Hindawi
Complexity
Volume 2022, Article ID 3221447, 7 pages
https://doi.org/10.1155/2022/3221447

Research Article

CTE Solvability, Nonlocal Symmetry, and Interaction Solutions of Coupled Integrable Dispersionless System

Jun Yu,[1] Bo Ren ,[2] Ping Liu,[3] and Jia-Li Zhou[2]

[1]*Institute of Nonlinear Science, Shaoxing University, Shaoxing 312000, China*
[2]*Department of Mathematics, Zhejiang University of Technology, Hangzhou 310014, China*
[3]*College of Electron and Information Engineering, University of Electronic Science and Technology of China Zhongshan Institute, Zhongshan 528402, China*

Correspondence should be addressed to Bo Ren; renbosemail@163.com

Received 11 June 2022; Revised 31 August 2022; Accepted 5 September 2022; Published 19 September 2022

Academic Editor: Ning Cai

The consistent tanh expansion (CTE) method is successfully applied to the coupled integrable dispersionless (CID) system. A nonauto-Bäcklund transformation (BT) theorem includes two fields f and v_1 is obtained by using the CTE method. One obtains the consistent condition in the nonauto-BT theorem by means of the relation between the fields f and v_1. The CID system possesses the CTE solvability property by some detailed analysis. Many interactions between one soliton and multiple resonant solitons, and between one soliton and cnoidal waves are generated by using the nonauto-BT theorem. The types of bright and gray two front waves are shown by some figures. In the meanwhile, the nonlocal symmetry is obtained by the truncated Painlevé method and the Möbious invariant form. The initial value problem and an auto-BT are constructed by the localization procedure.

1. Introduction

Finding soliton solutions of the nonlinear integrable systems is an important topic in nonlinear science. A large number of useful methods have been investigated, such as the inverse scattering transformation [1–3], Hirota's bilinear method [4], symmetry reductions [5–7], the Darboux transformation [8], the Painlevé analysis method [9], the Bäcklund transformation (BT) [10], the separated variable method [11], etc [12, 13]. The similarity solutions can be found by the Lie point symmetry method [14, 15]. To describe the complex physical phenomena, the interactions among different nonlinear excitations are worth studying compared with soliton solutions [16–19]. These interaction excitations cannot be obtained by the direct Lie point symmetry method. Recently, different types of excitations are valid by using the symmetry reductions related to nonlocal symmetry and a consistent tanh expansion (CTE) method [20–28]. The CTE method can be investigated not only for various different types of excitations but also for an integrable property of the nonlinear systems, including the supersymmetric extension of the nonlinear systems [29–32]. Many dispersionless integrable systems have been developed in various applications, such as quantum field theories, string theory, conformal field theory, and condensed matter theory [33–38]. The geometry of the dispersionless equation is systematically studied by using the bishop frames and Darboux frames [39, 40]. Multisoliton solutions of two components of the CID equation are constructed by the Darboux transformation [41]. The nonlocal symmetry of two components of the CID equation is obtained by the BT method [42]. A coupled dispersionless integrable system and its generalizations based on nonabelian Lie groups have many potential applications in diverse areas [43, 44]. Here, we shall apply the CTE method to study three components of the coupled integrable dispersionless (CID) system. The features of the CTE solvability, interaction solutions, and nonlocal symmetry are considered which might be sufficient to explain the relevant physical processes.

The layout of this paper is organized as follows: In Section 2, the CID system does accept a CTE solvable system by using the CTE method. The nonauto-BT theorem which consists of the consistent condition is given by the CTE method. In Section 3, interactions between one soliton and multiple resonant soliton solutions, between one soliton and cnoidal periodic waves are given by means of the nonauto-BT theorem. In Section 4, the nonlocal symmetry of the CID equation is obtained by the truncated Painlevé analysis. The initial value problem related to the nonlocal symmetry is solved by the localization procedure. Section 5 is a simple summary and discussion.

2. CTE Solvability of CID System

The CID system reads

$$\begin{aligned} u_{xt} + u_x w + u w_x &= 0, \\ v_{xt} - 2u_x v &= 0, \\ w_{xt} - 2u_x w &= 0, \end{aligned} \tag{1}$$

which is first given based on a point of view group theoretical [43]. The CID system (1) is solvable by the inverse scattering transform technique, the Lax pair, and the Painlevé property [43, 44]. The Darboux transformation of the CID system is studied based on a non-Abelian Lie group and expressed the matrix solutions in terms of quasideterminants [45]. The multisoliton solutions are constructed in terms of the Casorati determinants [46]. Based on the generalized Darboux transformation, the nth-order rogue wave solution of the CID equation is studied [47]. Lie symmetry analysis and group invariant solutions for the CID equation are given [48]. The multirotating loop soliton solutions are given by the perturbation technique and symbolic computation [49]. Traveling wave-guide channels of a CID system are investigated by using the fourth-order Runge-Kutta's computational scheme [50] and Hirota's bilinear method [51]. Algebraic structures of a general CID system are analyzed through the prolongation structure approach [52]. The multivalued loop soliton, chaotic soliton chain, and fractal pattern are studied by using the projective Riccati equation method [53].

To apply the CTE method in the CID equation, the consistent tanh expansion is written as the following form based on the leading order analysis [21]

$$\begin{aligned} u &= u_0 + u_1 \tanh(f), \\ v &= v_0 + v_1 \tanh(f), \\ w &= w_0 + w_1 \tanh(f), \end{aligned} \tag{2}$$

where u_0, u_1, v_0, v_1, w_0, w_1 and f are arbitrary functions of (t, x). By substituting equation (2) into the CID system (1) and vanishing the coefficients of powers of $\tanh^3(f)$ and $\tanh^2(f)$, we obtain the solutions of $\{u_1, v_1, w_1\}$ and $\{u_0, v_0, w_0\}$, respectively

$$u_1 = -f_t, v_1 = v_1, w_1 = f_t, \tag{3}$$

and

$$\begin{aligned} u_0 &= \frac{f_{tt}}{2f_t}, \\ v_0 &= -\frac{v_{1t}}{2f_t} - \frac{v_{1x}}{2f_x} + \frac{v_1 f_{xt}}{2f_x f_t}, \\ w_0 &= -\frac{f_{tt}}{2f_t}. \end{aligned} \tag{4}$$

Vanishing the coefficients of $\tanh^1(f)$ and $\tanh^0(f)$ with equations (3 and 4), the functions of f and v_1 satisfy

$$\{f; t\}_x - 4 f_t f_{xt} = 0, \tag{5a}$$

$$v_{1xt} - \left(\frac{v_1 f_{xt}}{f_t}\right)_t + \frac{f_{xt}}{f_t f_x}(v_1 f_t)_x = 0, \tag{5b}$$

where $\{f; t\} = (\partial/\partial t)(f_{tt}/f_t) - (1/2)(f_{tt}/f_t)^2$ is Schwarzian derivative. Two functions for f and v_1 satisfy equation (5). The expansion (2) is thus called a CTE and the CID system (1) is a CTE solvable system [20].

Nonauto-BT theorem. If the solution f and v_1 satisfies the consistent condition (5), then the solution of u, v'and w for equation (6) is also a solution of the CID system (1)

$$\begin{aligned} u &= -f_t \tanh(f) + \frac{f_{tt}}{2f_t}, \\ v &= v_1 \tanh(f) - \frac{v_{1t}}{2f_t} - \frac{v_{1x}}{2f_x} + \frac{v_1 f_{xt}}{2f_x f_t}, \\ w &= f_t \tanh(f) - \frac{f_{tt}}{2f_t}. \end{aligned} \tag{6}$$

The nonauto-BT theorem of the CID equation (1) can be constructed based on the aforementioned detail calculations. Some exact solutions including the interaction between soliton and other kinds of complicated waves can be obtained by means of aforementioned nonauto-BT theorem. In the next section, some concrete interesting examples are given via the nonauto-BT theorem.

3. Interaction Solutions for CID System

A quite trivial straight line solution of equation (5) has the form

$$f = kx + \omega t, v_1 = kx + \omega t, \tag{7}$$

where k and ω are the free constants. Substituting the trivial solution equations (7) into (6), the exact solution of the CID system yields

$$\begin{aligned} u &= -\omega \tanh(kx + \omega t), \\ v &= -1 + (kx + \omega t)\tan h(kx + \omega t), \\ w &= \omega \tanh(kx + \omega t). \end{aligned} \tag{8}$$

The nontrivial solution (8) of the CID equation is given from a quite trivial solution of equation (7).

To find the interaction between one soliton and other nonlinear waves, we can assume the solutions as one straight line (7) plus undetermined waves. The interaction between one soliton and multiple resonant soliton solutions f of equation (6) is given as

$$f = k_0 x + \omega_0 t + a_0 \ln\left[1 + \sum_{i=1}^{n} \exp\left(k_i x + \omega_i t\right)\right], \tag{9}$$

where k_0, k_i and ω_i are arbitrary constants while a_0 and ω_0 are determined by the relations

$$\begin{aligned} a_0 &= \pm\frac{1}{2}, \\ \omega_0 &= \mp\frac{\omega_n}{2}. \end{aligned} \tag{10}$$

The solution v_1 of equation (6) has the following form

$$v_1 = Af_t, \tag{11}$$

with the arbitrary constant A. Equations (5a) and (5b) will become one equation by substituting the relation (11) to (5). It demonstrates that equations (5a) and (5b) transform into a consistent system by means of the relation (11). The solution of the CID equation can be obtained by substituting equations (9) and (11) into (6). By selecting $n = 1, a_0 = (1/2), \omega_0 = -(\omega_1/2)$, the solution of the CID equation reads

$$\begin{aligned} u &= \frac{\omega_1\left[\tanh\left(k_0 x - (\omega_1/2)t + (1/2)\ln\left(k_1 x + \omega_1 t\right)\right) - \exp\left(k_1 x + \omega_1 t\right)\right]}{2\left[1 + \exp\left(k_1 x + \omega_1 t\right)\right]}, \\ v &= \frac{-\omega_1 A\left[\tanh\left(k_0 x - (\omega_1/2)t + (1/2)\ln\left(k_1 x + \omega_1 t\right)\right) + \exp\left(k_1 x + \omega_1 t\right)\right]}{2\left[1 + \exp\left(k_1 x + \omega_1 t\right)\right]}, \end{aligned} \tag{12}$$

and $w = -u$ from the nonauto-BT theorem. Figures 1 and 2 display the special interaction solution with the parameters selected as $k_0 = (1/8), k_1 = (1/3), A = 10$. The parameter ω_1 is chosen as 1/2 and −1/2 in Figures 1 and 2, respectively. The amplitude of v can be controlled with the parameter A. The amplitudes in Figures 1(b) and 2(b) are larger than Figures 1(a) and 2(a) due to $A = 10$. Figures 1(a) and 2(a) are the types of bright $(\omega_1 > 0)$ and gray $(\omega_1 < 0)$ two front waves for u, and vice versa for v.

Similar to the form of equation (9), the interaction between one soliton and cnoidal periodic waves reads as

$$f = k_0 x + \omega_0 t + F(X), X = kx + \omega t, \tag{13}$$

where k_0, ω_0, k and ω are all the free constants. Substituting the expression (13) into (5), we obtain the standard elliptic function equation of $F_1(X)$

$$F_{1X}^2 - 4F_1^4 + a_1 F_1^3 + a_2 F_1^2 + a_3 F_1 + a_4 = 0, F_1 = F_X, \tag{14}$$

with

$$\begin{aligned} a_1 &= \frac{8C_2}{\omega} - \frac{8\omega_0}{\omega}, \\ a_2 &= \frac{16\omega_0 C_2}{\omega^2} - \frac{4\omega_0^2}{\omega^2} - \frac{4C_2^2}{\omega^2} - \frac{C_1\omega^2}{4}, \\ a_3 &= \frac{8\omega_0^2 C_2}{\omega^3} - \frac{8\omega_0 C_2^2}{\omega^3} - \frac{C_1\omega\omega_0}{2}, \\ a_4 &= -C_1\omega_0^2 - \frac{16\omega_0^2 C_2^2}{\omega^4}, \end{aligned} \tag{15}$$

and C_1 and C_2 are arbitrary constants. The interaction between one soliton and cnoidal periodic waves f in equation (5) can be written as [22, 23].

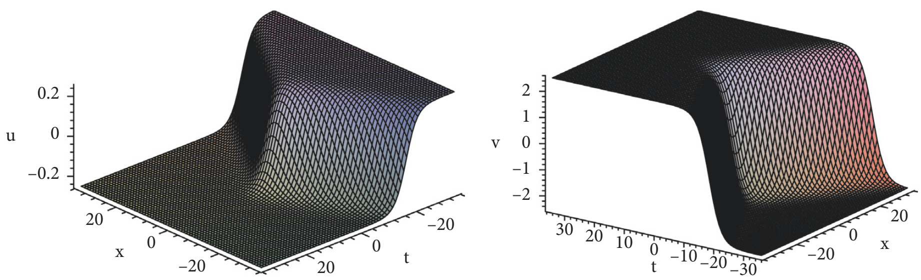
$$f = k_0 x + \omega_0 t + a_0 \,\text{EllipticPi}\left(\text{JacobiSN}\left(kx + \omega t, m\right), n, m\right), \tag{16}$$

where $k_0, \omega_0, a_0, k, \omega, n$ and m are constants and EllipticPi is the third incomplete elliptic integrals. The interaction between a soliton and cnoidal periodic waves f satisfies (5) with the parameter $n = 0$. Besides the solution v_1 of equation (11), one can also get other solutions of the field v_1 by the transformation $v_1 = V_1(X)$. The ordinary differential equation (ODE) is obtained from equation (5)

$$\begin{aligned} &V_{1XX} - \frac{a_0\left(2a_0 k\omega F_1 + k\omega_0 + k_0\omega\right)}{\left(a_0\omega F_1 + \omega_0\right)\left(a_0 k F_1 + k_0\right)} F_{1X} V_{1X} \\ &+ \left[\frac{a_0^2\omega\left(2a_0 k\omega F_1 + k\omega_0 + k_0\omega\right)}{\left(a_0\omega F_1 + \omega_0\right)^2\left(a_0 k F_1 + k_0\right)} F_{1X}^2 \right. \\ &\left. - \frac{a_0\omega}{a_0\omega F_1 + \omega_0} F_{1XX}\right] V_1 = 0. \end{aligned} \tag{17}$$

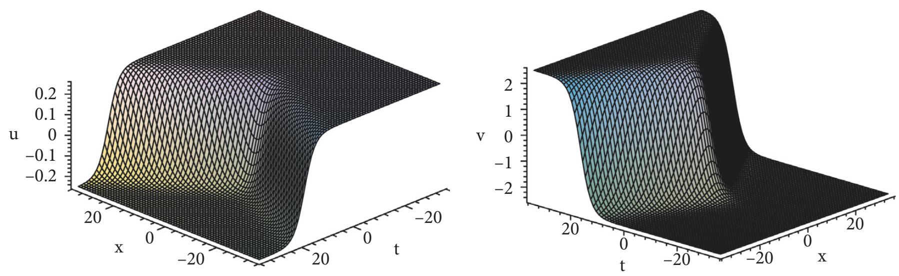
The solution of V_1 can be obtained directly by solving ODE (17)

$$\begin{aligned} v_1 = V_1(X) &= C_3\left(a_0\omega F_1 + \omega_0\right) \\ &+ C_4\left(a_0\omega F_1 + \omega_0\right)\int \frac{\left(a_0 k F_1 + k_0\right)}{\left(a_0\omega F_1 + \omega_0\right)} \mathrm{d}X. \end{aligned} \tag{18}$$



(a) (b)

Figure 1: The propagation for u (bright) and v (gray) is plotted in (a) and (b), respectively.



(a) (b)

Figure 2: The propagation for u (gray) and v (bright) is plotted in (a) and (b), respectively.

The interaction between solitons and cnoidal periodic waves can happen in the ocean [22]. The results are useful for explaining ocean phenomena.

4. Nonlocal Symmetry and BT for CID System

Based on the truncated Painlevé analysis of the CID equation, the Laurent series expansion of u, v and w reads [9]

$$u = \frac{u_1}{\phi} + u_0, v = \frac{v_1}{\phi} + v_0, w = \frac{w_1}{\phi} + w_0, \tag{19}$$

where ϕ, u_0, u_1, v_0, v_1, w_0, and w_1 are functions in a neighborhood of the noncharacteristic singular manifold. By substituting the expansion (19) into (1) and vanishing the coefficients of ϕ independently, one obtains

$$\begin{aligned} u_1 &= -\phi_t, \\ v_1 &= A\phi_t, \\ w_1 &= \phi_t. \end{aligned} \tag{20}$$

and

$$\begin{aligned} u_0 &= \frac{\phi_{tt}}{2\phi_t}, \\ v_0 &= -A\frac{\phi_{tt}}{2\phi_t}, \\ w_0 &= -\frac{\phi_{tt}}{2\phi_t}, \end{aligned} \tag{21}$$

with an arbitrary constant A. The field ϕ satisfies the following Schwarzian CID form

$$\{\phi; t\}_x = 0, \tag{22}$$

where $\{\phi; t\}$ is the Schwarzian derivative.

By the definition of residual symmetry [21], the nonlocal symmetry of the CID equation is given by the truncated Painlevé analysis (19).

$$\begin{aligned} \sigma^u &= -\phi_t, \\ \sigma^v &= A\phi_t, \\ \sigma^w &= \phi_t. \end{aligned} \tag{23}$$

The nonlocal symmetry (23) can be obtained by the Möbious transformation [54–56]. The Schwarzian form (22) is invariant under the Möbious transformation [9]

$$\phi \longrightarrow \frac{a\phi + b}{c\phi + d}, \quad ad \neq bc. \tag{24}$$

By selecting $d = 1, b = 0, c = -\epsilon$ in equations (24) and (22) possesses the symmetry

$$\sigma^\phi = a\phi + a\phi^2. \tag{25}$$

The nonlocal symmetry (23) will be given by substituting the Möbious transformation symmetry (25) into the symmetry equation of (21).

For Lie's first principle, the initial value problem related to the nonlocal symmetry (23) is

$$\frac{d\overline{u}}{d\epsilon} = -\phi_t, \quad \overline{u}|_{\epsilon=0} = u,$$
$$\frac{d\overline{v}}{d\epsilon} = A\phi_t, \quad \overline{v}|_{\epsilon=0} = v, \tag{26}$$
$$\frac{d\overline{w}}{d\epsilon} = \phi_t, \quad \overline{w}|_{\epsilon=0} = w.$$

The initial value problem of (26) exists in the function ϕ and its differentiation so it is difficult to solve [21]. To solve the initial value problem (26), one introduces a new field to remove the differentiation of ϕ. To eliminate the differentiation, the new field satisfies the following relation

$$\phi_t = g. \tag{27}$$

The symmetry for the prolonged systems (1), (21), and (27) gives

$$\sigma^u = -g, \sigma^v = Ag, \sigma^w = g, \sigma^\phi = -\phi^2, \sigma^g = -2\phi g. \tag{28}$$

The nonlocal symmetry (23) in the original space $\{x, t, u, v, w\}$ is localized to a Lie point symmetry (28) in the prolonged space $\{x, t, u, v, w, \phi, g\}$. The corresponding vector form is

$$V = -g\frac{\partial}{\partial u} + Ag\frac{\partial}{\partial v} + g\frac{\partial}{\partial w} - \phi^2\frac{\partial}{\partial \phi} - 2\phi g\frac{\partial}{\partial g}. \tag{29}$$

For the prolonged CID systems (1), (21), and (27), the initial value problem reads as

$$\frac{d\overline{u}}{d\epsilon} = -g, \quad \overline{u}|_{\epsilon=0} = u,$$
$$\frac{d\overline{v}}{d\epsilon} = Ag, \quad \overline{v}|_{\epsilon=0} = v,$$
$$\frac{d\overline{w}}{d\epsilon} = g, \quad \overline{w}|_{\epsilon=0} = w, \tag{30}$$
$$\frac{d\overline{\phi}}{d\epsilon} = -\phi^2, \quad \overline{\phi}|_{\epsilon=0} = \phi,$$
$$\frac{d\overline{g}}{d\epsilon} = -2\phi g, \quad \overline{g}|_{\epsilon=0} = g.$$

The auto-BT theorem of the enlarged CID systems is constructed by solving the aforementioned initial value problem (30).

Auto-BT Theorem. If u, w, v, ϕ and g are a solution to the enlarged CID systems, then $\overline{u}, \overline{v}, \overline{w}, \overline{\phi}$ and $\overline{g}$ are also a solution to the enlarged CID systems

$$\overline{u} = u - \frac{\epsilon g}{\beta(\epsilon\phi + 1)},$$
$$\overline{v} = v + \frac{\epsilon A g}{\beta(\epsilon\phi + 1)},$$
$$\overline{w} = w + \frac{\epsilon g}{\beta(\epsilon\phi + 1)}, \tag{31}$$
$$\overline{\phi} = \frac{f}{1 + \epsilon\phi},$$
$$\overline{g} = \frac{g}{(1 + \epsilon\phi)^2},$$

with an arbitrary group parameter ϵ. The new solution can be obtained by means of the old solution and the aforementioned auto-BT theorem.

5. Conclusions

In summary, the CTE method has been successfully applied to the CID equation. A nonauto-BT theorem is derived by using the CTE method. It demonstrates that the CID system possesses the CTE solvability property. Abundant interaction between solitons and other types of solitary waves for the CID system is studied by means of the nonauto-BT theorem. One obtains two relations of the solutions between v_1 and f for equations (11) and (18). The types of bright and gray two front waves for u and v are plotted in Figures 1 and 2. Based on the Painlevé analysis, the nonlocal symmetry is constructed. The initial value problem and the auto-BT related by the nonlocal symmetry are obtained by introducing a new field. In the meanwhile, interactions between solitons and other types of waves can be constructed by the symmetry reductions related to nonlocal symmetry. Besides the classical integrable systems, the study of nonlocal symmetry systems has become an important subject in nonlinear science [57, 58]. These aspects of the nonlocal CID system are worthy of study in the future.

Data Availability

The data that support the findings of this article are publicly available.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant (Nos. 11975156 and 11775146) and Key Research and Development Program of Zhejiang Province under Grant (Nos. 2022C01178 and 2022C01193).

References

[1] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Method for solving the Korteweg-deVries equation," *Physical Review Letters*, vol. 19, pp. 1095–1097, 1967.

[2] H. T. Tchokouansi, V. K. Kuetche, and T. C. Kofane, "Inverse scattering transform of a new optical short pulse system," *Journal of Mathematical Physics*, vol. 55, no. 12, Article ID 123511, 2014.

[3] H. T. Tchokouansi and T. C. Kofane, "On the propagation of solitons in ferrites: the inverse scattering approach," *Chaos, Solitons & Fractals*, vol. 86, pp. 64–74, 2016.

[4] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, UK, 2004.

[5] P. J. Olver, *Application of Lie Group to Differential Equation*, Springer-Verlag, Berlin, Germany, 1986.

[6] G. W. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, New York, USA, 2002.

[7] M. S. Velan and M. Lakshmanan, "Lie symmetries and infinite-dimensional Lie algebras of certain (1 + 1)-dimensional nonlinear evolution equations," *Journal of Physics A: Mathematical and General*, vol. 30, no. 9, pp. 3261–3271, 1997.

[8] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, Germany, 1991.

[9] J. Weiss, M. Tabor, and G. Carnevale, "The Painlevé property for partial differential equations," *Journal of Mathematical Physics*, vol. 24, no. 3, pp. 522–526, 1983.

[10] C. Rogers and W. K. Schief, "Bäcklund and Darboux transformations geometry and modern applications in soliton theory," *Cambridge texts in applied mathematics*, Cambridge University Press, Cambridge, United Kingdom, 2002.

[11] X. Y. Tang, S. Y. Lou, and Y. Zhang, "Localized excitations in (2+1)-dimensional systems," *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 66, no. 4, Article ID 046601, 2002.

[12] W. M. Moslem, S. Ali, P. K. Shukla, and X. Y. Tang, "Solitary, explosive, and periodic solutions of the quantum Zakharov-Kuznetsov equation and its transverse instability," *Physics of Plasmas*, vol. 14, no. 8, Article ID 082308, 2007.

[13] X. W. Jin, S. J. Shen, Z. Y. Yang, and J. Lin, "Magnetic lump motion in saturated ferromagnetic films," *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 105, no. 1, Article ID 014205, 2022.

[14] R. Kumar and A. Kumar, "Some invariant solutions of coupled Konno-Oono equations arising in electromagnetic and quantum fields," *Physica Scripta*, vol. 97, no. 7, Article ID 075501, 2022.

[15] R. Kumar and R. S. Verma, "Dynamics of some new solutions to the coupled DSW equations traveling horizontally on the seabed," *J. Ocean Eng. Sci.*In press.

[16] C. Q. Dai and J. F. Zhang, "Controlling effect of vector and scalar crossed double-Ma breathers in a partially nonlocal nonlinear medium with a linear potential," *Nonlinear Dynamics*, vol. 100, no. 2, pp. 1621–1628, 2020.

[17] C. Q. Dai, Y. Y. Wang, Y. Fan, and J. F. Zhang, "Interactions between exotic multi-valued solitons of the (2+1)-dimensional Korteweg-de Vries equation describing shallow water wave," *Applied Mathematical Modelling*, vol. 80, pp. 506–515, 2020.

[18] C. Q. Dai, Y. Fan, and Y. Y. Wang, "Three-dimensional optical solitons formed by the balance between different-order nonlinearities and high-order dispersion/diffraction in parity-time symmetric potentials," *Nonlinear Dynamics*, vol. 98, no. 1, pp. 489–499, 2019.

[19] P. F. Wei, C. X. Long, C. Zhu, Y. T. Zhou, H. Z. Yu, and B. Ren, "Soliton molecules, multi-breathers and hybrid solutions in (2+1)-dimensional Korteweg-de Vries-Sawada-Kotera-Ramani equation," *Chaos, Solitons & Fractals*, vol. 158, Article ID 112062, 2022.

[20] S. Y. Lou, "Consistent Riccati expansion for integrable systems," *Studies in Applied Mathematics*, vol. 134, no. 3, pp. 372–402, 2015.

[21] X. N. Gao, S. Y. Lou, and X. Y. Tang, "Bosonization, singularity analysis, nonlocal symmetry reductions and exact solutions of supersymmetric KdV equation," *Journal of High Energy Physics*, vol. 05, p. 029, 2013.

[22] X. R. Hu, S. Y. Lou, and Y. Chen, "Explicit solutions from eigenfunction symmetry of the Korteweg-de Vries equation," *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 85, no. 5, Article ID 056607, 2012.

[23] X. P. Cheng, S. Y. Lou, C. L. Chen, and X. Y. Tang, "Interactions between solitons and other nonlinear Schrödinger waves," *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, vol. 89, no. 4, Article ID 043202, 2014.

[24] X. W. Jin and J. Lin, "Rogue wave, interaction solutions to the KMM system," *Journal of Magnetism and Magnetic Materials*, vol. 502, Article ID 166590, 2020.

[25] J. C. Chen and Z. Y. Ma, "Consistent Riccati expansion solvability and soliton–cnoidal wave interaction solution of a (2+1)-dimensional Korteweg–de Vries equation," *Applied Mathematics Letters*, vol. 64, pp. 87–93, 2017.

[26] J. C. Chen, Z. Y. Ma, and Y. H. Hu, "Nonlocal symmetry, Darboux transformation and soliton-cnoidal wave interaction solution for the shallow water wave equation," *Journal of Mathematical Analysis and Applications*, vol. 460, no. 2, pp. 987–1003, 2018.

[27] J. C. Chen and S. D. Zhu, "Residual symmetries and soliton-cnoidal wave interaction solutions for the negative-order Korteweg-de Vries equation," *Applied Mathematics Letters*, vol. 73, pp. 136–142, 2017.

[28] J. W. Wu, J. T. He, and J. Lin, "Nonlocal symmetries and new interaction waves of the variable-coefficient modified Korteweg-de Vries equation in fluid-filled elastic tubes," *European Physical Journal A: Hadrons and Nuclei*, vol. 137, no. 7, p. 814, 2022.

[29] S. Y. Lou, X. P. Chen, and X. Y. Tang, "Dressed dark solitons of the defocusing nonlinear Schröinger equation," *Chinese Physics Letters*, vol. 31, Article ID 070201, 2014.

[30] C. L. Chen and S. Y. Lou, "CTE solvability and exact solution to the Broer-Kaup system," *Chinese Physics Letters*, vol. 30, no. 11, Article ID 110202, 2013.

[31] B. Ren, X. Z. Liu, and P. Liu, "Nonlocal symmetry reductions, CTE method and exact solutions for higher-order KdV equation," *Communications in Theoretical Physics*, vol. 63, no. 2, pp. 125–128, 2015.

[32] B. Ren, J. R. Yang, P. Liu, and X. Z. Liu, "Bosonization, Painlevé Property and exact solutions for the N = 1

supersymmetric mKdV equation," *Chinese Journal of Physics*, vol. 53, Article ID 080001, 2015.

[33] I. Krichever, "The dispersionless Lax equations and topological minimal models," *Communications in Mathematical Physics*, vol. 143, no. 2, pp. 415–429, 1992.

[34] R. Carroll and Y. Kodama, "Solution of the dispersionless Hirota equations," *Journal of Physics A: Mathematical and General*, vol. 28, no. 22, pp. 6373–6387, 1995.

[35] S. Aoyama and Y. Kodama, "Topologocal conformal field theory with a rational W potential and the dispersionless KP hierarchy," *Modern Physics Letters A*, vol. 09, no. 27, pp. 2481–2492, 1994.

[36] K. Takasaki and T. Takebe, "Integrable hierarchies and dispersionless limit," *Reviews in Mathematical Physics*, vol. 07, no. 05, pp. 743–808, 1995.

[37] K. Takasaki, "Dispersionless Toda hierarchy and two-dimensional string theory," *Communications in Mathematical Physics*, vol. 170, no. 1, pp. 101–116, 1995.

[38] P. B. Wiegmann and A. Zabrodin, "Conformal maps and integrable hierarchies," *Communications in Mathematical Physics*, vol. 213, no. 3, pp. 523–538, 2000.

[39] K. Eren, "Geometry of coupled dispersionless equations by using Bishop frames," in *TBILISI-MATHEMATICS*, I. Hvedri, Ed., pp. 38–47, Sciendo, Berlin, Germany, 2020.

[40] K. Eren and S. Ersoy, *Complex Coupled Dispersionless Equations in Minkowski 3-space, Complex Var*, 2022, In press.

[41] Q. L. Zha, Z. Yin-Long, and L. Zhi-Bin, "N-soliton solution of a coupled integrable dispersionless equation," *Chinese Physics B*, vol. 18, no. 5, pp. 1780–1786, 2009.

[42] H. C. Hu, X. Hu, and B. F. Feng, "Nonlocal symmetry and consistent tanh expansion method for the coupled integrable dispersionless equation," *Zeitschrift für Naturforschung A*, vol. 71, no. 3, pp. 235–240, 2016.

[43] H. Kakuhata and K. Konno, "A generalization of coupled integrable, dispersionless system," *Journal of the Physical Society of Japan*, vol. 65, no. 2, pp. 340-341, 1996.

[44] T. Alagesan and K. Porsezian, "Singularity structure analysis and Hirota's bilinearisation of the coupled integrable dispersionless equations," *Chaos, Solitons & Fractals*, vol. 8, no. 10, pp. 1645–1650, 1997.

[45] M. Hassan, "Darboux transformation of the generalized coupled dispersionless integrable system," *Journal of Physics A: Mathematical and Theoretical*, vol. 42, no. 6, Article ID 065203, 2009.

[46] S. Y. Lou and G. F. Yu, "A generalization of the coupled integrable dispersionless equations," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 14, pp. 4025–4034, 2016.

[47] Q. L. Zha, "On Nth-order rogue wave solution to nonlinear coupled dispersionless evolution equations," *Physics Letters A*, vol. 376, no. 45, pp. 3121–3128, 2012.

[48] Y. Zhang and B. Gao, "Lie symmetry analysis for the coupled integrable dispersionless equations," *Pramana - Journal of Physics*, vol. 93, no. 6, p. 100, 2019.

[49] S. Abbagari, S. Youssoufa, H. T. Tchokouansi, and V. K. Kuetche, "N-rotating loop-soliton solution of the coupled integrable dispersionless equation," *Journal of Applied Mathematics and Physics*, vol. 05, no. 06, pp. 1370–1379, 2017.

[50] A. Souleymanou, V. K. Kuetche, T. B. Bouetou, and T. C. Kofane, "Traveling wave-guide channels of a new coupled integrable dispersionless system," *Communications in Theoretical Physics*, vol. 57, no. 1, pp. 10–14, 2012.

[51] A. Souleymanou, V. K. Kuetche, T. B. Bouetou, and T. C. Kofane, "Scattering behavior of waveguide channels of a new coupled integrable dispersionless system," *Chinese Physics Letters*, vol. 28, no. 12, Article ID 120501, 2011.

[52] S. Abbagari, T. B. Bouetou, and T. C. Kofane, "Generating a new higher-dimensional coupled integrable dispersionless system: algebraic structures, Bäcklund transformation and hidden structural symmetries," *Communications in Theoretical Physics*, vol. 60, no. 2, pp. 145–149, 2013.

[53] C. Q. Dai and N. Zhang, "Re-observation on localized waves constructed by variable separation solutions of (1+1)-dimensional coupled integrable dispersionless equations via the projective Riccati equation method," *Applied Mathematics Letters*, vol. 96, pp. 20–26, 2019.

[54] X. P. Cheng, C. L. Chen, and S. Y. Lou, "Interactions among different types of nonlinear waves described by the Kadomtsev-Petviashvili equation," *Wave Motion*, vol. 51, no. 8, pp. 1298–1308, 2014.

[55] B. Ren, "Symmetry reduction related with nonlocal symmetry for Gardner equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 42, pp. 456–463, 2017.

[56] S. Y. Lou, X. R. Hu, and Y. Chen, "Nonlocal symmetries related to Bäcklund transformation and their applications," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, no. 15, Article ID 155209, 2012.

[57] X. Y. Wen, Z. Y. Yan, and Y. Q. Yang, "Dynamics of higher-order rational solitons for the nonlocal nonlinear Schrödinger equation with the self-induced parity-time-symmetric potential," *Chaos*, vol. 26, no. 6, Article ID 063123, 2016.

[58] C. L. Yuan and X. Y. Wen, "Soliton interactions and asymptotic state analysis in a discrete nonlocal nonlinear self-dual network equation of reverse-space type," *Chinese Physics B*, vol. 30, no. 3, Article ID 030201, 2021.