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Research Article

Analysis of the Fractional-Order Delay Differential Equations by the Numerical Method

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In this study, we implemented a new numerical method known as the Chebyshev Pseudospectral method for solving nonlinear delay differential equations having fractional order. The fractional derivative is defined in Caputo manner. The proposed method is simple, effective, and straightforward as compared to other numerical techniques. To check the validity and accuracy of the proposed method, some illustrative examples are solved by using the present scenario. The obtained results have confirmed the greater accuracy than the modified Laguerre wavelet method, the Chebyshev wavelet method, and the modified wavelet-based algorithm. Moreover, based on the novelty and scientific importance, the present method can be extended to solve other nonlinear fractional-order delay differential equations.

1. Introduction

Fractional calculus is used in various branches of mathematics due to its numerous applications in modeling different physical phenomena in engineering and science. The concept of fractional calculus has been derived from the fact $D^{\alpha}(f(x))$, where alpha is noninteger. Later on, different scientists such as Riemann–Liouville, Euler, Leibniz, L'Hospital, Bernoulli, and Wallis have devoted their work to this research area. Fractional calculus has numerous applications in different field of sciences. For example dynamic of viscoelastic materials [1], electromagnetism [2], fluid mechanics [3], propagation of spherical flames [4], and viscoelastic materials [5].

In our real life, DEs are used to develop a different number of physical problems. Some are more complex and cannot be modeled with the help of simple differential equations. For these complex problems, a new technique has been used by the researchers known as fractional differential equations (FDEs). In the mathematical modeling of real-world physical problems, FDEs have been widespread due to their numerous applications in engineering and real-life sciences problems [6–9], such as economics [10], solid mechanics [11], continuum and statistical mechanics [12], oscillation of earthquakes [13], dynamics of interfaces between soft-nanoparticles and rough substrates [14], fluid-dynamic traffic model [15], colored noise [16], solid mechanics [11], anomalous transport [17], and bioengineering [18–20].

Delay differential equations (DDEs) have a wide range of applications in engineering and science. Delay differential equation simplifies the ordinary differential equation, depends on the past data, and is suitable for physical systems. Nowadays, researchers pay more attention to FDDEs as compared to DEs because a slight delay has a large effect. In this regard, numerous papers have been dedicated to the study of the numerical solution of FDDEs. FDDs have been widespread in mathematical modelings, such as population dynamics, epidemiology, immunology, physiology, and neural networks [21–25].

In literature, there is no precise technique for finding an exact or analytical solution for every FDDEs; the researcher's effort is to find the numerical solution of FDDEs. Various methods have been implemented for solving these problems numerically. The well-known among these methods are new predictor corrector method (NPCM) [26], adomian decomposition method (ADM) [27], Legendre pseudospectral method (LSM) [28], kernel method (KM) [29], LMS method (LMSM) [30], Adams–Bashforth–Moulton algorithm (ABMA) [31], extend predictor corrector method (EPCM) [32], simplified reproducing kernel method (SRKM) [29], variation iteration method (VIM) [33], homotopy perturbation method (HPM) [34], Galerkin method (GM) [35], Runge–Kutta-type methods (RKM) [36], Bernoulli wavelet method (BWM) [37], and modified Laguerre wavelet method [38] have been used for the analytical and numerical solution of FDDEs.

In the present work, CPM is extended for the solutions of FDDEs. The results we obtained are compared with other methods, which show that CPM has good convergence rate than other methods. We focus on FDDE of the form

$$D_u^{\gamma} f(u) = g(u, f(u), f(h(u))), \quad c \le u \le d,\ m < \gamma \le m+1,\ m = 1, 2, 3, \ldots, \tag{1}$$

with the following boundary conditions:

$$f(c) = \alpha_0, \quad f(d) = \alpha_1,\ f(u) = \zeta(u),\ u \in [c_0, c], \tag{2}$$

where h is the delay function which is to be assumed continues in the interval [c, d] and satisfies the inequality $c_0 \le h(u) \le u$ for some fix real constant c_0, for $u \in [c, d]$ and $\zeta \in C[c_0, c]$

The following is a summary of the paper's structure. In Section 2, we introduce some fundamental fractional calculus definitions and mathematical techniques that will be useful in our later study. The approximation of the fractional derivative $D_u^{\gamma} f(u)$ is obtained in Section 3. Section 4 describes the Chebyshev collocation method's application to the solution of eq. (1). As a result, a set of algebraic equations is created, and the solution to the problem in question is presented. Section 5 provides some numerical results to help clarify the method.

2. Basic Definitions of Fractional Derivatives

Definition 1. A real function, $g(u), u > 0$, is said to be in the space $C_{\mu}, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $g(u) = u^p g_1(u)$, where $g_1(u) \in [0, \infty)$, and it is said to be in the space C_{μ}^{m} if and only if $g^{(m)} \in C_{\mu}$, $m \in \mathbb{N}$.

Definition 2. In Caputo manner, the derivative having fractional-order $D^{\gamma} g(u)$ is given as below:

$$D^{\gamma} g(u) = \frac{1}{\Gamma(j-\gamma)} \int_0^u (u-t)^{j-\gamma-1} g^{(n)}(t) \mathrm{d}t, \quad u > 0,\ j-1 < \gamma < j. \tag{3}$$

The order of the derivative is $\gamma > 0$, and the lowest integer greater than γ is $j \in \mathbb{N}$ and $g \in C_{-1}^{n}$.

We have the Caputo derivative [39]:

$$D^{\gamma} C = 0, \quad C \text{ is a constant}, \tag{4}$$

$$D^{\gamma} u^{\alpha} = \begin{cases} 0 \text{ for } \alpha \in \mathbb{N}_0 \text{ and } \alpha < \lceil \gamma \rceil \\ \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)} u^{\alpha-\gamma} \text{ for } \alpha \in \mathbb{N}_0 \text{ and } \alpha \ge \lceil \gamma \rceil \end{cases}, \tag{5}$$

where the lowest integer larger than or equal to γ is denoted by the ceiling function $\lceil \gamma \rceil$ and $\mathbb{N}_0 = 1, 2, \ldots$. Remember that the Caputo differential operator is the same as the normal differential operator of the integer order for $\gamma \in \mathbb{N}$. Fractional differentiation is a linear operation, just like integer-order differentiation:

$$D^{\gamma} (\phi g(u) + \mu h(u)) = \phi D^{\gamma} g(u) + \mu D^{\gamma} h(u), \tag{6}$$

where ϕ and μ are constants.

3. Chebyshev Series Expansion Is Used to Approximate a Caputo Derivative

On the interval $[-1, 1]$, Chebyshev polynomials are defined and, with the help of recurrence formulae, explained as [40, 41]

$$T_{j+1}(u) = 2u T_j(u) - T_{j-1}(u), \quad j = 1, 2, \ldots, \tag{7}$$

where $T_0(u) = 1$ and $T_1(u)u$. The Chebyshev polynomial analytical form for degree j is defined as [41]

$$T_j(u) = \frac{j}{2} \sum_{r=0}^{\lfloor j/2 \rfloor} (-1)^r \frac{(j-r-1)!}{r!(j-2r)!} (2u)^{j-2r}. \tag{8}$$

If we apply the Chebyshev polynomials over the $[0, 1]$ interval, we explain the Chebyshev shifted polynomials $\widehat{T}_j(u)$. These are described in the sense of Chebyshev polynomials $T_j(u)$ as [41]

$$\widehat{T}_j(u) = T_j(2u-1). \tag{9}$$

And recurrence formula is as follows:

$$\widehat{T}_{j+1}(u) = 2(2u-1)\widehat{T}_j(u) - \widehat{T}_{j-1}(u), \quad j = 1, 2, \ldots, \tag{10}$$

where $\widehat{T}_0(u) = 1$ and $\widehat{T}_1(u) = 2u - 1$. The orthogonality condition is [42]

$$\int_0^1 \frac{\widehat{T}_j(u)\widehat{T}_m(u)}{\sqrt{u-u^2}} \mathrm{d}u = \begin{cases} 0 \, m \ne j, \\ \dfrac{\pi}{2} \, m = j \ne 0, \\ \pi \, m = j = 0. \end{cases} \tag{11}$$

Now, we can use the well-known relation,

$$\widehat{T}_j(u) = T_{2j}(\sqrt{u}), \tag{12}$$

and equation (8) to get shifted Chebyshev polynomials analytical form having order j as

$$\widehat{T}_j(u) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r} \frac{j(2j-r-1)!}{r!(2j-2r)!} (x)^{j-2r}. \quad (13)$$

A function $f(u) \in L_2[0,1]$ may be described in terms of Chebyshev shifted polynomials as

$$f(u) = \sum_{j=1}^{\infty} c_j \widehat{T}_j(u), \quad (14)$$

where the coefficients c_j, $j = 1, 2, \ldots$, are given by

$$c_0 = \frac{1}{\pi} \int_0^1 \frac{g(u)\widehat{T}_0(u)}{\sqrt{u-u^2}} du \text{ and } c_n = \frac{2}{\pi} \int_0^1 \frac{g(u)\widehat{T}_j(u)}{\sqrt{u-u^2}} du. \quad (15)$$

Only Chebyshev shifted polynomials first $(m+1)$-terms are considered in practice. Thus,

$$f_m(u) = \sum_{j=0}^{m} c_j \widehat{T}_j(u). \quad (16)$$

3.1. Chebyshev Truncation Theorem [43]. The sum of the absolute values of all the disregarded coefficients limits the inaccuracy in approximating $f(u)$ by the sum of its first m terms. That is, assuming

$$f_m(u) = \sum_{k=0}^{m} c_k T_k(u), \quad (17)$$

then, for all $f(u)$, all m, and all $u \in [-1,1]$, we obtain

$$E_T(m) = |f(u) - f_m(u)| \leq \sum_{k=m+1}^{\infty} |c_k|. \quad (18)$$

Proof. For any $u \in [-1,1]$ and all k, the Chebyshev polynomials are bounded by 1, $|T_k(u)| \leq 1$. As a result, the kth term is restricted by $|c_k|$. By subtracting the reduced series from the infinite series, bounding each term in the difference, and then summing the bounds, the theorem can be derived.

The following theorem contains the main approximate formula for the fractional derivative of $f(u)$. □

3.2. Theorem [44]. Assume $\alpha > 0$ and that $f(u)$ is estimated by the shifted Chebyshev polynomials as in (16). Then,

$$D^\alpha(f_m(u)) = \sum_{j=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_j b_{j,r}^\alpha u^{j-r-\alpha}, \quad (19)$$

where $b_{j,r}^\alpha$ is given by

$$b_{j,r}^\alpha = (-1)^r 2^{2j-2r} \frac{j(2j-r-1)!(j-r)!}{r!(2j-2r)!\Gamma(j-r+1-\alpha)}. \quad (20)$$

Proof. Since Caputo fractional differentiation is a linear operation, we have

$$D^\alpha(f_m(u)) = \sum_{j=0}^{m} c_n D^\alpha\left(\widehat{T}_j(u)\right). \quad (21)$$

Now, to evaluate $D^\alpha(\widehat{T}_j(u))$, applying to equations (4) and (5)–(13),

$$D^\alpha\left(\widehat{T}_j(u)\right) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r} \frac{j(2j-r-1)!}{r!(2j-2r)!} D^\alpha(u)^{j-r}, \quad j = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \ldots m. \quad (22)$$

Since $\widehat{T}_j(u)$ is a polynomial having degree j, we obtain

$$D^\alpha\left(\widehat{T}_j(u)\right) = 0 \text{ forall } \quad j = 0, 1, 2, \ldots, \lceil\alpha\rceil - 1, \ \alpha > 0. \quad (23)$$

The following is the result of combining (21)–(23):

$$D^\alpha(f_m(u)) = \sum_{j=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} \frac{c_j(-1)^r 2^{2j-2r} j(2j-r-1)!(j-r)!}{r!(2j-2r)!\Gamma(j-r+1-\alpha)} u^{j-r-\alpha} = \sum_{j=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_j b_{j,r}^\alpha u^{j-2r-\alpha}, \quad (24)$$

which is the desired result.

Test example: consider formula (19) with $f(u) = u^2, m = 2$. The shifted series of u^2 is

$$u^2 = c_0\widehat{T}_0(u) + c_1\widehat{T}_1(u) + c_2\widehat{T}_2(u) = \frac{3}{8}\widehat{T}_0(u) + \frac{1}{2}\widehat{T}_1(u) + \frac{1}{8}\widehat{T}_2(u) \quad (25)$$

and

$$D^{\frac{1}{2}}\left(u^2\right) = \sum_{j=1}^{2}\sum_{r=0}^{n-1} c_n b_{n,r}^{\left(\frac{1}{2}\right)} u^{j-r-\frac{1}{2}} = c_1 b_{1,0}^{\left(\frac{1}{2}\right)} u^{\frac{1}{2}} + c_2 b_{2,0}^{\left(\frac{1}{2}\right)} u^{\frac{3}{2}} + c_2 b_{2,1}^{\left(\frac{1}{2}\right)} u^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} u^{\frac{1}{2}} + \frac{8}{3\sqrt{\pi}} u^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}} u^{\frac{1}{2}} = \frac{8}{3\sqrt{\pi}} u^{\frac{3}{2}}, \tag{26}$$

which yields the same result as evaluating $D^{1/2}(u^2)$ by relation (5). □

4. Chebyshev Collocation Method

We solve the FDDE (1) with the boundary conditions (2) stated in the next part using Chebyshev's collocation method in this section. Assume that the approximate solution $f(u)$ is defined in terms of a finite number m of shifted Chebyshev polynomials, i.e.,

$$f_m(u) = \sum_{j=0}^{m} c_j \widehat{T}_j(u). \tag{27}$$

We can use theorem (4.1) and equation (22) to solve equation (1):

$$\sum_{j=\lceil\alpha\rceil}^{m}\sum_{r=0}^{n-\lceil\alpha\rceil} c_j b_{j,r}^{\alpha} u^{j-2r-\alpha} = g\left(u, \sum_{j=0}^{m} c_j \widehat{T}_j(u), \sum_{j=0}^{m} c_j \widehat{T}_j(h(u))\right), \quad 0<u<1,\ m+1<\alpha<m. \tag{28}$$

Now, we collocate (23) at points u_p, $p = 0, 1, 2, \ldots, m-\lceil\alpha\rceil$:

$$\sum_{j=\lceil\alpha\rceil}^{m}\sum_{r=0}^{n-\lceil\alpha\rceil} c_j b_{j,r}^{\alpha} u_p^{j-2r-\alpha} = g\left(u_p, \sum_{j=0}^{m} c_j \widehat{T}_j(u_p), \sum_{j=0}^{m} c_j \widehat{T}_j(h(u_p))\right), \quad u_p,\ p = 0, 1, \ldots m-\lceil\alpha\rceil,\ m+1<\alpha<m. \tag{29}$$

Using (22) in the boundary conditions (2), we may construct the following $\lceil\alpha\rceil$ equations:

$$\sum_{i=0}^{m} (-1)^i c_i = \alpha_0, \quad \sum_{i=0}^{m} c_i = \alpha_1. \tag{30}$$

We get $(m+1-\lceil\alpha\rceil)$ algebraic equations from (24) and $\lceil\alpha\rceil)$ algebraic equations from (26). As a result, we have total $(m+1)$ linear or nonlinear algebraic equations that can be easily solved using matrices for unknowns c_j, $j = 0, 1, 2, \ldots m$, to find out an estimated solution $\mu_m(\psi)$.

5. Numerical Representation

In this section, we solve some delay problems. The results we obtained are compared with other methods. All the numerical results are obtained using MAPLE.

Problem 1. Consider the FDDE:

$$\frac{d^\alpha f(u)}{du} = \frac{1}{2}\exp^{\frac{u}{2}} f\left(\frac{u}{2}\right) + \frac{1}{2} f(u), \quad 0<\alpha\le 1, \tag{31}$$

subject to the initial conditions $f(0) = 1$, having accurate solution $f(u) = \exp^u$ at $\alpha = 1$.

The exact solution and CPM solution are given in Table 1. Table 2 shows CPM and CWM error comparison at $m = 4$ which confirm that CPM converges quickly as compared to CWM. We illustrate the accurate and estimated solutions for $m = 4$ in Figure 1, while Figure 2 shows the error comparison of both methods. Also, Figure 3 provides the graphical layout of the solution of example 1 at various fractional orders. It can be seen that the solutions of CPM are in good agreement to the actual solution than that of CWM.

Problem 2. Consider the nonlinear DDE,

$$\frac{d^\alpha f(u)}{du} = 1 - 2f^2\left(\frac{u}{2}\right), \quad 0\le u\le 1,\ 1<\alpha\le 2, \tag{32}$$

subjects to the initial condition $f(0) = 1$, $f'(0) = 0$.

The accurate solution of this equation for $\alpha = 2$ is $f(u) = \cos(u)$. The exact solution and CPM solution are shown in Table 3. Table 4 shows the error comparison of CPM at $m = 3$ and MWBA at $m = 20$ which confirm that CPM converges quickly as compare to MLWM. The estimated and accurate solutions are illustrated in Figure 4, whereas Figure 5 shows the error comparison of both methods. In addition, the convergence phenomena of the solutions at different fractional orders can be seen in Figure 6. The results of the presented method are better than those of the MWBA method for example 2.

Problem 3. Consider the fractional DDE of the form

$$\frac{d^\alpha f(u)}{du} = f\left(\frac{u}{2}\right) + \frac{3}{4} f(u) - u^2 + 2, \quad 0\le u\le 1,\ 1<\alpha\le 2, \tag{33}$$

with initial conditions $f(0) = f'(0) = 0$.

Table 1: Exact, CPM solution, and CPM A.E of problem 1 for $m = 4$.

u	Exact	CPM	CPM error
0	1.000 000 000 000 000	1.000 000 000 000 000	$0.0\,000\,000\,000E+00$
0.01	1.010 050 167 084 170	1.010 050 167 197 000	$1.1\,283\,000\,000E-10$
0.02	1.020 201 340 026 760	1.020 201 341 818 670	$1.7\,919\,100\,000E-09$
0.03	1.030 454 533 953 520	1.030 454 542 957 020	$9.0\,035\,000\,000E-09$
0.04	1.040 810 774 192 390	1.040 810 802 432 060	$2.8\,239\,670\,000E-08$
0.05	1.051 271 096 376 020	1.051 271 164 791 820	$6.8\,415\,800\,000E-08$
0.06	1.061 836 546 545 360	1.061 836 687 312 300	$1.4\,076\,694\,000E-07$
0.07	1.072 508 181 254 220	1.072 508 439 997 560	$2.5\,874\,334\,000E-07$
0.08	1.083 287 067 674 960	1.083 287 505 579 630	$4.3\,790\,467\,000E-07$
0.09	1.094 174 283 705 210	1.094 174 979 518 540	$6.9\,581\,333\,000E-07$
0.10	1.105 170 918 075 650	1.105 171 970 002 350	$1.0\,519\,267\,000E-06$

Table 2: Absolute error (A.E) comparison of CPM and other different methods of problem 1 at $m = 4$.

u	CPM A.E	CWM A.E
0	$0.0\,000\,000\,000E+00$	$0.0\,000\,000\,000E+00$
0.01	$1.1\,283\,000\,000E-10$	$1.82\,567E-04$
0.02	$1.7\,919\,100\,000E-09$	$3.68\,446E-04$
0.03	$9.0\,035\,000\,000E-09$	$5.57\,408E-04$
0.04	$2.8\,239\,670\,000E-08$	$7.49\,213E-04$
0.05	$6.8\,415\,800\,000E-08$	$9.43\,609E-04$
0.06	$1.4\,076\,694\,000E-07$	$1.14\,034E-03$
0.07	$2.5\,874\,334\,000E-07$	$1.33\,912E-03$
0.08	$4.3\,790\,467\,000E-07$	$1.53\,969E-03$
0.09	$6.9\,581\,333\,000E-07$	$1.74\,174E-03$
0.10	$1.0\,519\,267\,000E-06$	$1.94\,497E-03$

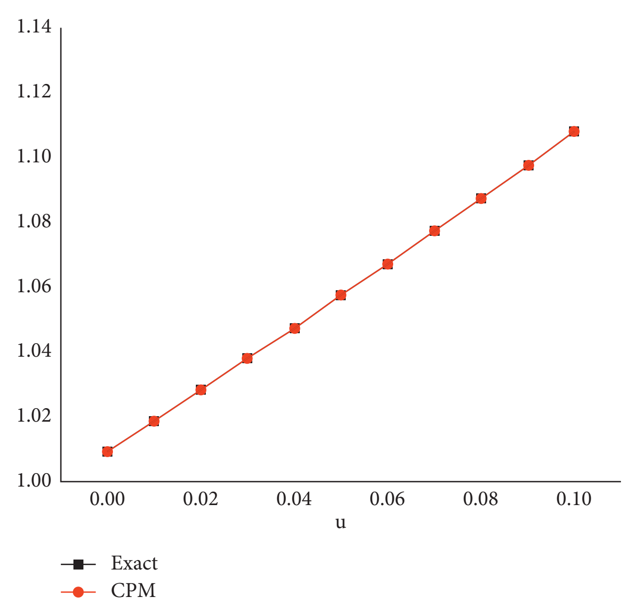

Figure 1: The exact and CPM solution graph for problem 1.

The exact solution of this equation for $\alpha = 2$ is $f(u) = u^2$. The exact solution and CPM solution are shown in Table 5. Table 6 shows the error comparison of CPM at $m = 3$ and MWLM at $m = 5$ which confirm that CPM converges quickly as compare to MLWM. We illustrate the accurate and estimated solutions for $m = 3$ in Figure 7, while Figure 8 shows the error comparison of both methods. The results of the presented method are better than those of the MWBA method for example 3.

Problem 4. Consider the following nonlinear delay differential equation with boundary conditions $f(0) = 1$ and $f(1) = 1$:

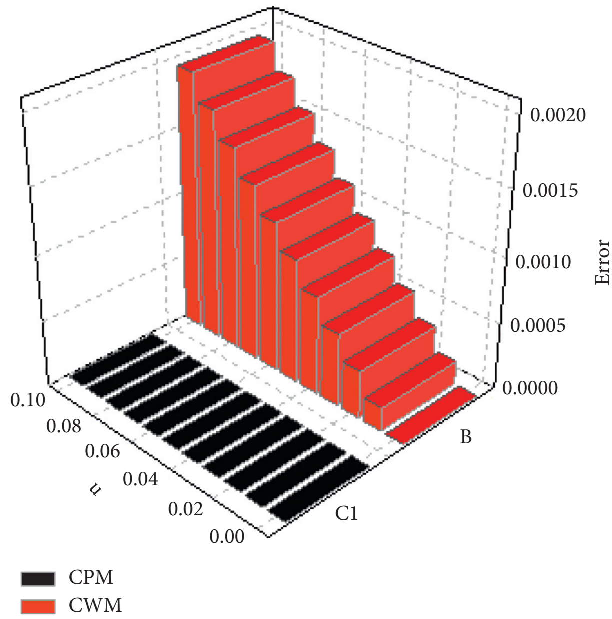

Figure 2: Error graph of CWM and CPM for problem 1.

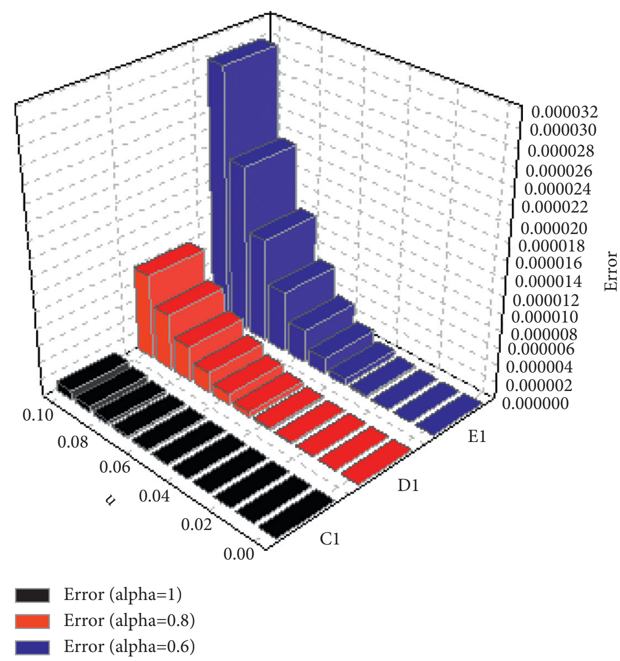

Figure 3: The graph of the absolute error at various fractional orders for problem 1.

$$\frac{d^{\alpha}}{du^{\alpha}} f(u) = \frac{8}{3}\frac{d}{du}\left(f\left(\frac{u}{2}\right)\right) f(u) + 8u^{2} f\left(\frac{u}{2}\right) - \frac{4}{3} - \frac{22}{3}u - 7u^{2} - \frac{5}{3}u^{3}, \quad 1 < \alpha \leq 2. \tag{34}$$

The accurate solution of this equation for $\alpha = 2$ is $f(u) = 1 + u - u^{3}$. The exact and CPM solution are shown in Table 7. Table 8 shows the error comparison of CPM at $m = 4$ and MWBA at $m = 8$ which confirm that CPM converges quickly as compare to MWBA. The estimated and accurate solutions are illustrated in Figure 9, while Figure 10 shows the error comparison of both methods. It can be seen that our method is more accurate.

Table 3: Exact, CPM solution, and CPM A.E of problem 2 at $m = 10$.

u	Exact	CPM	CPM error
0	1.000 000 000 000 000	1.000 000 000 000 000	0.0 000 000 000E + 00
0.10	0.995 004 165 278 026	0.995 004 165 278 026	2.9 000 000 000E – 19
0.20	0.980 066 577 841 242	0.980 066 577 841 242	2.7 250 000 000E – 16
0.30	0.955 336 489 125 606	0.955 336 489 125 621	1.5 097 140 000E – 14
0.40	0.921 060 994 002 885	0.921 060 994 003 138	2.5 278 786 000E – 13
0.50	0.877 582 561 890 373	0.877 582 561 892 544	2.1 711 958 200E – 12
0.60	0.825 335 614 909 678	0.825 335 614 921 738	1.2 059 545 750E – 11
0.70	0.764 842 187 284 488	0.764 842 187 333 195	4.8 707 056 530E – 11
0.80	0.696 706 709 347 165	0.696 706 709 498 894	1.5 172 838 626E – 10
0.90	0.621 609 968 270 664	0.621 609 968 640 609	3.6 994 453 098E – 10
1.0	0.540 302 305 868 140	0.540 302 306 536 394	6.6 825 465 360E – 10

Table 4: Absolute error (A.E) comparison of CPM and other different methods for problem 2.

u	CPM A.E at ($m = 3$)	MLWM A.E at ($m = 20$)
0	0	2.10 000E – 08
0.10	2.9 000 000 000E – 19	2.11 000E – 08
0.20	2.7 250 000 000E – 16	2.09 000E – 08
0.30	1.5 097 140 000E – 14	2.09 000E – 08
0.40	2.5 278 786 000E – 13	2.08 000E – 08
0.50	2.1 711 958 200E – 12	2.06 000E – 08
0.60	1.2 059 545 750E – 11	2.04 000E – 08
0.70	4.8 707 056 530E – 11	2.03 000E – 08
0.80	1.5 172 838 626E – 10	2.00 000E – 08
0.90	3.6 994 453 098E – 10	1.99 000E – 08
1.0	6.6 825 465 360E – 10	1.97 000E – 08

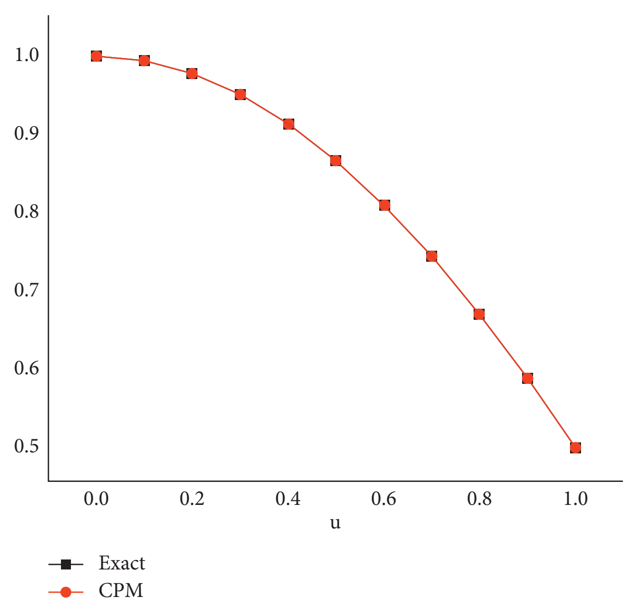

Figure 4: The graph of absolute error at various fractional orders for problem 2.

Problem 5. Consider the FDDE

$$\frac{d^{\alpha} f(u)}{du} + f(u) + f(u - 0.3) = \exp^{-u+0.3}, \quad 1 < \alpha < 2,\ 0 < \alpha < 1, \tag{35}$$

having initial conditions $f(0) = 1, f'(0) = -1$, and $f''(0) = 1$.

The accurate solution of this problem for $\alpha = 3$ is $f(u) = \exp^{-u}$. The exact and CPM solutions are shown in

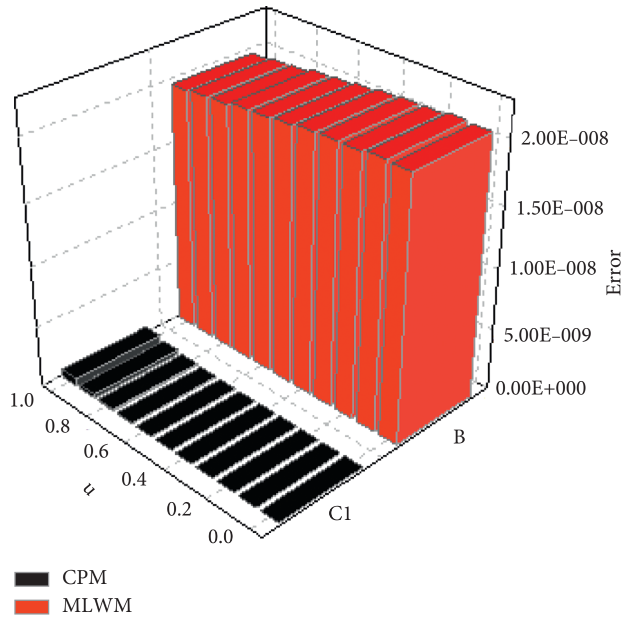

Figure 5: Exact and CPM solution graph for problem 2.

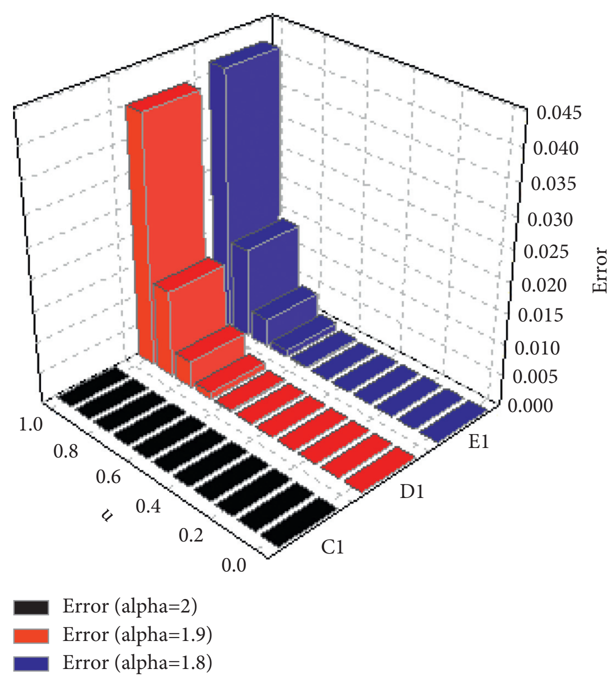

Figure 6: Error graph of MLWM and CPM for problem 2.

Table 9. Table 10 shows the error comparison of CPM and CWM at $m = 6$ which confirm that CPM converges quickly as compared to CWM. We illustrate the accurate and estimated solutions for $m = 6$ in Figure 11, while Figure 12 shows the error comparison of both methods. In Figure 13, the solution for example 4.5 at different fractional orders is calculated. It is confirmed that the solution at various fractional order approaches towards the integer-order solution. The results of the presented method are better than those of the CWM method for this problem.

Table 5: Exact, CPM solution, and CPM A.E of problem 3 at $m = 3$.

u	Exact	CPM	CPM error
0	0.000 000 000 000 00	0.000 000 000 000 00	$4.0\,000\,000\,000E+00$
0.10	0.010 000 000 000 00	0.010 000 000 000 00	$1.2\,100\,000\,000E-29$
0.20	0.040 000 000 000 00	0.040 000 000 000 00	$2.4\,800\,000\,000E-29$
0.30	0.090 000 000 000 00	0.090 000 000 000 00	$3.7\,400\,000\,000E-29$
0.40	0.160 000 000 000 00	0.160 000 000 000 00	$5.0\,000\,000\,000E-29$
0.50	0.250 000 000 000 00	0.250 000 000 000 00	$6.4\,000\,000\,000E-29$
0.60	0.360 000 000 000 00	0.360 000 000 000 00	$7.8\,000\,000\,000E-29$
0.70	0.490 000 000 000 00	0.490 000 000 000 00	$9.4\,000\,000\,000E-29$
0.80	0.640 000 000 000 00	0.640 000 000 000 00	$1.1\,100\,000\,000E-28$
0.90	0.810 000 000 000 00	0.810 000 000 000 00	$1.2\,800\,000\,000E-28$
1.0	1.000 000 000 000 00	1.000 000 000 000 00	$1.4\,000\,000\,000E-28$

Table 6: Absolute error (A.E) comparison of CPM and other different methods of problem 3.

u	CPM A.E at $(m = 3)$	MLWM A.E at $(m = 5)$
0	$4.0\,000\,000\,000E+00$	$1.41\,421E-09$
0.10	$1.2\,100\,000\,000E-29$	$4.75\,800E-08$
0.20	$2.4\,800\,000\,000E-29$	$9.69\,300E-08$
0.30	$3.7\,400\,000\,000E-29$	$1.47\,010E-07$
0.40	$5.0\,000\,000\,000E-29$	$1.98\,200E-07$
0.50	$6.4\,000\,000\,000E-29$	$2.50\,900E-07$
0.60	$7.8\,000\,000\,000E-29$	$3.05\,500E-07$
0.70	$9.4\,000\,000\,000E-29$	$3.62\,400E-07$
0.80	$1.1\,100\,000\,000E-28$	$4.22\,000E-07$
0.90	$1.2\,800\,000\,000E-28$	$4.84\,800E-07$
1.0	$1.4\,000\,000\,000E-28$	$5.51\,000E-07$

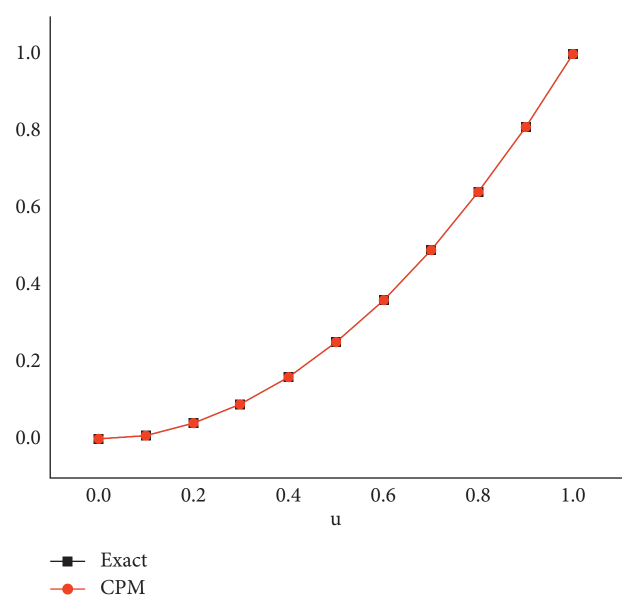

Figure 7: The exact and CPM solution graph for problem 3.

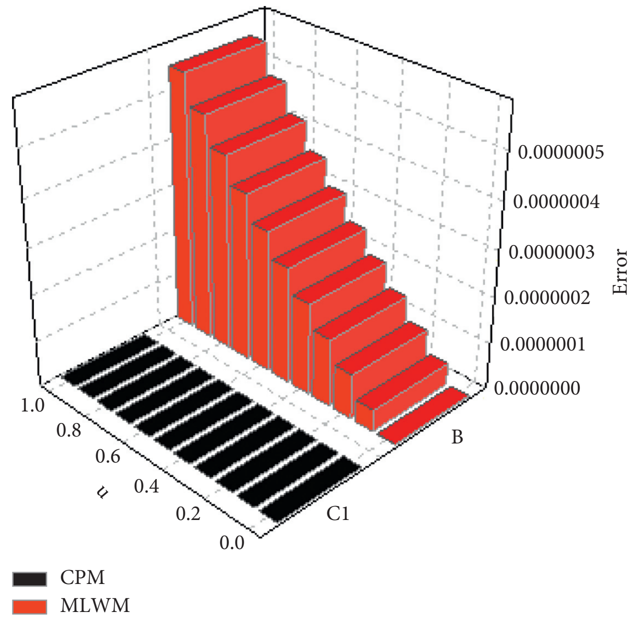

Figure 8: Error graph of MLWM and CPM for problem 3.

Table 7: Exact, CPM solution, and CPM A.E at $m = 4$ of problem 4.

u	Exact	CPM	CPM error
0	1.000 000 000 000 000	1.000 000 000 000 000	$2.0\,000\,000\,000E-40$
0.10	1.099 000 000 000 000	1.099 000 000 000 000	$3.0\,000\,000\,000E-39$
0.20	1.192 000 000 000 000	1.192 000 000 000 000	$6.0\,000\,000\,000E-39$
0.30	1.273 000 000 000 000	1.273 000 000 000 000	$7.0\,000\,000\,000E-39$
0.40	1.336 000 000 000 000	1.336 000 000 000 000	$1.0\,000\,000\,000E-38$
0.50	1.375 000 000 000 000	1.375 000 000 000 000	$1.3\,000\,000\,000E-38$
0.60	1.384 000 000 000 000	1.384 000 000 000 000	$1.6\,000\,000\,000E-38$
0.70	1.357 000 000 000 000	1.357 000 000 000 000	$1.9\,000\,000\,000E-38$
0.80	1.288 000 000 000 000	1.288 000 000 000 000	$2.3\,000\,000\,000E-38$
0.90	1.171 000 000 000 000	1.171 000 000 000 000	$2.5\,000\,000\,000E-38$
1.0	1.000 000 000 000 000	1.000 000 000 000 000	$2.9\,000\,000\,000E-38$

Table 8: Absolute error (A.E) comparison of CPM and other different methods for problem 4.

u	CPM error at ($m = 4$)	MWBA error at ($m = 8$)
0	$2.0\,000\,000\,000E-40$	$1.20\,000E-29$
0.10	$3.0\,000\,000\,000E-39$	$1.00\,000E-29$
0.20	$6.0\,000\,000\,000E-39$	$1.00\,000E-29$
0.30	$7.0\,000\,000\,000E-39$	$1.00\,000E-29$
0.40	$1.0\,000\,000\,000E-38$	$1.00\,000E-29$
0.50	$1.3\,000\,000\,000E-38$	$1.00\,000E-29$
0.60	$1.6\,000\,000\,000E-38$	$1.00\,000E-29$
0.70	$1.9\,000\,000\,000E-38$	$1.00\,000E-29$
0.80	$2.3\,000\,000\,000E-38$	$2.00\,000E-29$
0.90	$2.5\,000\,000\,000E-38$	$2.00\,000E-29$
1.0	$2.9\,000\,000\,000E-38$	$1.50\,000E-29$

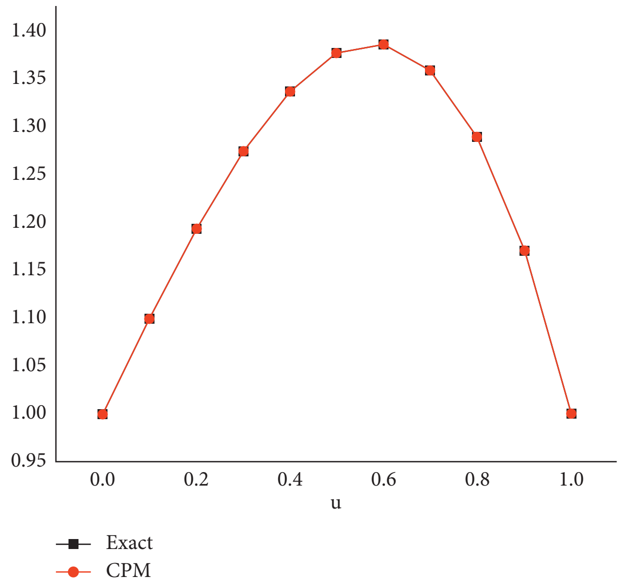

Figure 9: The exact and CPM solution graph for problem 4.

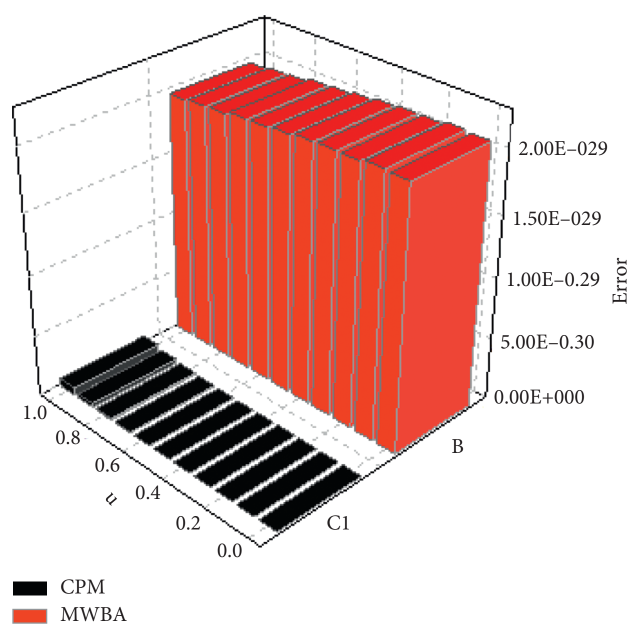

Figure 10: Error graph of MWBA and CPM for problem 4.

Table 9: Exact, CPM solution, and CPM error of problem 5 for $m = 6$.

u	Exact	CPM	CPM (A.E)
0	1.000 000 000 000 000	1.000 000 000 000 000	$0.0\,000\,000\,000E+00$
0.01	0.990 049 833 749 168	0.990 049 833 749 168	$2.6\,506\,231\,572E-16$
0.02	0.980 198 673 306 755	0.980 198 673 306 738	$1.6\,837\,479\,038E-14$
0.03	0.970 445 533 548 508	0.970 445 533 548 318	$1.9\,035\,198\,323E-13$
0.04	0.960 789 439 152 323	0.960 789 439 151 262	$1.0\,614\,659\,655E-12$
0.05	0.951 229 424 500 714	0.951 229 424 496 695	$4.0\,185\,154\,936E-12$
0.06	0.941 764 533 584 249	0.941 764 533 572 341	$1.1\,907\,912\,782E-11$
0.07	0.932 393 819 905 948	0.932 393 819 876 151	$2.9\,797\,628\,776E-11$
0.08	0.923 116 346 386 636	0.923 116 346 320 752	$6.5\,884\,200\,594E-11$
0.09	0.913 931 185 271 228	0.913 931 185 138 694	$1.3\,253\,379\,871E-10$
0.10	0.904 837 418 035 960	0.904 837 417 788 512	$2.4\,744\,798\,291E-10$

Table 10: Absolute error (A.E) comparison of CPM and other different methods of problem 5 at $m = 6$.

u	CPM A.E	CWM A.E
0	$0.0000000000E+00$	$0.0000000000E+00$
0.01	$2.6506231572E-16$	$8.20000E-09$
0.02	$1.6837479038E-14$	$6.68000E-08$
0.03	$1.9035198323E-13$	$2.28800E-07$
0.04	$1.0614659655E-12$	$5.50500E-07$
0.05	$4.0185154936E-12$	$1.09130E-06$
0.06	$1.1907912782E-11$	$1.91420E-06$
0.07	$2.9797628776E-11$	$3.08520E-06$
0.08	$6.5884200594E-11$	$4.67410E-06$
0.09	$1.3253379871E-10$	$6.75420E-06$
0.10	$2.4744798291E-10$	$9.40260E-06$

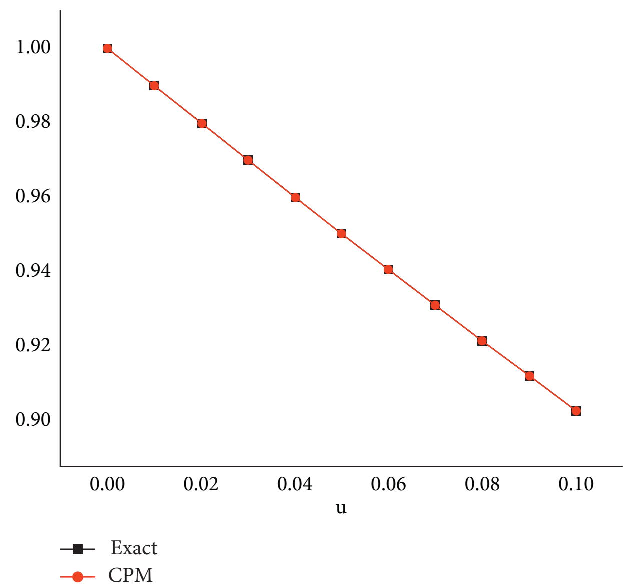

Figure 11: The exact and CPM solution graph for problem 5.

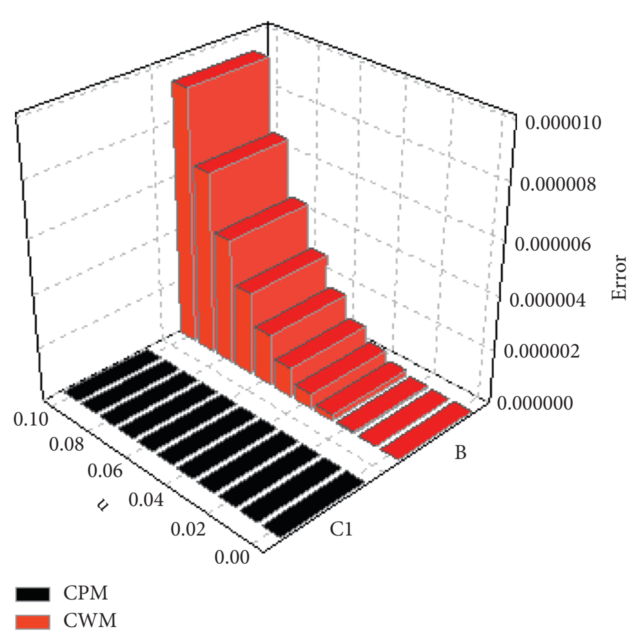

Figure 12: Error graph of CWM and CPM for problem 5.

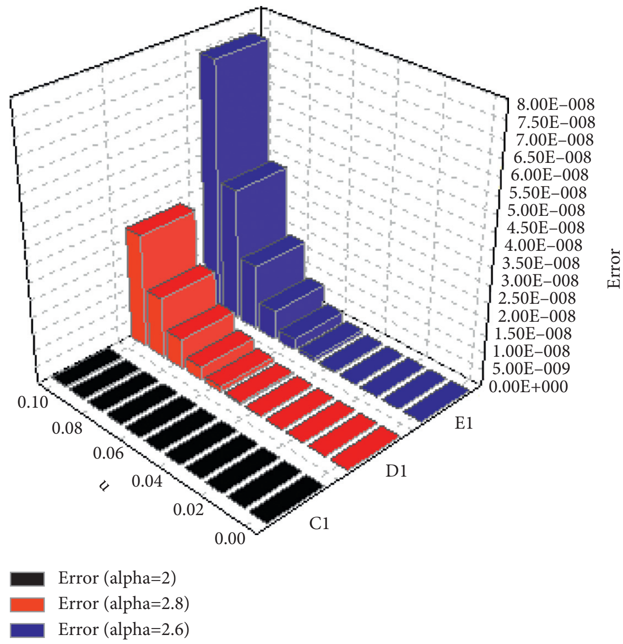

Figure 13: The graph of absolute error at various fractional orders for problem 5.

6. Conclusion

In this study, we applied the Chebyshev pseudospectral method for solving fractional delay differential equations. The technique is easy to implement and show good convergence rate than other methods. Some examples are solved which shows the effectiveness of the present method. The results we obtained are compared with other methods such as modified wavelet-based algorithm (MWBA), modified Laguerre wavelet method (MLWM), Chebyshev wavelet method (CWM). It is clear from comparison that CPM has higher accuracy than all these methods. Although, CPM can easily be extended to other fractional delay or nondelay models of physics and real-life sciences.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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