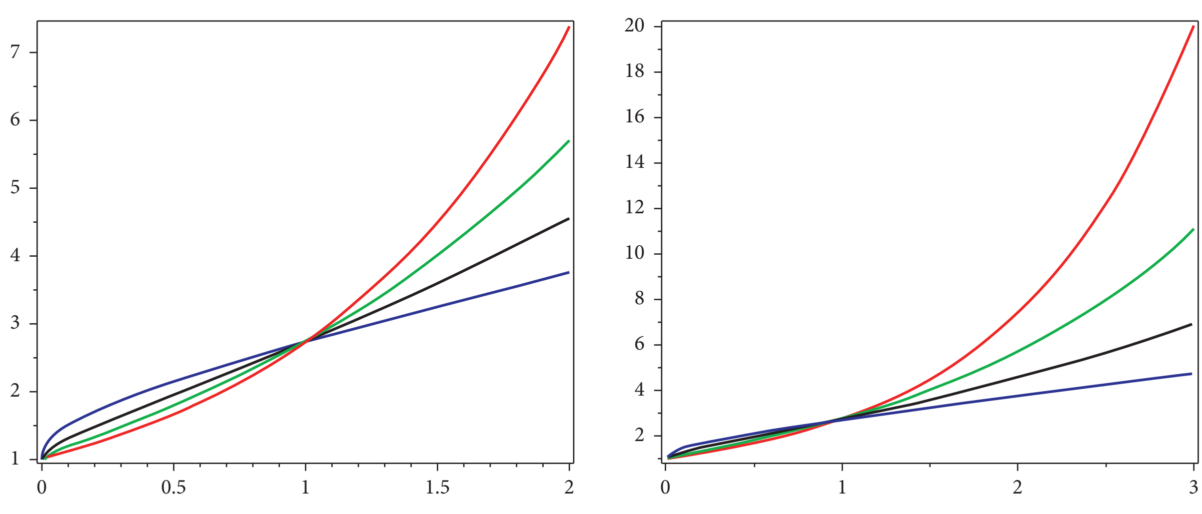

Figure 6: The different fractional-order graphs of problem 4.2, with respect to time.

Applying inverse Yang transform,

$$\mu(\zeta,\eta) = \mathbb{Y}^{-1}\left[s\mu(\zeta,0) - s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta)\right\}\right]. \tag{39}$$

Using ADM procedure, we get

$$\begin{aligned}
\mu_0(\zeta,\eta) &= \mathbb{Y}^{-1}[s\mu(\zeta,0)] = \zeta^2,\\
\sum_{\ell=0}^{\infty}\mu_{\ell+1}(\zeta,\eta) &= \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\sum_{\ell=0}^{\infty}C_{\ell}\left(\mu_{\zeta\zeta}\mu_{\zeta}\right) - \frac{1}{8}\sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta) - \sum_{\ell=0}^{\infty}\mu_{\ell}(\zeta,\eta)\right\}\right], \quad \ell = 0,1,2,\\
C_0\left(\mu_{\zeta\zeta}\mu_{\zeta}\right) &= \frac{\partial^2\mu_0}{\partial\zeta^2}\frac{\partial\mu_0}{\partial\zeta},\\
C_1\left(\mu_{\zeta\zeta}\mu_{\zeta}\right) &= \frac{\partial^2\mu_0}{\partial\zeta^2}\frac{\partial\mu_1}{\partial\zeta} + \frac{\partial^2\mu_1}{\partial\zeta^2}\frac{\partial\mu_0}{\partial\zeta},\\
C_2\left(\mu_{\zeta\zeta}\mu_{\zeta}\right) &= \frac{\partial^2\mu_0}{\partial\zeta^2}\frac{\partial\mu_2}{\partial\zeta} + \frac{\partial^2\mu_1}{\partial\zeta^2}\frac{\partial\mu_1}{\partial\zeta} + \frac{\partial^2\mu_2}{\partial\zeta^2}\frac{\partial\mu_0}{\partial\zeta},
\end{aligned} \tag{40}$$

for $\ell = 0$,

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$$\begin{aligned}
\mu_1(\zeta,\eta) &= \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta)\right\}\right],\\
\mu_1(\zeta,\eta) &= -\zeta^2\frac{\eta^{\delta}}{\Gamma(\delta+1)},\\
\mu_2(\zeta,\eta) &= \mathbb{Y}^{-1}\left[s^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta)\right\}\right],\\
\mu_2(\zeta,\eta) &= \frac{\zeta\left(2^{1-\delta} + 2^2\zeta + 1\right)\eta^{2\delta}}{2\Gamma(2\delta+1)},\\
\mu_3(\zeta,\eta) &= \mathbb{Y}^{-1}\left[ss^{\delta}\mathbb{Y}\left\{\frac{\partial^2\mu(\zeta/2,\eta/2)}{\partial\zeta^2}\frac{\partial\mu(\zeta/2,\eta/2)}{\partial\zeta} + \frac{1}{8}\mu(\zeta,\eta) + \mu(\zeta,\eta)\right\}\right],\\
\mu_3(\zeta,\eta) &= \frac{\eta^{3\delta}}{2\Gamma(3\delta+1)}\left(-1 - 2\zeta^2 - 2^4 + \frac{1}{2^{\delta}} + \frac{1}{2^{2\delta}} + \frac{2^{-3}}{2^{\delta}} + \frac{2^{-2}}{2^{3\delta}} + \zeta\frac{2^{-1}}{2^{2\delta}}\frac{2\Gamma(1+2\delta)}{2^{\delta}\Gamma(1+\delta)^2}\right).
\end{aligned} \tag{41}$$

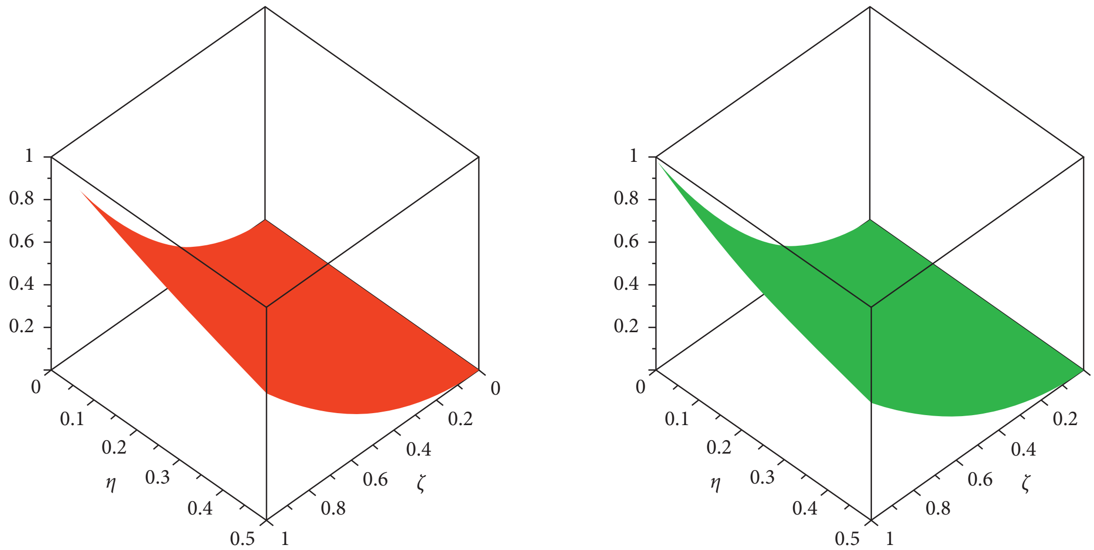

FIGURE 7: The approximate result graph of problem 4.3, at $\delta = 1$ and 0.8.

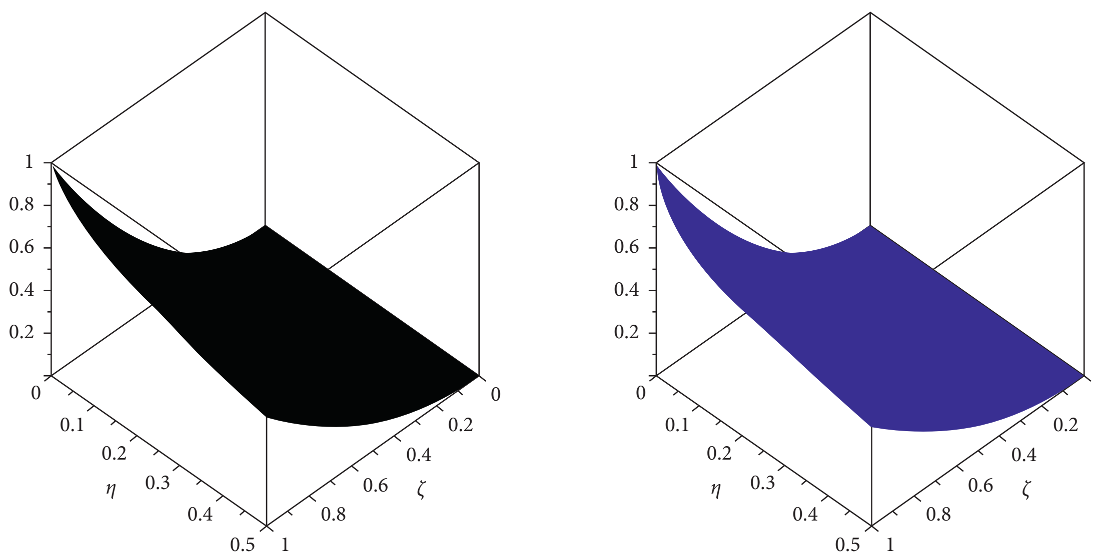

FIGURE 8: The approximate result graphs of problem 4.3, at $\delta = 0.6$ and 0.4.

The YDM result of problem 4.3 is

$$\begin{aligned}
\mu(\zeta,\eta) &= \mu_0(\zeta,\eta) + \mu_1(\zeta,\eta) + \mu_2(\zeta,\eta) + \mu_3(\zeta,\eta) + \mu_4(\zeta,\eta)\ldots, \\
\mu(\zeta,\eta) &= \zeta^2 - \zeta^2\frac{\eta^{\delta}}{\Gamma(\delta+1)} + \frac{\zeta\left(2^{1-\delta} + 2^2\zeta + 1\right)\eta^{2\delta}}{2\Gamma(2\delta+1)} \\
&\quad + \frac{\eta^{3\delta}}{2\Gamma(3\delta+1)}\left(-1 - 2\zeta^2 - 2^4 + \frac{1}{2^{\delta}} + \frac{1}{2^{2\delta}} + \frac{2^{-3}}{2^{\delta}} + \frac{2^{-2}}{2^{3\delta}} + \zeta\frac{2^{-1}}{2^{2\delta}}\frac{2\Gamma(1+2\delta)}{2^{\delta}\Gamma(1+\delta)^2}\right) + \ldots,
\end{aligned} \tag{42}$$

when $\delta = 1$, then YDM result is

$$\mu(\zeta,\eta) = \zeta^2\left(1 - \eta + \frac{\eta^2}{2!} - \frac{\eta^3}{3!} + \frac{\eta^4}{4!} - \frac{\eta^5}{5!} + \ldots\right). \tag{43}$$

The exact result is

$$\mu(\zeta,\eta) = \zeta^2 e^{-\eta}. \tag{44}$$

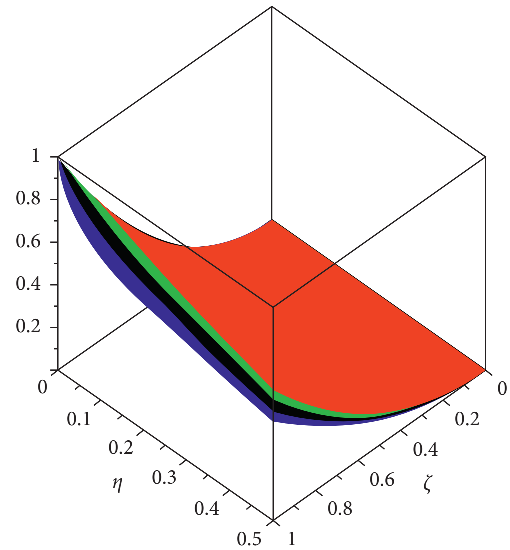

Figure 9: The approximate solution graph of problem 4.3, at different fractional-order of δ.

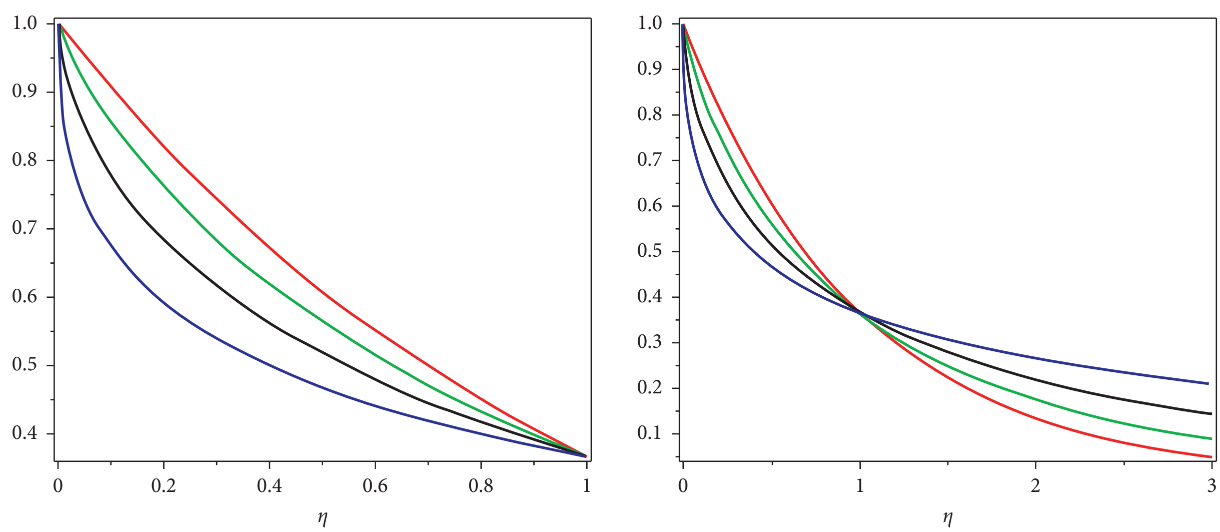

Figure 10: The approximate solution graph of problem 4.3, at different fractional-order of δ with respect to η.

Figure 7 shows the approximate result graph of problem 4.3, at $\delta = 1$ and 0.8. Figure 8 shows the approximate result graphs of problem 4.3, at $\delta = 0.6$ and 0.4. Figure 9 shows the approximate solution graph of problem 4.3, at different fractional-order of δ. Figure 10 shows the approximate solution graph of problem 4.3, at different fractional-order of δ with respect to η.

5. Conclusions

In this article, the approximate results of fractional-order delay partial differential equations are calculated, applying Yang decomposition method. The Yang decomposition method solutions are calculated for both fractional and integer order models. The suggested results are in close contact with homotopy perturbation technique [42], reduced differential transform technique [43], and homotopy perturbation transformation technique. The Yang decomposition method results have seen the highest concurrence with the actual results of the models. Moreover, the applicability and validity of the suggested technique confirm with the aid of three numeric problems. Yang decomposition method, results for fractional-order models, will prove the better understanding of the real world models represented by fractional partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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