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Research Article

The Analysis of Fractional-Order System Delay Differential Equations Using a Numerical Method

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To solve fractional delay differential equation systems, the Laguerre Wavelets Method (LWM) is presented and coupled with the steps method in this article. Caputo fractional derivative is used in the proposed technique. The results show that the current procedure is accurate and reliable. Different nonlinear systems have been solved, and the results have been compared to the exact solution and different methods. Furthermore, it is clear from the figures that the LWM error converges quickly when compared to other approaches. When compared with the exact solution to other approaches, it is clear that LWM is more accurate and gets closer to the exact solution faster. Moreover, on the basis of the novelty and scientific importance, the present method can be extended to solve other nonlinear fractional-order delay differential equations.

1. Introduction

In 1965, a mathematician named L'Hopital asked Leibniz what would be the solution to the problem if the derivatives and integrals were fractional order. This L'Hopital question has resulted in the creation of new mathematical knowledge, but no one has been able to deal with it for a long time [1]. Mathematicians began to conduct study in the field of fractional derivatives, integration, and the development of a new field of fractional calculus after a period of time. In mathematics, this domain is known as fractional calculus, and it is a significant branch of mathematics that deals with the study of fractional derivatives and integration. Mathematicians have recently started working on fractional calculus because of its wide applications in all fields of research such as economics [2], viscoelastic materials [3], dynamics of interfaces between soft nanoparticles and rough substrates [4], continuum and statistical mechanics [5], solid mechanics [6], and many other topics.

Many natural problems can be solved using mathematical formulations by transforming physical facts into equation form. Differential equations (DEs) are a type of equation that is used to model a variety of phenomena. However, certain cases are too complicated to be solved using a differential equation. In this case, the researchers used fractional differential equations (FDEs), which are more accurate than differential equations with order integers in modelling the phenomenon. FDEs have realised the importance of real-world modelling challenges in recent years. Such as electrochemistry of corrosion [7], electrode-electrolyte polarization [8], heat conduction [9], optics and signal processing [10], diffusion wave [11], circuit systems [12], control theory of dynamical systems [6], probability and statistics [14, 15], fluid flow [16], and so on.

Equations with delayed arguments are known as fractional delay differential equations (FDDEs). Time delay, spatial delay, step size delay, constant delay, and so on are examples of delayed arguments. Due to various delay arguments found in nature, FDDEs are classified into distinct types. FDDEs are time delay DDEs, which are equations in which the current time derivatives are dependent on the solution and possibly its derivatives at a previous time. In the last few decades, mathematicians have paid more attention to FDDEs for modelling than simple ODEs, because a small delay has a big impact. FDDEs are employed in a variety of domains of mathematics, including infection diseases, navigation control, population dynamics, circulating blood, and the body's reaction to carbon dioxide [17–19], as well as some additional applications in advanced research studies.

It is necessary to develop accurate, time-efficient, and computationally efficient numerical algorithms for solving FDDEs. Xu and Ma [20] investigated the SEIRS epidemic model with a saturation incidence rate and a time delay that defined the latent period. Rihan et al. [21] investigated a delay differential model, numerically analysed it, and established an effective method of combining chemotherapy with therapeutic immunotherapy in 2014. The global stability of the Lotka–Volterra autonomous model with diffusion and time delay was studied by Beretta and Takeuchi [22]. Lv and Gao [23] used the well-known reproducing kernel Hilbert space approach to solve neutral functional proportional delay differential equations (RKHSM). Galach [24] investigated the time delay in the model presented by Kuznetsov and Taylor, where the time delay was included to gain better compatibility with reality. Furthermore, some researchers discussed the behaviour of delay fractional differential equations or a system of delay fractional differential equations, as well as their stability and analysis. Some works, such as in [25, 26], demonstrate this style of research.

In a number of situations, exact FDDEs solutions are difficult to get. As a result, the researchers' key goal is to develop a numerical or analytical solution to FDDEs. As a result, many strategies have been employed such as the New Predictor Corrector Method (NPCM) [27], New Iterative Method (NIM) [28], Adomian Decomposition Method (ADM) [29], Backward Differentiation Formula (BDF) [30], Chebyshev Pseudospectral Method (CPM) [31], Legendre–Gauss Collocation Method (LGCM) [32], Adams–Bashforth–Moulton Algorithm (ABMA) [33], operational matrix based on poly-Bernoulli polynomials (OMM) [34], and Runge Kutta-type Method (RKM) [35]. Overall, some of the approaches used to obtain numerical or analytical solutions to FDDEs have low accuracy of convergence, while others have great accuracy. Among all of these approaches, the wavelet approximation family is one of the more recent methods for locating FDDE solutions. For the approximate solution of FDDEs systems in the current study, we implement Laguerre Wavelets Method (LWM) in combination with the steps method. The proposed solution is shown to be entirely compatible with the complexity of such problems and to be extremely user-friendly. The error comparison shows that the suggested technique has a very high level of accuracy.

The structure of remaining paper is summarized as follows. Section 2 defines some basic definitions related to our present work. The general methodology for solving FDDEs is provided in Section 3. Section 4 presents the main results, numerical simulations, and graphical representations. The conclusion along with future research directions is drawn in Section 5.

2. Preliminaries Concept

This section introduces the basic concept and several important definitions from fractional calculus, which we will apply in our current research.

2.1. Definition. The following mathematical statement demonstrates Caputo's definition for fractional derivatives of order δ [36, 37].

$$D^{\delta}\xi(\psi) = \frac{1}{\Gamma(m-\delta)}\int_0^{\psi}(\psi-\tau)^{m-\delta-1}\xi^{(m)}(\tau)\mathrm{d}\tau, \qquad (1)$$

for $n-1<\delta\leq m$, $m\in\mathbb{N}$, $\psi>0$, $\xi\in\mathbb{C}_{-1}^{n}$.

2.2. Definition. The Riemann–Liouville integral operator for order δ is given as [36, 37].

$$I^{\delta}\xi(\psi) = \frac{1}{\Gamma(\delta)}\int_0^{\psi}(\psi-\tau)^{\delta-1}\xi(\tau)\mathrm{d}\tau. \qquad (2)$$

The following are the properties of the Caputo derivative and Riemann–Liouville integral operators.

$$D^{\delta}I^{\delta}\xi(\psi) = \xi(\psi),$$

$$I^{\delta}D^{\delta}\xi(\psi) = \xi(\psi) - \sum_{k=0}^{n-1}\frac{\xi^{(k)}(0^{+})}{k!}\psi^{k}, \quad \psi\geq 0\, n-1<\delta<n. \qquad (3)$$

3. Laguerre Wavelets

Wavelets [38–40] are a family of functions made up of dilation and translation of a single function called the mother wavelet, $\varphi(\psi)$. The family of continuous wavelets [41] is formed when the dilation parameter a and the translation parameter b vary continuously.

$$\varphi_{a,b}(\psi) = |a|^{-1/2}\varphi\left(\frac{\psi-b}{a}\right), \quad a,b\in\mathbf{R}, a\neq 0. \qquad (4)$$

The following family of discrete wavelets results from restricting the parameters a and b to discrete values as $a=a_0^{-p}, a=nb_0a_0^{-p}, a_0>1, b_0>0$,

$$\varphi_{p,n}(\psi) = |a|^{-p/2}\varphi\left(a_0^{p}(\psi)-nb_0\right), \quad p,n\in\mathbb{Z}, \qquad (5)$$

Table 1: Comparison of the exact and MLWM solution for example 1 at $m = 9$.

ψ	Exact $\xi(\psi)$	Exact $\zeta(\psi)$	MLWM solution $\xi(\psi)$	MLWM solution $\zeta(\psi)$
0	1.000000000000000	0.000000000000000	1.000000000000000	0.000000000000000
0.1	0.900316999845194	0.998334166468281	0.900316999845194	0.998334166468281
0.2	0.802410647342520	0.198669330795061	0.802410647342527	0.198669330795061
0.3	0.707730678026351	0.295520206661339	0.707730678025662	0.295520206661339
0.4	0.617405647901646	0.389418342308650	0.617405647901653	0.389418342308637
0.5	0.532280730215671	0.479425538604203	0.532280730215273	0.479425538604203
0.6	0.452953789145250	.5646424733950353	0.452953789145497	.5646424733947035
0.7	0.379809389925154	0.644217687237691	0.37980938992536	0.644217687236876
0.8	0.313050504004480	0.717356090899522	0.313050504004346	0.717356090898501
0.9	0.252727753291169	0.783326909627483	0.252727753291868	0.783326909628249
1.0	0.198766110346413	0.841470984807896	0.198766110346480	0.841470984813237

Table 2: Error estimation of proposed method with FBPs for example 1 at $m = 9$.

ψ	Error (ξ_{MLWM})	Error (ζ_{MLWM})	Error (ξ_{FBPs})	Error (ζ_{FBPs})
0.2	7.1240361137E-15	2.1190342156276E-17	1.22E-11	3.55E-12
0.4	3.1090504913E-13	1.2920241091118E-14	9.91E-12	1.04E-11
0.6	4.0077803050E-12	3.3177676221433E-13	7.20E-12	1.59E-11
0.8	2.5602591191E-12	1.0209754593019E-12	6.56E-12	2.06E-11
1.0	4.9538867166E-11	5.3406078485367E-12	7.58E-10	1.47E-11

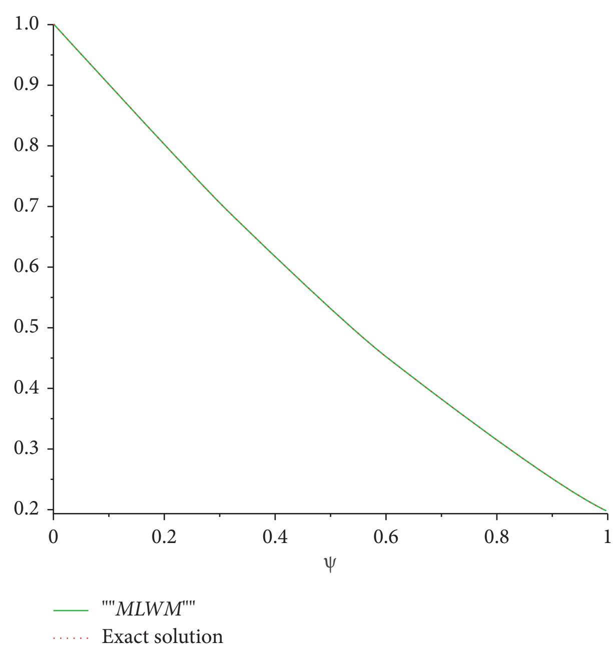

Figure 1: Behaviour of the exact solution and proposed method solution for $\xi(\psi)$ of problem 1.

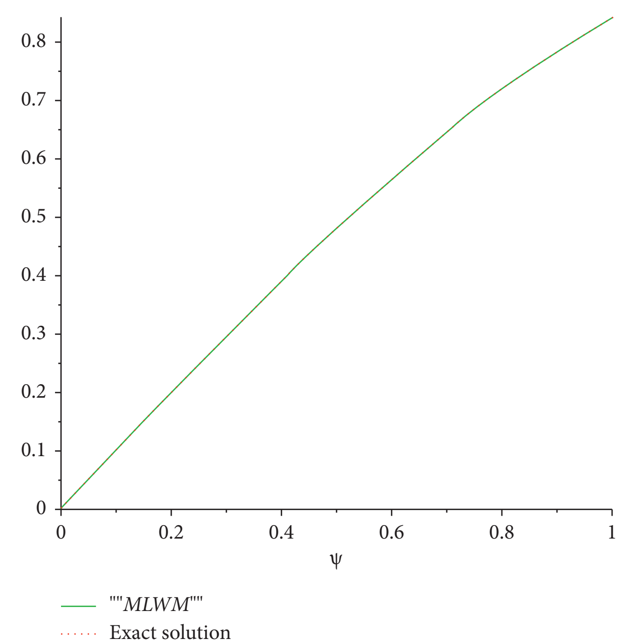

Figure 2: Behaviour of the exact solution and proposed method solution for $\zeta(\psi)$ of problem 1.

where the wavelet basis for $L^2(\mathbf{R})$ is $\varphi_{p,n}$. When $a_0 = 2$ and $b_0 = 1$, for instance, $\varphi_{p,n}(\psi)$ forms an orthonormal basis. There are four arguments in the Laguerre wavelets $\Phi_{n,m}(\psi) = \varphi(k, n, m, \psi)$, $n = 1, 2, \ldots, 2^{k-1}$, where k is non-negative integer, m represents the Laguerre polynomials degree, and represents normalized time. Over the interval $[0, 1)$, they are defined as

$$\varphi_{n,m} = \begin{cases} 2^{p/2}\widetilde{\mathscr{L}}_m\left(2^p\psi - 2n + 1\right), & \dfrac{n-1}{2^{p-1}} \leq \psi < \dfrac{n}{2^{p-1}}, \\ 0, & \text{Otherwise}, \end{cases} \tag{6}$$

where

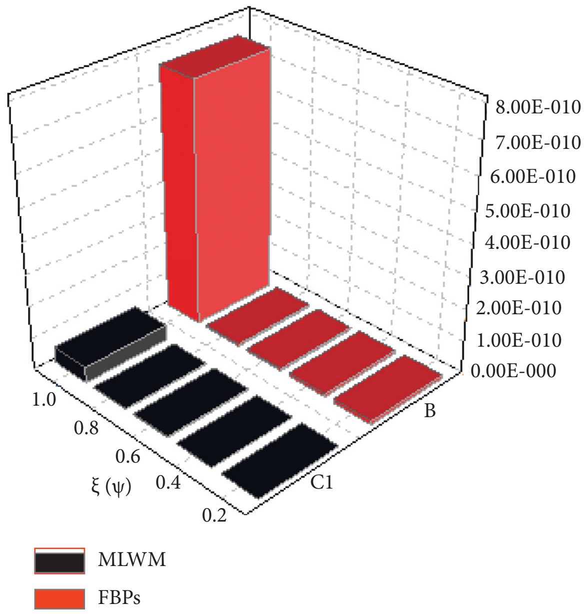

FIGURE 3: FBPs and proposed method error analysis for $\xi(\psi)$ of example 1.

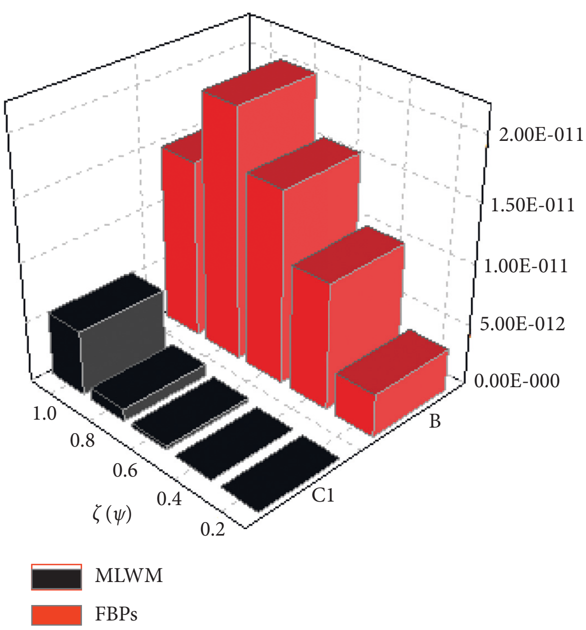

FIGURE 4: FBPs and proposed method error analysis for $\zeta(\psi)$ of example 1.

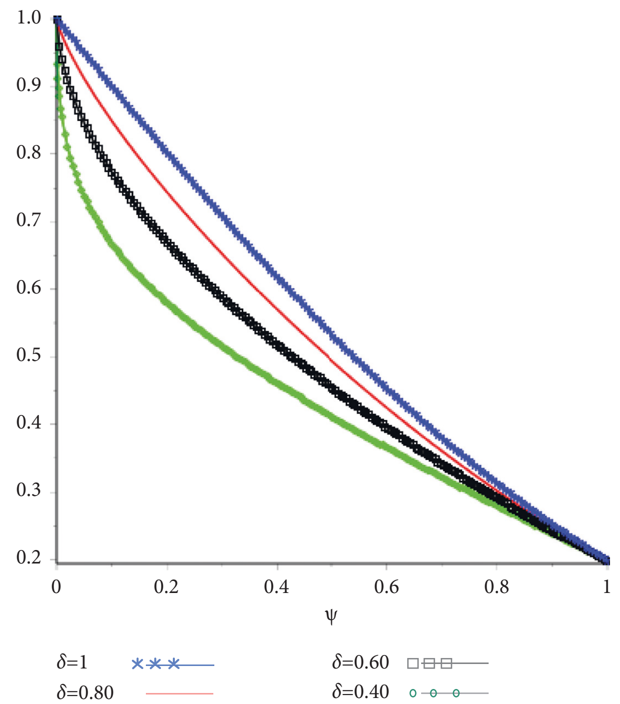

FIGURE 5: The error comparison at various fractional-orders of for $\xi(\psi)$ example 1.

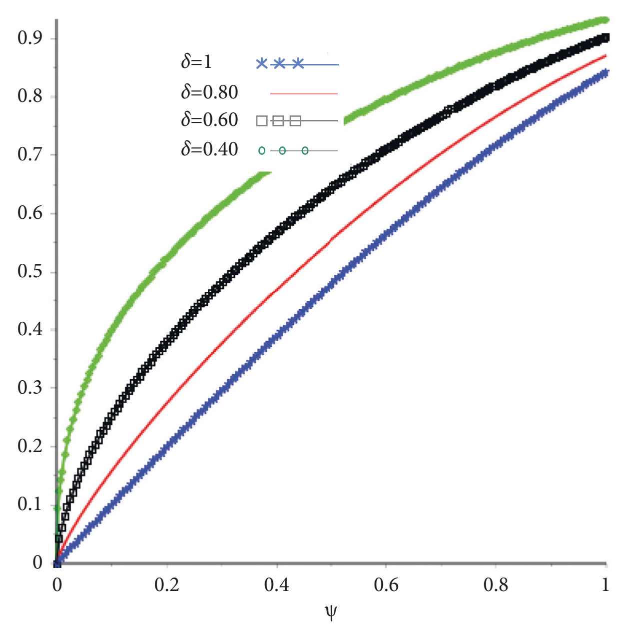

FIGURE 6: The error comparison at various fractional-orders of for $\zeta(\psi)$ example 1.

$$\widetilde{\mathscr{L}}_m = \frac{1}{m!}\mathscr{L}_m(\psi) \quad m = 0, 1, 2, \ldots A - 1. \tag{7}$$

$m = 0, 1, 2, \ldots, M - 1$. The coefficients are utilised in (10) to determine orthonormality. The Laguerre polynomials having degree m with regard to $w(\psi) = 1$ weight function on the interval $[0, \infty]$ are $L_m(\psi)$ and satisfy the recursive formula:

$$\mathscr{L}_0(\psi) = 1,\ \mathscr{L}_1(\psi) = 1 - \psi,$$
$$\mathscr{L}_{m+2} = \frac{((2m + 3 - x)\mathscr{L}_{m+1}(\psi) - (m + 1)\mathscr{L}_m)}{m + 2} \quad m = 0, 1, 2, 3, 4, \ldots. \tag{8}$$

where

Modified Laguerre wavelets method (MLWM): Here, we consider the delay differential equation of the form:

$$y^{\alpha}(\psi) = f(\psi) + g(\psi)y\left(\frac{\psi}{a} - c\right), \quad 0 < \psi < b, 0 < \alpha \leq 1,$$
$$y(\psi) = p(\psi), \quad -b \leq \psi \leq 0, \tag{9}$$

Table 3: Comparison at different fractional-order of δ on the basis of error for example 2.

δ	ψ	Exact	*MLWM* solution	*MLWM* error	Spline functions
	0.01	0.0001	0.00009999986375	1.3625E-10	8.2E-4
	0.02	0.0004	0.0003999998553	1.447E-10	2.5E-3
0.1	0.03	0.0009	0.0008999998624	1.376E-10	4.7E-3
	0.04	0.0016	0.001599999883	1.17E-10	7.3E-3
	0.05	0.0025	0.002499999813	1.87E-10	1.0E-2
	0.01	0.0001	0.0001000000111	1.11E-11	4.4E-4
	0.02	0.0004	0.0004000001035	1.035E-10	1.4E-3
0.2	0.03	0.0009	0.0008999999845	1.55E-11	2.7E-3
	0.04	0.0016	0.001600000057	5.7E-11	4.4E-3
	0.05	0.0025	0.002500000022	2.2E-11	6.1E-3
	0.01	0.0001	0.0001000001798	1.798E-10	2.1E-4
	0.02	0.0004	0.0004000002303	2.303E-10	7.1E-4
0.3	0.03	0.0009	0.0009000001717	1.717E-10	1.4E-3
	0.04	0.0016	0.001600000106	1.06E-10	2.4E-3
	0.05	0.0025	0.002500000136	1.36E-10	3.5E-3
	0.01	0.0001	0.0001000000189	1.89E-11	8.1E-5
	0.02	0.0004	0.0003999999502	4.98E-11	2.9E-4
0.4	0.03	0.0009	0.0008999998916	1.084E-10	6.1E-4
	0.04	0.0016	0.001599999941	5.9E-11	1.0E-3
	0.05	0.0025	0.002499999997	3.000E-12	1.0E-3
	0.01	0.0001	0.00009999988074	1.1926E-10	4.5E-6
	0.02	0.0004	0.0003999997780	2.220E-10	2.6E-5
0.5	0.03	0.0009	0.0008999996906	3.094E-10	7.0E-5
	0.04	0.0016	0.001599999717	3.83E-10	1.4E-4
	0.05	0.0025	0.002499999657	3.43E-10	2.5E-4

Table 4: Comparison at different fractional-orders of δ on the basis of error for example 2.

δ	ψ	Exact	*MLWM* solution	*MLWM* error	Spline functions
	0.01	0.0001	0.00009999986375	1.3625E-10	8.2E-4
	0.02	0.0004	0.0003999998553	1.447E-10	2.5E-3
	0.03	0.0009	0.0008999998624	1.376E-10	4.7E-3
	0.04	0.0016	0.001599999883	1.17E-10	7.3E-3
0.1	0.05	0.0025	0.002499999813	1.87E-10	1.0E-2
	0.01	0.0001	0.0001000000111	1.11E-11	4.4E-4
	0.02	0.0004	0.0004000001035	1.035E-10	1.4E-3
	0.03	0.0009	0.0008999999845	1.55E-11	2.7E-3
	0.04	0.0016	0.001600000057	5.7E-11	4.4E-3
0.2	0.05	0.0025	0.002500000022	2.2E-11	6.1E-3
	0.01	0.0001	0.0001000001798	1.798E-10	2.1E-4
	0.02	0.0004	0.0004000002303	2.303E-10	7.1E-4
	0.03	0.0009	0.0009000001717	1.717E-10	1.4E-3
	0.04	0.0016	0.001600000106	1.06E-10	2.4E-3
0.3	0.05	0.0025	0.002500000136	1.36E-10	3.5E-3
	0.01	0.0001	0.0001000000189	1.89E-11	8.1E-5
	0.02	0.0004	0.0003999999502	4.98E-11	2.9E-4
	0.03	0.0009	0.0008999998916	1.084E-10	6.1E-4
	0.04	0.0016	0.001599999941	5.9E-11	1.0E-3
0.4	0.05	0.0025	0.002499999997	3.000E-12	1.0E-3
	0.01	0.0001	0.00009999988074	1.1926E-10	4.5E-6
	0.02	0.0004	0.0003999997780	2.220E-10	2.6E-5
	0.03	0.0009	0.0008999996906	3.094E-10	7.0E-5
0.5	0.04	0.0016	0.001599999717	2.83E-10	1.4E-4
	0.05	0.0025	0.002499999657	3.43E-10	2.5E-4

where $f(\psi)$ is a provided continuous linear or nonlinear function and $g(\psi)$ is a source term function. Using the proposed method, transform the delay differential equation (12) to an inhomogeneous ordinary differential equation by using the initial source, $p(\psi)$, as shown in (12):

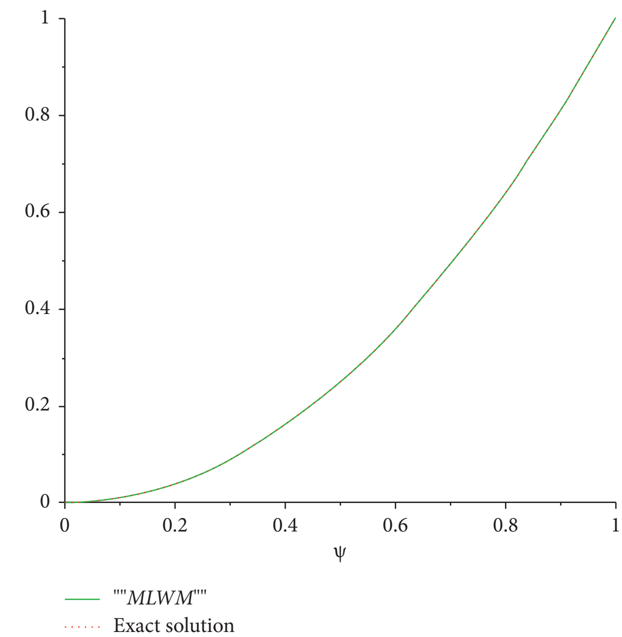

Figure 7: Analysis of the exact and proposed method solution for $\xi(\psi)$ of problem 2.

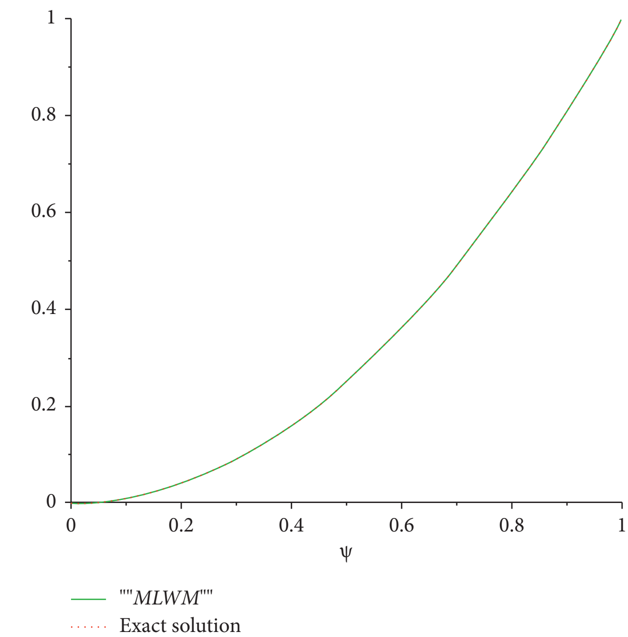

Figure 8: Analysis of the exact and proposed method solution for $\zeta(\psi)$ of problem 2.

$$y^{\alpha}(\psi) = f(\psi) + g(\psi)p\left(\frac{\psi}{a} - c\right), \quad 0 < \psi < b, 1 < \alpha \leq 2. \tag{10}$$

Equation (14) can be expanded as a Laguerre wavelets series as follows:

$$y(\psi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{n,m}\varphi_{n,m}(\psi), \tag{11}$$

where $\varphi_{n,m}(\psi)$ is determined by (9). The truncated series is used to approximate $y(\psi)$.

$$y_p, A = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{n,m}\varphi_{n,m}(\psi), \tag{12}$$

Then, there should be a total of $2^{p-1}A$ conditions for determining the $2^{p-1}A$ coefficient:

$$c_{10}, c_{11} \ldots c_{A-1} \ldots c_{20}, c_{2A-1} \ldots c_{2^{p-1}1} \ldots c_{2^{p-1}A-1}. \tag{13}$$

Since the initial and boundary conditions, respectively, provide the conditions.

$$y_p, A(0) = \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-1} d_{n,m}\varphi_{n,m}(0) = q(0). \tag{14}$$

$$\frac{d}{d\psi} y_p, A(1) = \frac{d}{d\psi} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-1} d_{n,m}\varphi_{n,m}(1) = \acute{q}(1). \tag{15}$$

We see that there should be $2^{p-1}A - 2$ extra condition to recover the unknown coefficient $d_{n,m}$. These conditions can be obtained by substituting (14) in (12):

$$\frac{d^{\alpha}}{d\psi^{\alpha}} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi) = f\left(\sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi)\right) + g(\psi)p\left(\frac{\psi}{a} - c\right). \tag{16}$$

We, now assume equation (18) is exact at $2^{p-1}A - 3$ points ψ_i as follows:

$$\frac{d^{\alpha}}{d\psi^{\alpha}} \sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi_i) = f\left(\sum_{n=1}^{2^{p-1}} \sum_{m=0}^{A-3} d_{n,m}\varphi_{n,m}(\psi_i)\right) + g(\psi_i)p\left(\frac{\psi_i}{a} - c\right). \tag{17}$$

The best choice of the ψ_i points are the zeros of the shifted Laguerre polynomials of degree $2^{p-1}A - 2$ in the interval [0, 1] that is $\psi_i = s_i - 1/2$, where $s_i = \cos((2_i - 1)\pi/2^{p-1}A - 1), i = 1, 2, 3, \ldots 2^{p-1}A - 2$. Since the initial and boundary conditions, respectively, provide the conditions. Combining equations (9) and (12) yields $2^{p-1}A$ linear equations from which the unknown coefficients, $d_{n,m}$, can be computed. The same technique is followed for first- and second-order delay differential equations.

4. Numerical Representation

4.1. Example. Consider the system of fractional ordinary delay differential equations [42],

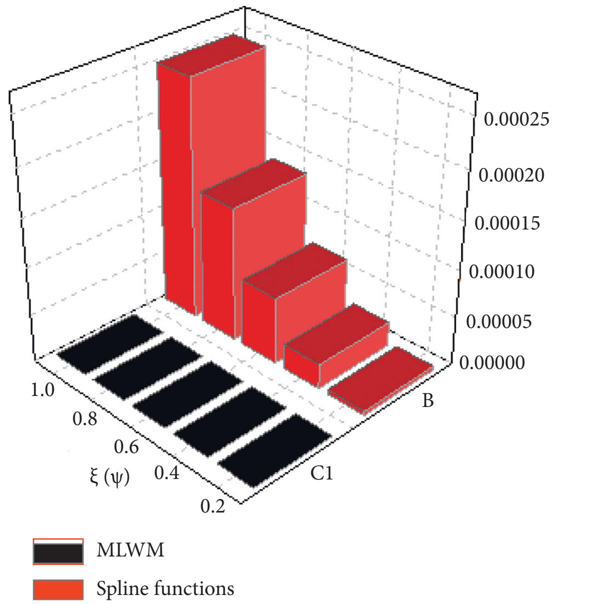

FIGURE 9: Spline functions and proposed method error analysis for $\xi(\psi)$ of example 2.

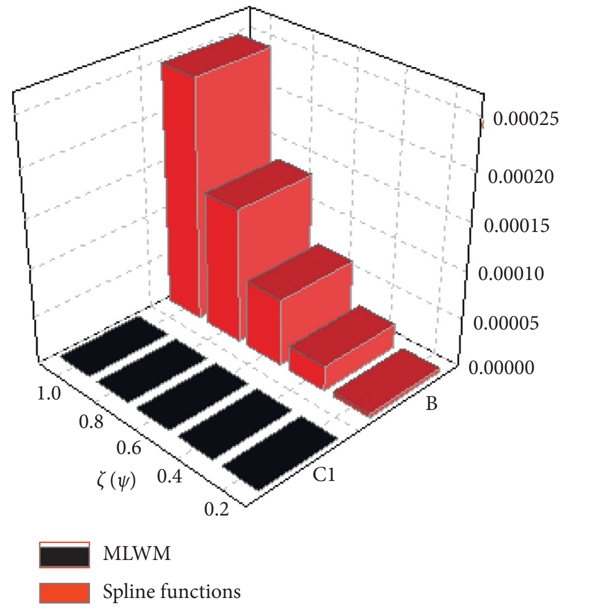

FIGURE 10: Spline functions and proposed method error analysis for $\zeta(\psi)$ of example 2.

$$D^{\delta}\xi(\psi) = -\zeta(\psi) - 2e^{-\frac{3}{4}\psi}\cos\left(\frac{1}{2}\psi\right)\sin\left(\frac{1}{4}\psi\right)\xi(0.25\psi) - e^{-\psi}\cos\left(\frac{1}{2}\psi\right)\zeta(0.5\psi),$$
$$D^{\delta}\zeta(\psi) = e^{\psi}\xi^{2}(0.5\psi) - \zeta^{2}(0.5\psi), \tag{18}$$

with the initial sources $\xi(0) = 1$, $\zeta(0) = 0$, and having exact solution at $\delta = 1$ as $\xi(\psi) = e^{-\psi}\cos(\psi)$, $\zeta(\psi) = \sin(\psi)$.

Table 1 shows the exact solution and numerical results achieved using the proposed method. Table 2 shows the comparison on the basis of absolute error between our technique and those derived from FBPs. When $\delta = 1$, the behaviour of the exact solution and proposed method solution of this problem is shown in Figures 1 and 2, respectively, whereas the error comparison of CPM and FBPs is shown in Figures 3 and 4. Figures 5 and 6 show graphical representations for different fractional orders of δ, confirming that the proposed method solution converges to the exact solution as the value of δ approaches from fractional-order towards integer-order.

4.2. Example. Consider the system of fractional ordinary delay differential equations [43].

$$D^{\delta}\xi(\psi) = -\xi(\psi) + \zeta\left(\frac{\psi}{2}\right) + \frac{3}{4}\psi^{2} + \frac{2}{\Gamma(3-\delta)}\psi^{2-\delta},$$
$$D^{\delta}\zeta(\psi) = \zeta(\psi) - \xi\left(\frac{\psi}{2}\right) - \frac{3}{4}\psi^{2} + \frac{2}{\Gamma(3-\delta)}\psi^{2-\delta}. \tag{19}$$

The exact solution is given by $\xi(\psi) = \psi^{2}$ and $\zeta(\psi) = \psi^{2}$.

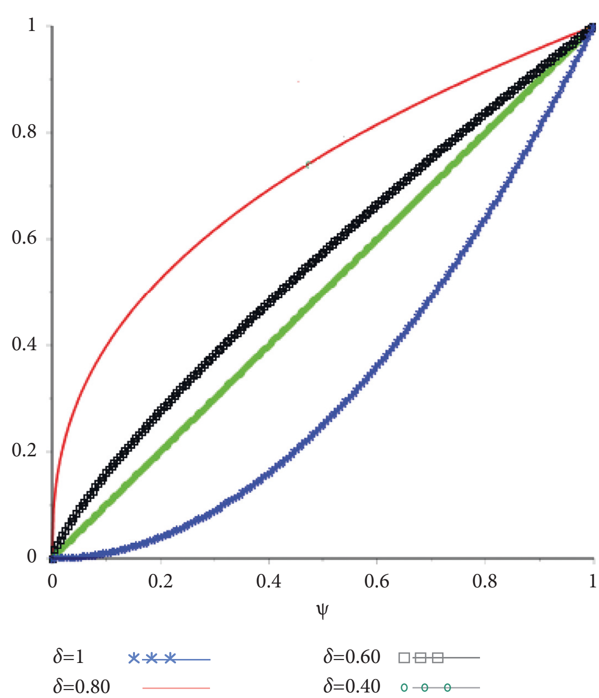

FIGURE 11: The error comparison at various fractional-orders for $\xi(\psi)$ of example 2.

The comparison among the exact solution and the Spline function polynomial technique solution are shown in Table 3. In Table 4, the errors acquired by the current technique are compared to those obtained by the Spline function polynomial method. In Figures 7 and 8, we compare the exact and approximated solutions, which shows that they are very close to each other. In addition, Figures 9 and 10 show the MLWM and Spline function error comparisons,

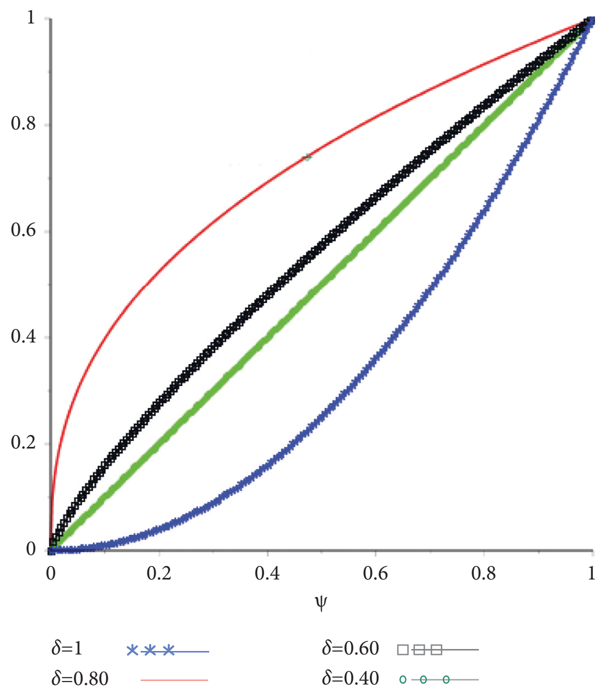

Figure 12: The error comparison at various fractional-orders for $\zeta(\psi)$ of example 2.

demonstrating that suggested approach is in best agreement with the exact solution.

5. Conclusion

We used the MLWM to solve fractional delay differential equations systems in this research. The proposed method's convergence is given special consideration. As demonstrated in Figures 1–12, the fractional-order delay differential equation solution approaches towards the solution of the integer-order delay differential equation. The results obtained by implementing the proposed method are in great agreement with the exact solution and are more accurate than those obtained by implementing other techniques. The proposed method (MLWM) is extremely user-friendly but extremely accurate, according to computational effort and numerical results. The computations work in this article are done using Maple.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors jointly worked on the results, and they read and approved the final manuscript.

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