

Research Article

Modeling and Dynamics of the Fractional Order SARS-CoV-2 Epidemiological Model

Tahir Khan ^{1,2}, Roman Ullah ², Ali Yousef ³, Gul Zaman ⁴, Qasem M. Al-Mdallal ⁵,
and Yasser Alraey ⁶

¹Department of Mathematics and Statistics, Woman University Swabi, Khyber Pakhtunkhwa, Pakistan

²Department of Computing, Muscat College Muscat, Muscat, Oman

³Department of Mathematics, Kuwait College of Science and Technology, Safat 13133, P. O. Box 27235, Kuwait

⁴Department of Mathematics, University of Malakand, Chakdara Dir Lower, Khyber Pakhtunkhwa, Pakistan

⁵Department of Mathematical Sciences, UAE University, P. O. Box 15551, Al-Ain, UAE

⁶Department of Clinical Laboratory Sciences, Central Research Laboratory, College of Applied Medical Sciences, King Khalid University, Abha, Saudi Arabia

Correspondence should be addressed to Qasem M. Al-Mdallal; q.almdallal@uaeu.ac.ae

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We propose a theoretical study to investigate the spread of the SARS-CoV-2 virus, reported in Wuhan, China. We develop a mathematical model based on the characteristic of the disease and then use fractional calculus to fractionalize it. We use the *Caputo-Fabrizio* operator for this purpose. We prove that the considered model has positive and bounded solutions. We calculate the threshold quantity of the proposed model and discuss its sensitivity analysis to find the role of every epidemic parameter and the relative impact on disease transmission. The threshold quantity (reproductive number) is used to discuss the steady states of the proposed model and to find that the proposed epidemic model is stable asymptotically under some constraints. Both the global and local properties of the proposed model will be performed with the help of the mean value theorem, *Barbalat's lemma*, and linearization. To support our analytical findings, we draw some numerical simulations to verify with graphical representations.

1. Introduction

The coronavirus family causes infections in humans, beginning with a common cold and progressing to SARS. Two coronavirus epidemics have been recorded in the preceding twenty years [1–3]. SARS was one of them, and it created a large-scale outbreak in several nations. Approximately, 8000 people were affected by this outbreak, with 800 of them dying. In December 2019, a serious respiratory sickness outbreak began in Wuhan, China [4]. In early January 2020, the causal agent, a new coronavirus, was identified and isolated from a single patient (COVID-19). According to scientific evidence, animals were the earliest source of virus transmission, although the majority of cases are caused by infected

humans contacting susceptible humans. The spread of this virus is a hot topic that has touched practically every corner of the globe and has been reported in over 200 countries. According to current records, there have been over 401,288,380 confirmed cases, with 5,783,182 deaths occurring till February 9th 2022. According to the WHO, this is a public health emergency. The World Health Organization (WHO) has designated it a public health emergency of worldwide concern due to the severity of the condition. This virus appears to be highly contagious, spreading rapidly to nearly every country on the planet, prompting the declaration of a global pandemic. It signifies that it is a highly major public health threat, with symptoms such as cough, fever, lethargy, and breathing problems after infection.

Fractional computing is an emerging area of mathematics and attracted the attention of researchers. Because of the wide applications to express the axioms of heritage and recall different physical situations that occur in various fields of applied science. Many classical models have been shown with less accuracy in prediction about the temporal dynamics of the disease, while on the other hand, models with noninteger order provide better information in allocating and preserving data for large-scale analysis [5–7]. Moreover, the derivative of integer order does not find the dynamics between two various points [8, 9]. Furthermore, the comparison of integer and noninteger order epidemic models reveals that models with noninteger order are the generalization of integer order and provide more and accurate dynamics rather than the classical order, (for detail see [9–11]). A model with noninteger order demonstrating the complex dynamics of a biological system has been proposed by Asma et al. [12]. A fractional-order epidemic model has been investigated to explore the dynamics of toxoplasmosis in feline and human populations [13]. Another study has been reported and studied the stability analysis of pests in tea with fractional order [14]. Similarly, many authors studied dynamics of infectious diseases with fractional-order derivatives, for e.g., Hadamard and Caputo and Riemann and Liouville [15–19]. For the solution of Caputo fractional order, epidemiological models of many iterative and numerical methods have been developed; however, the complications of singular kernel arise. So, Caputo and Fabrizio presented an idea based on the nonsingular kernel to overcome the limitation that arises in the above fractional-order derivatives [20].

Coronavirus disease 2019 (COVID-19) is one of the top infectious diseases among other ones and therefore has been recognized as a global threat by World Health Organization (WHO). Due to novel characteristics of the coronavirus disease various researchers have taken a keen interest. Several researchers formulated various epidemiological models to study the dynamics of communicable diseases (see for instance [21–25]). The current pandemic of novel coronavirus disease is also a burning issue and many biologists and mathematicians reported different studies. For example, Wu et al. introduced a model to describe the transmission of the disease based on reported data from 31.12.2019 to 28.01.2020 [26]. Imai et al. studied the transmission of the disease with the help of computational modeling to estimate the disease outbreak in Wuhan, whose main focus was on the human-to-human transmission [27]. Another study has been investigated by Zhu et al. [28] to analyze the infectivity of the novel coronavirus. The dynamical analysis of the novel disease of COVID-19 under the effect of the carrier with environmental contamination has been performed by Hattaf et al. [29]. All the reported studies indicate that bats and minks may be two animal hosts of the novel coronavirus. Similarly, many more studies have been reported on the dynamics of a novel coronavirus, for instance, see [30, 31]. Nevertheless, the literature reveals that the work proposed is an excellent contribution, however, it could be possible to improve further by incorporating some

interesting and important factors related to the novel coronavirus disease.

The spreading of coronavirus disease globally rises from the human-to-human transmission, while the initial source of the disease was an animal/reservoir. The characteristic of SARS-CoV-2 confirms that various infection phases are significant and affect the transmission. The role of asymptomatic is notable because with no symptoms it becomes the major source of transmission of the infection. So, a small number of this population leads to a big disaster. We develop a mathematical model according to the novel disease of coronavirus and keeping in view the aesthetic of the virus. To do this, first, we formulate the model and then fractionalize it to perform the fractional type analysis of the proposed model. The fractional derivative used in this study is a particular case of the new generalized Hattaf fractional (GHF) derivative [32, 33]. We show that the proposed fractional-order epidemiological model is bounded and possesses positive solutions. We also find the steady states of the epidemic problem and discuss asymptotic stabilities. For this, we use the dynamical systems theory. Particularly, we utilize the *linearization*, *mean value theorem*, and *Barbalat's Lemma*. Moreover, the sensitivity analysis will be performed for the threshold parameter to find the impact of each epidemic parameter involved in the model mechanism. We use the sensitivity index formula for this purpose. We perform the numerical visualization of the analytical results to verify the theoretical part and show the effectiveness of the control strategy. We also show the difference between integer and noninteger order epidemiological cases.

2. Formulation of the Model with Fractional Analysis

The proposed problem is formulated by taking into account the characteristics of the novel coronavirus illness. We divide the total human population $N_h(t)$ into four various compartments and assume that $M(t)$ represents the reservoir. In the proposed study, we also consider several transmission routes, such as from human-to-human and from a reservoir-to-human. Before we show the model, we make the following assumption:

- (i) The parameters and variables involved in the model are positive or non-negative values
- (ii) The inflow of newborn are susceptible
- (iii) The novel disease is transmitted by several routes, such as from latent and symptomatic individuals as well as from reservoirs and so accordingly incorporated.
- (iv) Those who have a strong immune system got natural recovery
- (v) Two types of recoveries i.e., from latent and symptomatic populations are taken
- (vi) The death rate due to disease is taken in the symptomatic infected compartment

As a result of combining all the above assumptions, the following system of nonlinear differential equations emerges:

$$\left\{ \begin{array}{l} \frac{dS_h(t)}{dt} = \Lambda - \beta_1 S_h(t) L_h(t) - \gamma \beta_2 S_h(t) I_h(t) - \psi \beta_3 S_h(t) M(t) - d S_h(t), \\ \frac{dL_h(t)}{dt} = \beta_1 S_h(t) L_h(t) + \gamma \beta_2 S_h(t) I_h(t) + \psi \beta_3 S_h(t) M(t) - (\gamma_1 + \gamma_2 + d) L_h(t), \\ \frac{dI_h(t)}{dt} = \gamma_1 L_h(t) - (\gamma_3 + d + d_1) I_h(t), \frac{dR_h(t)}{dt} = \gamma_3 I_h(t) + \gamma_2 L_h(t) - d R_h(t), \\ \frac{dM(t)}{dt} = \eta_2 I_h(t) + \eta_1 L_h(t) - \alpha M(t). \end{array} \right. \quad (1)$$

and the initial population sizes are assumed to be as follows:

$$S_h(0) > 0, L_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0, M(0) \geq 0. \quad (2)$$

In the proposed epidemiological model, the parameters described as Λ is the new birth rate, and the disease transmission rates are symbolized by β_1 , β_2 , and β_3 , which represent the transmission from latent, symptomatic, and reservoir, respectively. We also denote the reduced transmission coefficient by γ and ψ , while γ_1 is the moving ratio of latent to infected and γ_2 denotes the recovery rate. We also denote the recovery rate under treatment by γ_3 , while d is the natural mortality. The disease-induced rate is d_1 . Furthermore, η_1 and η_2 are the two ratios that contribute production of the virus in the seafood market. We denote the removing rate of the virus with α .

2.1. Fractional-Order Epidemiological Model. Let σ be the fractional-order parameter $0 < \sigma < 1$. We will extend the model to its associate fractional order. First, we give some fundamental concepts that will be used in getting our findings.

Definition 1. (see [9]). Let $T > 0$ and assume that $\phi \in H^1(0, T)$, if $n - 1 < \sigma < n$ and $\sigma > 0$ such that $n \in \mathbb{N}$, then the derivative in the sense of *Caputo* as well as the *Caputo-Fabrizio* with σ order are given as follows:

$$CFD_{0,t}^\sigma \{\varphi(t)\} = \frac{K(\sigma)}{(1-\sigma)} \int_0^t \varphi'(u) \exp\left(\frac{(u-t)\sigma}{1-\sigma}\right) du, \quad (3)$$

$$CD_{0,t}^\sigma \{\varphi(t)\} = \frac{1}{\Gamma(-\sigma+n)} \int_0^t (t-u)^{-1+n-\sigma} \varphi^n(u) du,$$

where CF and C are used for the representation of *Caputo-Fabrizio* and *Caputo*, respectively, while $t > 0$ and $K(\sigma)$ are the normalization function, and $K(0) = 0 = K(1)$.

Definition 2 see [9]). (If $0 < \sigma < 1$ and $\varphi(t)$ varies with time t , then the integral is described as follows:

$$RLJ_{0,t}^\sigma \{\varphi(t)\} = \frac{1}{\Gamma(\sigma)} \int_0^t (t-z)^{\sigma-1} \varphi(u) du. \quad (4)$$

The above integral is known as the Riemann–Liouville integral.

$$CFJ_{0,t}^\sigma \{\varphi(t)\} = \frac{2}{(2-\sigma)K(\sigma)} \left\{ (1-\sigma)\varphi(t) + \sigma \int_0^t \varphi(u) du \right\}. \quad (5)$$

The integral defined by Equation (5) is said to be the *Caputo-Fabrizio-Caputo* (CF) integral.

Since σ is the fractional order, and using the notion $\gamma_1 = \gamma_1^\sigma + \gamma_2^\sigma + d^\sigma$ and $\varrho_2 = \gamma_3^\sigma + d^\sigma + d_1^\sigma$ for the shake of simplicity, therefore the fractional order model looks like the following equation:

$$\left\{ \begin{array}{l} CFD_{0,t}^\sigma S_h(t) = \Lambda^\sigma - \beta_1^\sigma L_h(t) S_h(t) - \gamma^\sigma \beta_2^\sigma I_h(t) S_h(t) - \psi^\sigma \beta_3^\sigma M(t) S_h(t) - d^\sigma S_h(t), \\ CFD_{0,t}^\sigma L_h(t) = \beta_1^\sigma L_h(t) S_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) S_h(t) + \psi^\sigma \beta_3^\sigma M(t) S_h(t) - \varrho_1 L_h(t), \\ CFD_{0,t}^\sigma I_h(t) = \gamma_1^\sigma L_h(t) - \varrho_2 I_h(t), \\ CFD_{0,t}^\sigma R_h(t) = \gamma_3^\sigma I_h(t) + \gamma_2^\sigma L_h(t) - d^\sigma R_h(t), \\ CFD_{0,t}^\sigma M(t) = \eta_2^\sigma I_h(t) + \eta_1^\sigma L_h(t) - \alpha^\sigma M(t). \end{array} \right. \quad (6)$$

We show that the proposed fractional-order epidemic model as reported by the above system is both biologically and mathematically feasible. For this, we discuss the positivity and boundedness of the model (6), which proves that the underconsidered problem is well-posed. We also investigate that the dynamics of the proposed model are confined to a certain region invariant positively. The following *Lemmas* is established for this purpose.

Lemma 1. *Since $(S_h(t), L_h(t), I_h(t), R_h(t), M(t))$ are the proposed model (6) solutions and let us consider that it possessing non-negative initial sizes of population, then $(S_h(t), L_h(t), I_h(t), R_h(t), M(t))$ are non-negative for all $t \geq 0$.*

Proof. Since, σ is the fractional order and assuming that G represents the fractional operator with order σ , then system (1) leads to

$$\begin{cases} GD_{0,t}^\sigma(S_h(t)) = \Lambda^\sigma - (\beta_1^\omega L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) + d^\sigma)S_h(t), \\ GD_{0,t}^\sigma(L_h(t)) = (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t))S_h(t) - \varrho_1 L_h(t), \\ GD_{0,t}^\sigma(I_h(t)) = \gamma_1^\sigma L_h(t) - \varrho_2 I_h(t), \\ GD_{0,t}^\sigma(R_h(t)) = \gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) - d^\sigma R_h(t), GD_{0,t}^\sigma(M(t)) = \eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) - \alpha^\sigma M(t). \end{cases} \quad (7)$$

This implies that

$$\begin{cases} GD_{0,t}^\sigma S_h(t)|_{\kappa(S_h)} = \Lambda^\sigma > 0, \\ GD_{0,t}^\sigma L_h(t)|_{\kappa(L_h)} = (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t))S_h(t) \geq 0, \\ GD_{0,t}^\sigma I_h(t)|_{\kappa(I_h)} = \gamma_1^\sigma L_h(t) \geq 0, GD_{0,t}^\sigma R_h(t)|_{\kappa(R_h)} = \gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) \geq 0, \\ GD_{0,t}^\sigma M(t)|_{\kappa(M)} = \eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) \geq 0, \end{cases} \quad (8)$$

where $\kappa(\xi) = \{\xi = 0 \text{ and } S_h, L_h, I_h, R_h, M \text{ are in } C(R_+ \times R_+)\}$ and $\xi \in \{S_h, L_h, I_h, R_h, M\}$, respectively. Following the methodology proposed in [34] and consequently used by Qureshi et al. [35], we reach to the conclusion that the solutions are non-negative for all non-negative t . \square

Lemma 2. *Let us assume that the Ω is the feasible region of the model (6), then within it, the model that is under consideration is invariant and the feasible region is given by*

$$\Omega = \left\{ \begin{array}{l} (S_h(t), L_h(t), I_h(t), R_h(t), M(t)) \in R_+^5: S_h + L_h \\ \quad + I_h + R_h \leq \left(\frac{\Lambda}{d}\right)^\sigma, \\ M(t) \leq \frac{\Lambda^\sigma (\eta_1^\sigma + \eta_2^\sigma)}{d^\sigma \alpha^\sigma} \end{array} \right\}. \quad (9)$$

Proof. Let $N_h(t)$ represent the total human population, then the use of the proposed fractional model leads to the assertion is given by the following equation:

$$GD_{0,t}^\sigma N_h(t) + d^\sigma N_h(t) \leq \Lambda^\sigma N_h(t). \quad (10)$$

Solving equation (10), we get the following equation:

$$N_h(t) \leq N_h(0)E_\omega(-d^\sigma t^\sigma) + \left(\frac{\Lambda}{d}\right)^\sigma (1 - E_\sigma(-d^\sigma t^\sigma)). \quad (11)$$

It could be also noted that $L_h, I_h \leq N_h$, so the last equation of the fractional model (6) looks like the following equation:

$$GD_{0,t}^\sigma M(t) + \alpha^\sigma M(t) \leq \frac{(\eta_1^\sigma + \eta_2^\sigma)\Lambda^\sigma}{\alpha^\sigma d^\sigma}. \quad (12)$$

The solution of (12) leads to the following equation:

$$M(t) \leq M(0)E_\sigma(-\alpha^\sigma t^\sigma) + \left(\frac{\Lambda^\sigma (\eta_1^\sigma + \eta_2^\sigma)}{d^\sigma \alpha^\sigma}\right) (1 - E_\sigma(-\alpha^\sigma t^\sigma)). \quad (13)$$

In (11) and (13), $E(\cdot)$ denotes the *Mittag-Leffler* function and $E_\sigma(Z) = \sum_{n=0}^{\infty} \Gamma(\sigma n + 1)/Z^n$. Furthermore, it is obvious that when times grows without bound then (11) and (13) gives that $N_h(t) \rightarrow (\Lambda/d)^\sigma$ and $M(t) \rightarrow (\Lambda(\eta_1^\sigma + \eta_2^\sigma)/d^\sigma \alpha^\sigma)$. Thus, if $N_h(0) \leq (\Lambda/d)^\sigma$ and $M(0) \leq (\Lambda^\sigma (\eta_1^\sigma + \eta_2^\sigma)/d^\sigma \alpha^\sigma)$, then $N_h(t) \leq (\Lambda/d)^\sigma$ and $M(t) \leq (\Lambda^\sigma (\eta_1^\sigma + \eta_2^\sigma)/d^\sigma \alpha^\sigma)$ for every $t > 0$, while if $N_h(0) > (\Lambda/d)^\sigma$ and $M(0) > (\Lambda^\sigma (\eta_1^\sigma + \eta_2^\sigma)/d^\sigma \alpha^\sigma)$, then N_h and M contained in Ω and will never leave. So, the dynamics of the fractional epidemic model can be investigated in feasible region Ω . \square

3. Stability Analysis

In this section, we will examine the stability of fractional epidemiological model (6). We find the steady states first and threshold parameter (*basic reproductive number*) of the fractional model to investigate the stability conditions. We use the notion X_1 for disease-free equilibrium calculating at steady state with $L_h = I_h = R_h = M = 0$. It is easily stated

that the component of X_1 looks like $S_{h0} = \Lambda^\omega/d^\omega$ and $L_{h0} = I_{h0} = R_{h0} = M = 0$. We now use the disease-free state and find the threshold parameter. This quantity represents the maximum epidemic potential of a pathogen, which describes what would happen if an infectious agent were to enter a susceptible community, and therefore is an estimate based on an idealized scenario. The effective threshold quantity depends on the nature of the population's current susceptibility. This measure the potential transmission, which is likely lower than the basic reproduction number, depends on various factors e.g., whether some individuals have immunity due to prior exposure to the pathogen or whether some individuals are vaccinated against the disease. Therefore, this quantity is effective and changes over time and is an estimate based on a more realistic situation within the population. We calculate this quantity i.e., the threshold quantity (R_0) of the proposed model by following [36], therefore following the next generation matrix approach, we calculate the associated matrices i.e., F and V as given by the following equation:

$$F = \begin{pmatrix} \beta_1^\sigma S_{h0} & \gamma^\sigma \beta_2^\sigma S_{h0} & \psi^\sigma \beta_3^\sigma S_{h0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} \varrho_1 & 0 & 0 \\ 0 & \varrho_2 & 0 \\ -\eta_1^\sigma & -\eta_2^\sigma & \alpha^\sigma \end{pmatrix}. \quad (14)$$

The associated threshold quantity of model (6) is the spectral radius of the matrix ($\bar{K} = FV^{-1}$) is as $R_0 = R_1 + R_2 + R_3$, where

$$R_1 = \frac{\Lambda^\sigma \beta_1^\sigma}{d^\sigma \varrho_1}, R_2 = \frac{\Lambda^\sigma \beta_2^\sigma \gamma^\sigma \eta_1^\sigma}{d^\sigma \varrho_1 \varrho_2}, R_3 = \frac{\Lambda^\sigma \beta_3^\sigma \psi^\sigma \eta_1^\sigma}{\alpha^\sigma d^\sigma \varrho_1} + \frac{\Lambda^\sigma \beta_3^\sigma \psi^\sigma \eta_2^\sigma \gamma_1^\sigma}{\alpha^\sigma d^\sigma \varrho_1 \varrho_2}. \quad (15)$$

It could be noted that the threshold quantity consists of three parts that describe various transmission routes. One may observe, that there is a transmission from an infected human, while the other from reservoirs.

Similarly, we use the above quantity (R_0) and assume that X_2 is the endemic equilibrium of the fractional order model, then the components are calculated by solving system (6) simultaneously at steady state. We also set $S_h = S_h^*$, $L_h = L_h^*$, $I_h = I_h^*$, $R_h = R_h^*$, and $M = M^*$ for the sake of convenience, then the corresponding endemic equilibrium leads to $X_2 = (S_h^*, L_h^*, I_h^*, R_h^*, M^*)$, whose components are defined by the following equation:

$$S_h^* = \frac{\varrho_1 \varrho_2 \alpha^\sigma}{\beta_1^\sigma \varrho_2 \alpha^\sigma + \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma},$$

$$L_h^* = \frac{\alpha^\sigma d^\sigma \varrho_2 \varrho_1 (R_0 - 1)}{q_2 (\beta_1^\sigma \varrho_2 \alpha^\sigma + \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma)},$$

$$I_h^* = \frac{\gamma_1^\sigma}{\varrho_2} L_h^*, R_h^* = \frac{1}{d^\sigma \varrho_2} (\gamma_2^\sigma \varrho_2 + \gamma_1^\sigma \gamma_3^\sigma) L_h^*,$$

$$M^* = \frac{1}{\alpha^\sigma \varrho_2} (\eta_1^\sigma \varrho_2^\sigma + \gamma_1^\sigma \eta_2^\sigma) L_h^*. \quad (16)$$

The endemic equilibrium reveals that X_2 exists only if $R_0 > 1$. For this, we state the following result.

Lemma 3. *The endemic equilibrium $X_2 = (S_h^*, L_h^*, I_h^*, R_h^*, M^*)$ for the proposed problem (6) exists only whenever, the threshold quantity (R_0) is greater than unity.*

We use the linear stability analysis to discuss the temporal dynamics of the fractional model (6) around X_1 and X_2 . So we have the following results.

Theorem 1. *If the threshold quantity (R_0) is less than unity, then the local, as well as global dynamics of the problem, is asymptotically stable around $X_1 = (S_{h0}, 0, 0, 0, 0)$.*

Proof. Following Theorem 3 reported in [37] to obtain the required results. Since it is clear that all other compartments of the proposed model do not depend explicitly on the recovered class, so we study the dynamics of the model for only the three-compartment, which will be enough for the whole model. Let $A(X_1)$ be the Jacobian matrix of the proposed model (6) around X_1 ; then,

$$A(X_1) = \begin{pmatrix} -d^\sigma & \frac{\beta_1^\sigma \Lambda^\sigma}{d^\sigma} & \frac{\gamma^\sigma \beta_2^\sigma \Lambda^\sigma}{d^\sigma} & \frac{\psi^\sigma \beta_3^\sigma \Lambda^\sigma}{d^\sigma} \\ 0 & \frac{\beta_1^\sigma \Lambda^\sigma}{d^\sigma} - \varrho_1 & \frac{\gamma^\sigma \beta_2^\sigma \Lambda^\sigma}{d^\sigma} & \frac{\psi^\sigma \beta_3^\sigma \Lambda^\sigma}{d^\sigma} \\ 0 & \gamma_1^\sigma & -\varrho_2 & 0 \\ 0 & \eta_1^\sigma & \eta_2^\sigma & -\alpha^\sigma \end{pmatrix}. \quad (17)$$

The calculation shows that $A(X_1)$, obviously has two negative eigenvalue i.e., $\lambda_1 = -d^\sigma$ and $\lambda_2 = -\alpha^\sigma$. To find the nature of the remaining, we take the matrix given by the following equation:

$$A(X_1) = \begin{pmatrix} \frac{\beta_1^\sigma \Lambda^\sigma}{d^\sigma} - \varrho_1 & \frac{\gamma^\sigma \beta_2^\sigma \Lambda^\sigma}{d^\sigma} \\ \gamma_1^\sigma & -\varrho_2 \end{pmatrix}. \quad (18)$$

It is sufficient for the Routh–Hurwitz criteria that H_1 : $\text{trace}(A(x_1)) < 0$, and $\det(A(X_1)) > 0$ holds. We calculate the trace ($A(x_1)$) and $\det(A(X_1))$, such that

$$\text{trace}(A(X_1)) = -\varrho_1 (1 - R_1) - \varrho_2, \quad (19)$$

$$\det(A(X_1)) = q_2 q_3 (1 - (R_1 + R_2)). \quad (20)$$

It can be noted from the above equations (19)–(20) that $\text{trace}(A(X_1)) < 0$ and $\det(A(X_1)) > 0$, if $R_1 + R_2 < 1$. So the Routh–Hurwitz criteria are satisfied if $R_0 < 1$. It proves the conclusion that the local dynamics of the model (6) is asymptotically stable, if $R_0 < 1$. \square

The application of linear stability analysis is utilized to find the dynamics of the proposed model (6) around its

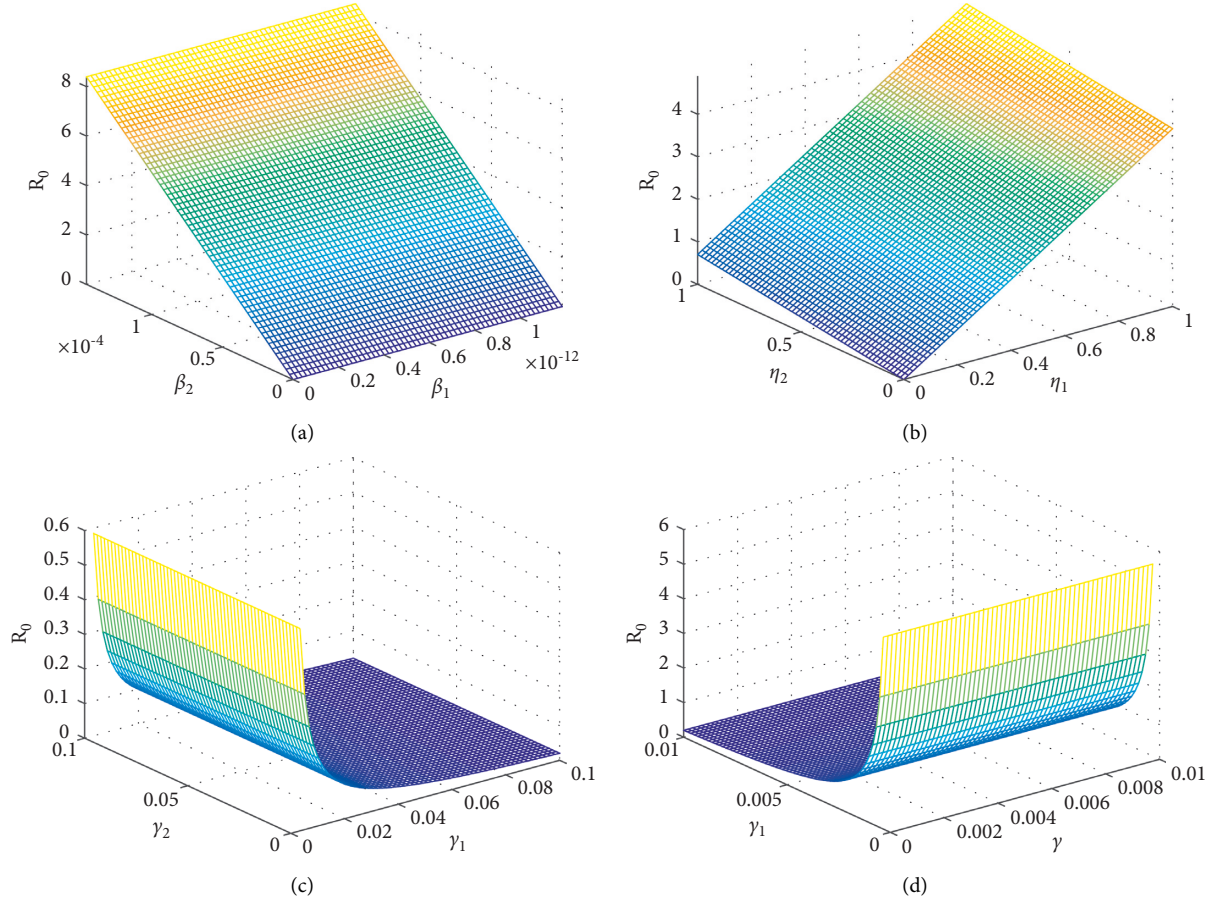


FIGURE 1: The graphical results demonstrate the local sensitivity analysis of the threshold quantity (R_0) with respect to the model parameters β_1 , β_2 , η_1 , η_2 , γ_1 , γ_2 , and γ . For this we use the parametric values as follows: $\Pi = 0.2453$, $\psi = 0.00180$, $\beta_1 = 0.0001847$, $\gamma_3 = 0.09960$, $\gamma = 0.8631$, $\gamma_4 = 0.01470$, $\beta_2 = 0.0004615$, $\mu_1 = 0.00130$, $\mu_0 = 0.00002$, $\mu_2 = 0.00230$, $\gamma_1 = 0.2381$, $\eta_1 = 0.00200$, $\gamma_2 = 1.3184$, and $\eta_2 = 0.00005$. (a) R_0 versus β_1 and β_2 . (b) R_0 versus η_1 and η_2 . (c) R_0 versus γ_1 and γ_2 . (d) R_0 versus γ and γ_1 .

associated endemic equilibrium (16). For this, we describe the result as follows. \square

Theorem 2. *If the threshold quantity (R_0) is greater than unity i.e., $R_0 > 1$, then the local as well as the global dynamics of the endemic equilibrium, $X_2 = (S_h^*, I_h^*, I_h^*, R_h^*, M^*)$ is asymptotically stable.*

Proof. Using the theory of a dynamical system, we discuss the local dynamics of the proposed system around the endemic equilibrium. Let $A(X_2)$ be the Jacobian matrix of system (6) around X_2 ; then,

$$A(X_*) = \begin{pmatrix} -(d^\sigma + \beta_1^\sigma L_h^* + \gamma^\sigma \beta_2^\sigma I_h^* + \psi^\sigma \beta_3^\sigma M^*) & -\beta_1^\sigma S_h^* & -\gamma^\sigma \beta_2^\sigma S_h^* & -\psi^\sigma \beta_3^\sigma S_h^* \\ \beta_1^\sigma L_h^* + \gamma^\sigma \beta_2^\sigma I_h^* + \psi^\sigma \beta_3^\sigma M^* & \psi^\sigma \beta_3^\sigma S_h^* - \varrho_1 & \beta_2^\sigma S_h^* & 0 \\ 0 & \gamma_1^\sigma & -\varrho_2 & 0 \\ 0 & \eta_1^\sigma & \eta_2^\sigma & -\alpha^\sigma \end{pmatrix}. \quad (21)$$

We find the characteristic polynomial of the matrix (21), such that

$$p(\lambda) = \lambda^4 + k_1 \lambda^3 + k_2 \lambda^2 + k_3 \lambda + k_4, \quad (22)$$

where

$$\begin{aligned}
k_1 &= ((\alpha^\sigma + d^\sigma + \varrho_1)(\beta_1^\sigma \varrho_2 \alpha^\sigma + \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma) + \varrho_1(\alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 \\
&\quad + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + (\beta_1^\sigma \alpha^\sigma d^\sigma \varrho_2 + \alpha^\sigma d^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma (\eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma d^\sigma))(R_0 - 1), \\
k_2 &= \alpha^{2\sigma} d^\sigma \beta_1^\sigma \varrho_2 + d^\sigma \varrho_2 \beta_1^\sigma \alpha^\sigma + \alpha^{2\sigma} \varrho_2 \beta_1^\sigma + \alpha^\sigma d^\sigma \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \alpha^\sigma d^\sigma \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + \alpha^\sigma \varrho_1 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma \\
&\quad + \alpha^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + d^\sigma \varrho_1 \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + d^\sigma \varrho_1 \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + d^\sigma \varrho_1 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + d^\sigma \varrho_2 \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + d^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + \varrho_1 \varrho_2 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + \alpha^{2\sigma} d^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \alpha^{2\sigma} \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \alpha^{2\sigma} \varrho_2 \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + \alpha^\sigma \varrho_2 \psi^\sigma \beta_3^\sigma \eta_1^\sigma + d^\sigma \varrho_2 \psi^\sigma \beta_3^\sigma \eta_1^\sigma + \varrho_2 \varrho_3 \psi^\sigma \beta_3^\sigma \eta_1^\sigma + d^\sigma (\alpha^{3\sigma} \beta_1^{2\sigma} \varrho_2 + \varrho_2^3 \beta_1^{2\sigma} \alpha^{2\sigma} + \varrho_2^4 \alpha^{4\sigma} \beta_1^{2\sigma} \\
&\quad + 2\beta_3^\sigma \psi^\sigma \varrho_2^3 \eta_1^\sigma \beta_1^\sigma \alpha^\sigma + 2\beta_3^\sigma \psi^{2\sigma} \varrho_2^2 \eta_1^\sigma \eta_2^\sigma \gamma_1^\sigma + 2\alpha^{3\sigma} \beta_1^\sigma \varrho_2 \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + 2\varrho_2^2 \beta_1^\sigma \alpha^{2\sigma} \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + 2\varrho_2^3 \alpha^{4\sigma} \beta_1^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + 2\varrho_2^4 \alpha^{3\sigma} \beta_1^\sigma \psi^\sigma \beta_3^\sigma \eta_1^\sigma + \varrho_2^2 \alpha^{2\sigma} \eta_2^{2\sigma} \gamma_1^{2\sigma} \psi^{2\sigma} \beta_3^{2\sigma} + \gamma_1^\sigma \alpha^{2\sigma} \varrho_2 \varrho_1 \beta_2^{2\sigma} \gamma^\sigma \\
&\quad + \alpha^\sigma \beta_3^{2\sigma} \psi^{2\sigma} \eta_1^{2\sigma} \varrho_2 + \alpha^\sigma \beta_3^{2\sigma} \psi^{2\sigma} \gamma_1^{2\sigma} \eta_2^{2\sigma} + \beta_3^{2\sigma} \psi^{2\sigma} \varrho_1 \eta_1^{2\sigma} \varrho_2 + \beta_3^{2\sigma} \psi^{2\sigma} \varrho_1 \gamma_1^{2\sigma} \eta_2^{2\sigma} + \beta_3^{2\sigma} \psi^{2\sigma} \varrho_2 \gamma_1^{2\sigma} \eta_2^{2\sigma} \\
&\quad + \varrho_2^2 \alpha^{4\sigma} \gamma_1^{2\sigma} \gamma^{2\sigma} \beta_2^{2\sigma} + \varrho_2^4 \alpha^{2\sigma} \psi^{2\sigma} \beta_3^{2\sigma} \eta_1^{2\sigma} + \varrho_1 \alpha^{2\sigma} \gamma_1^{2\sigma} \gamma^{2\sigma} \beta_2^{2\sigma} + \varrho_3 \alpha^{2\sigma} \gamma_1^{2\sigma} \gamma^{2\sigma} \beta_2^{2\sigma} + 2\alpha^{2\sigma} \beta_3^\sigma \psi^\sigma \eta_1^\sigma \varrho_3^2 \beta_1^\sigma \\
&\quad + 2\alpha^{2\sigma} \beta_3^\sigma \psi^\sigma \eta_1^\sigma \varrho_2 \gamma_1^\sigma \beta_2^\sigma + 2\alpha^\sigma \beta_3^{2\sigma} \psi^{2\sigma} \eta_1^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma + 2\alpha^{2\sigma} \beta_3^\sigma \psi^\sigma \gamma_1^\sigma \eta_2^\sigma \beta_1^\sigma \varrho_2 + 2\alpha^{2\sigma} \beta_3^\sigma \psi^\sigma \gamma_1^\sigma \eta_2^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + \beta_3^\sigma \psi^\sigma \varrho_1 \eta_1 \varrho_2^2 \beta_1^\sigma \alpha^\sigma + 2\beta_3^\sigma \psi^\sigma \varrho_1 \eta_1^\sigma \varrho_2 \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + 2\beta_3^{2\sigma} \psi^{2\sigma} \varrho_1 \eta_1^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma + \beta_3^\sigma \psi^\sigma \varrho_1 \gamma_1^\sigma \eta_2^\sigma \beta_1^\sigma \varrho_2 \alpha^\sigma \\
&\quad + 2\beta_3^\sigma \psi^\sigma \varrho_1 \gamma_1^\sigma \eta_2^\sigma \alpha^\sigma \gamma^\sigma \beta_2^\sigma + 2\beta_3^\sigma \psi^\sigma \varrho_2^2 \eta_1^\sigma \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + 2\beta_3^\sigma \psi^\sigma \varrho_2^2 \gamma_1^\sigma \eta_2^\sigma \beta_1^\sigma \alpha^\sigma + 2\beta_3^\sigma \psi^\sigma \varrho_2 \gamma_1^\sigma \eta_2^\sigma \alpha^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + \varrho_1 \beta_1^\sigma \varrho_2 \alpha^{2\sigma} \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + 2\varrho_2^3 \alpha^{3\sigma} \beta_1^\sigma \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + 2\varrho_2^3 \alpha^{3\sigma} \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \psi^\sigma \beta_3^\sigma \eta_1^\sigma + 2\varrho_2^2 \alpha^{3\sigma} \gamma_1^{2\sigma} \gamma^\sigma \beta_2^\sigma \eta_2^\sigma \psi^\sigma \beta_3^\sigma \\
&\quad + 2\varrho_2^3 \alpha^{2\sigma} \psi^{2\sigma} \beta_3^{2\sigma} \eta_1^\sigma \eta_2^\sigma \gamma_1^\sigma + \varrho_1 \varrho_2^2 \alpha^\sigma \beta_2^\sigma \beta_3^\sigma \psi^\sigma \eta_1^\sigma + \varrho_2 \varrho_3 \alpha^\sigma \beta_2^\sigma \beta_3^\sigma \psi^\sigma \gamma_1^\sigma \eta_2^\sigma + \beta_3^{2\sigma} \psi^{2\sigma} \varrho_2^3 \eta_1^{2\sigma} \\
&\quad + \alpha^{3\sigma} \gamma_1^{2\sigma} \gamma^{2\sigma} \beta_2^{2\sigma} (R_0 - 1), \\
k_3 &= d(\beta_1^\sigma \varrho_2 \alpha^\sigma + \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma)(\psi^\sigma \varrho_1 \eta_1^\sigma \beta_3^\sigma \varrho_2 + \psi^\sigma \varrho_2 \beta_3^\sigma \alpha^\sigma \varrho_1 \eta_1^\sigma + \psi^\sigma \varrho_2 \beta_3^\sigma \varrho_1 \eta_2^\sigma \gamma_1^\sigma \\
&\quad + \psi^\sigma \beta_3^\sigma \varrho_1 \alpha^\sigma \gamma_1^\sigma \eta_2^\sigma + \psi^\sigma \beta_3^\sigma \alpha^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma + \varrho_2^2 \varrho_1 \alpha^\sigma \beta_2^\sigma + \alpha^{2\sigma} \varrho_2^2 \beta_1^\sigma + \varrho_2 \varrho_1 \alpha^\sigma \gamma_1^\sigma \beta_3^\sigma - \varrho_1 \varrho_2 \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + \varrho_2 \alpha^{2\sigma} \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma + \varrho_1 \varrho_2 \alpha^{2\sigma} \beta_2^\sigma + \gamma_1^\sigma \alpha^{2\sigma} \varrho_1 \beta_2^\sigma \gamma^\sigma + \gamma_1^\sigma \alpha^\sigma (R_0 - 1) + (d^\sigma \varrho_1 \varrho_2 \psi^\sigma \beta_3^\sigma \eta_1^\sigma \\
&\quad + d^\sigma \varrho_1 \varrho_2 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + \alpha^{2\sigma} d^\sigma \varrho_2 \beta_1^\sigma + \gamma_1^\sigma \alpha^{2\sigma} d^\sigma \beta_2^\sigma \gamma^\sigma \varrho_2 + \alpha^\sigma d^\sigma \varrho_2 \psi^\sigma \beta_3^\sigma \eta_1^\sigma + \alpha^\sigma d^\sigma \varrho_2 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma \\
&\quad + \gamma_1^\sigma \alpha^{2\sigma} d^\sigma \beta_2^\sigma \gamma^\sigma \varrho_1 + \alpha^\sigma d^\sigma \varrho_1 \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma + \alpha^\sigma \varrho_1 \varrho_2 \psi^\sigma \beta_3^\sigma \eta_1^\sigma - \varrho_1 \varrho_2 \alpha^\sigma (\beta_1^\sigma \varrho_2 \alpha^\sigma + \alpha^\sigma \gamma_1^\sigma \gamma^\sigma \beta_2^\sigma \\
&\quad + \psi^\sigma \beta_3^\sigma \eta_1^\sigma \varrho_2 + \eta_2^\sigma \gamma_1^\sigma \psi^\sigma \beta_3^\sigma), \text{ and} \\
k_4 &= d^\sigma \varrho_2 \varrho_3 \alpha^{2\sigma} (\varrho_2^2 \beta_2^\sigma \beta_3^\sigma \gamma_1^\sigma \psi^\sigma \alpha^{3\sigma} + \varrho_2^2 \beta_1^\sigma \beta_2^\sigma \alpha^\sigma + \varrho_2^2 \beta_2^\sigma \beta_3^\sigma \psi^\sigma \eta_1^\sigma + \varrho_2 \gamma_1^\sigma \beta_2^{2\sigma} \gamma^\sigma \alpha^\sigma + \varrho_2 \beta_2^\sigma \beta_3^\sigma \psi^\sigma \gamma_1^\sigma \eta_2^\sigma \\
&\quad + \varrho_2 \gamma_1^\sigma \beta_2^\sigma \beta_3^\sigma \eta_1^\sigma \psi^\sigma + \varrho_2 \beta_3^{2\sigma} \gamma^\sigma \gamma_1^\sigma \psi^\sigma \eta_1^\sigma + \gamma_1^{2\sigma} \beta^\sigma - 2\beta_3^\sigma \gamma^{2\sigma} \alpha^\sigma + \beta_3^{2\sigma} \gamma_1^{2\sigma} \psi^\sigma \eta_2^\sigma \\
&\quad + \gamma_1^{2\sigma} \beta_2^\sigma \beta_3^\sigma \eta_2^\sigma \gamma^\sigma \psi^\sigma (R_0 - 1).
\end{aligned} \tag{23}$$

Eigenvalues (roots) of (22) are negative or having negative real parts, if (H_0) : $k_i > 0$, $i = 1, 2, \dots, 4$ and $\delta > 0$ holds, where

$$\Delta_4 = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ k_3 & k_2 & k_1 & 1 \\ 0 & k_4 & k_3 & k_2 \\ 0 & 0 & 0 & k_4 \end{pmatrix}. \tag{24}$$

It could be noted, from the above equations, that obviously H_0 holds, if $R_0 > 1$. Thus, we conclude that the local dynamics of the proposed model (6) at endemic equilibrium (X_2) is asymptotically stable, if the threshold quantity is greater than unity. \square

Theorem 3. *If $R_0 \leq 1$, then the point X_1 of the model (6) is globally asymptotically stable, while the same holds for X_2 whenever $R_0 > 1$.*

Proof. To perform the global analysis of the proposed fractional order epidemiological model, first we assume that $X(t) = (S_h(t), L_h(t), I_h(t), R_h(t), M(t))$ and if $t \rightarrow \infty$, then it has finite limit, therefore by following the result 3.1 in [38], then from first equation of the model (7), we may write the following equation:

$$CFD_{0,t}^\alpha S_h(t) \leq \Lambda^\sigma - d^\sigma S_h(t). \tag{25}$$

Since for every $\phi \leq \phi e^t$, so by following the result Theorem 1 in [38] with the application of *mean value theorem*, the above (25) may leads to the following equation:

$$\|S_h(t)\| \leq aU \exp[-(-d)t] + 1, t \geq T, \tag{26}$$

TABLE 1: The results represent the sensitivity indices of epidemic parameters.

Parameter	Index value	Parameter	Index value
β_1	0.1075×10^{-7}	β_2	0.999999
γ_1	-0.995739	η_2	0.004176
γ	0.3489×10^{-8}	γ_2	-0.00387
γ_3	-0.00029	η_1	0.995823

where $a = \|S_h(0)\|e^{-T} + KT^\sigma e^{-T}/\sigma\Gamma(\sigma) + \Lambda^\sigma$, and $U > 0$. Consequently, we may derive the following equation:

$$\lim_{t \rightarrow \infty} S_h(t) \leq U\Lambda^\sigma. \quad (27)$$

Similarly, \lim of $L_h(t)$, $I_h(t)$, $R_h(t)$, and $M(t)$ can be proved. We also assume that

$$\lim_{t \rightarrow \infty} X(t) = (S_h^\infty, L_h^\infty, I_h^\infty, R_h^\infty, M^\infty),$$

$$\vartheta(X) = \begin{pmatrix} \psi_1(X) \\ \psi_2(X) \\ \psi_3(X) \\ \psi_4(X) \\ \psi_5(X) \end{pmatrix} = \begin{pmatrix} \Lambda^\sigma - (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) + d^\sigma)S_h(t), \\ (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t))S_h(t) - \varrho_1 L_h(t), \\ \gamma_1^\sigma L_h(t) - \varrho_2 I_h(t), \\ \gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) - dR_h(t), \\ \eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) - \alpha^\sigma M(t) \end{pmatrix}. \quad (28)$$

Thus, in the light of *mean value theorem*, there exists constants $C_1 > 0, C_2 > 0$, such that

$$\|\vartheta(X)\| \leq C_1 + C_2\|X\|. \quad (29)$$

So, Theorem 2.1 and 1 in [39] implies that $CFD_{0,t}^\alpha(S_h(t), L_h(t), I_h(t), R_h(t), M(t))$ is uniformly

continuous. Thus, the application of *Barbalat's Lemma* (see for detail, [40]) gives the following equation:

$$\lim_{t \rightarrow \infty}^{CF} D_{0,t}^\alpha(X(t)) = (0, 0, 0, 0, 0). \quad (30)$$

Consequently,

$$\begin{cases} \Lambda^\sigma - (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) + d^\sigma)S_h(t) = 0, \\ (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t))S_h(t) - \varrho_1 L_h(t) = 0, \gamma_1^\sigma L_h(t) - \varrho_2 I_h(t) = 0, \\ \gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) - d^\sigma R_h(t) = 0, \eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) - \alpha^\sigma M(t) = 0. \end{cases} \quad (31)$$

Therefore, $(S_h^\infty, L_h^\infty, I_h^\infty, R_h^\infty, M^\infty)$ is the equilibrium of model (6) and by the similar procedure as adopted in [41], we reach to the following equation:

$$\lim_{t \rightarrow \infty} (X(t)) = X_1 \text{ or } \lim_{t \rightarrow \infty} (X(t)) = X_2. \quad (32)$$

So, it could be concluded that the disease endemic state X_2 does not exist if $R_0 < 1$, and so $\lim X(t) = X_1$ whenever t approaches ∞ and if $R_0 = 1$ then $X_2 = X_1$, and $\lim X(t) = X_1$ as t approaches ∞ , while on the other hand whenever $R_0 > 1$, then X_2 exists and thus $\lim X(t) = X_2$ as t tend to ∞ . \square

4. Numerical Simulation

In this section, we present the numerical simulation of the proposed epidemic problem. We divided the section into

two subsections in which we discuss the sensitivity analysis of every epidemic parameter and its relative impact on disease transmission. We also discuss the temporal dynamics for the long run and present the significance of fractional parameters.

4.1. Sensitivity Analysis. We discuss the local sensitivity analysis of the model parameters to define the relation between threshold quantity and the epidemic parameters. This allows us to measure the relative impact of every epidemic parameter on disease transmission. We follow the work presented in [42] to perform the sensitivity analysis. Using the sensitivity index formula, we get the sensitivity indices as given in Table 1. It may be noted from the sensitivity indices that the set of parameters $S_1 = \{\beta_1, \beta_2, \eta_1, \eta_2, \gamma\}$ has a direct relation with threshold quantity, which means that increase in the value of these

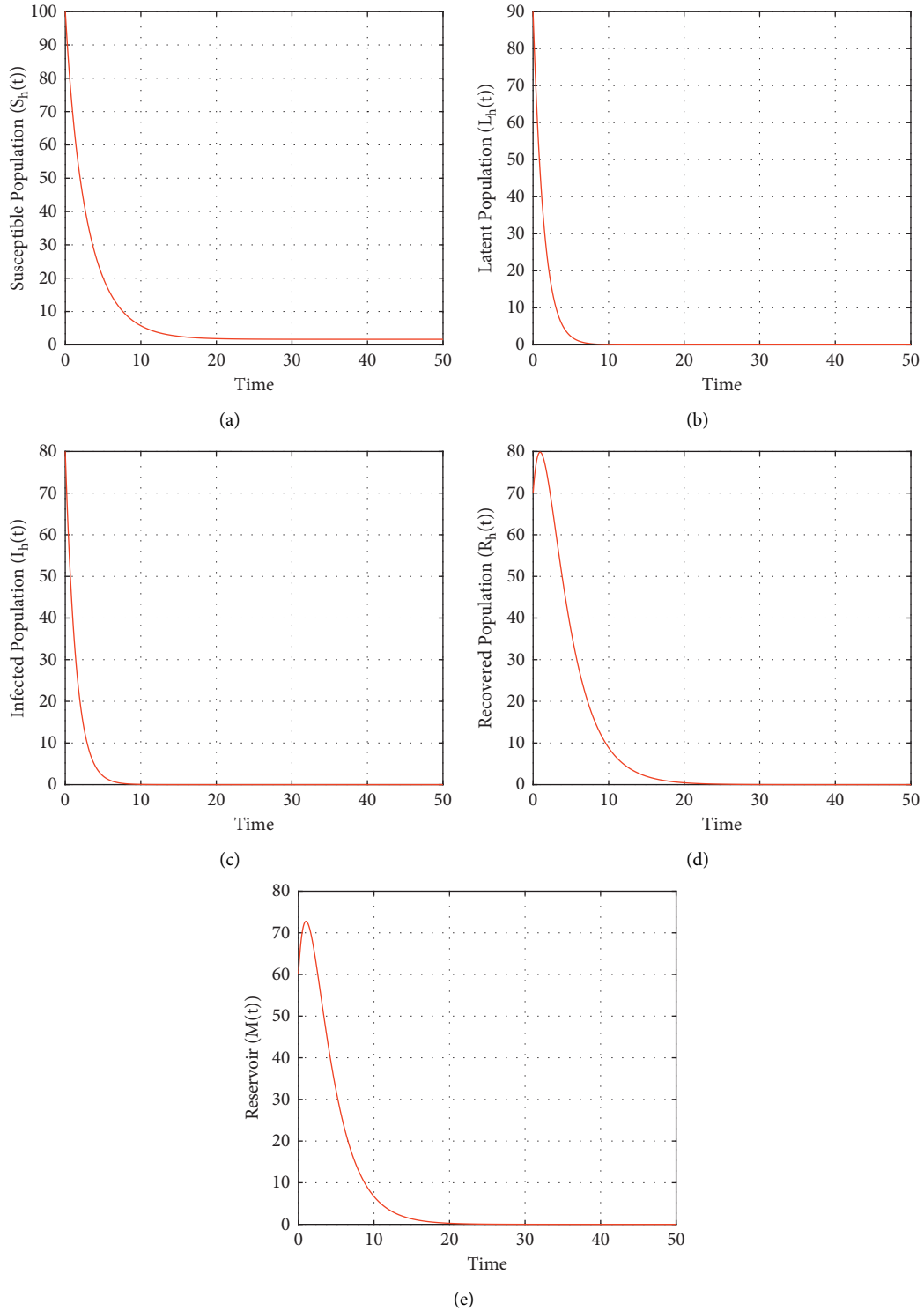


FIGURE 2: The results visualize the dynamics of the fractional order model around the disease-free state (X_1), where the value of epidemic parameters are chosen as: $\Lambda = 0.5$, $\beta_1 = 0.0011$, $\beta_2 = 0.00000005$, $\beta_3 = 0.0001$, $d = 0.3$, $\gamma_1 = 0.0000001$, $\gamma_2 = 0.5$, $\gamma_3 = 0.005$, $d_1 = 0.02$, $\eta_1 = 0.2$, $\eta_2 = 0.6$, and $\alpha = 0.6$. In this case, the value of the threshold parameter is less than unity i.e., $R_0 < 1$. (a) Susceptible- $S_h(t)$. (b) Latent- $L_h(t)$. (c) Infected- $I_h(t)$. (d) Recovered- $R_h(t)$. (e) Reservoir- $M(t)$.

parameters causes an increase in the value of the threshold quantity of the model. On the other hand, there is an inverse relationship between the set of parameters $\{\gamma_1, \gamma_2, \gamma_3\}$ and so an increase in these parameters will cause a decrease in the

value of threshold quantity. The highest sensitivity index parameter is β_2 having the sensitivity index 0.999999, which means that an increase in the value of this parameter say by 10% would increase the value of the threshold quantity by

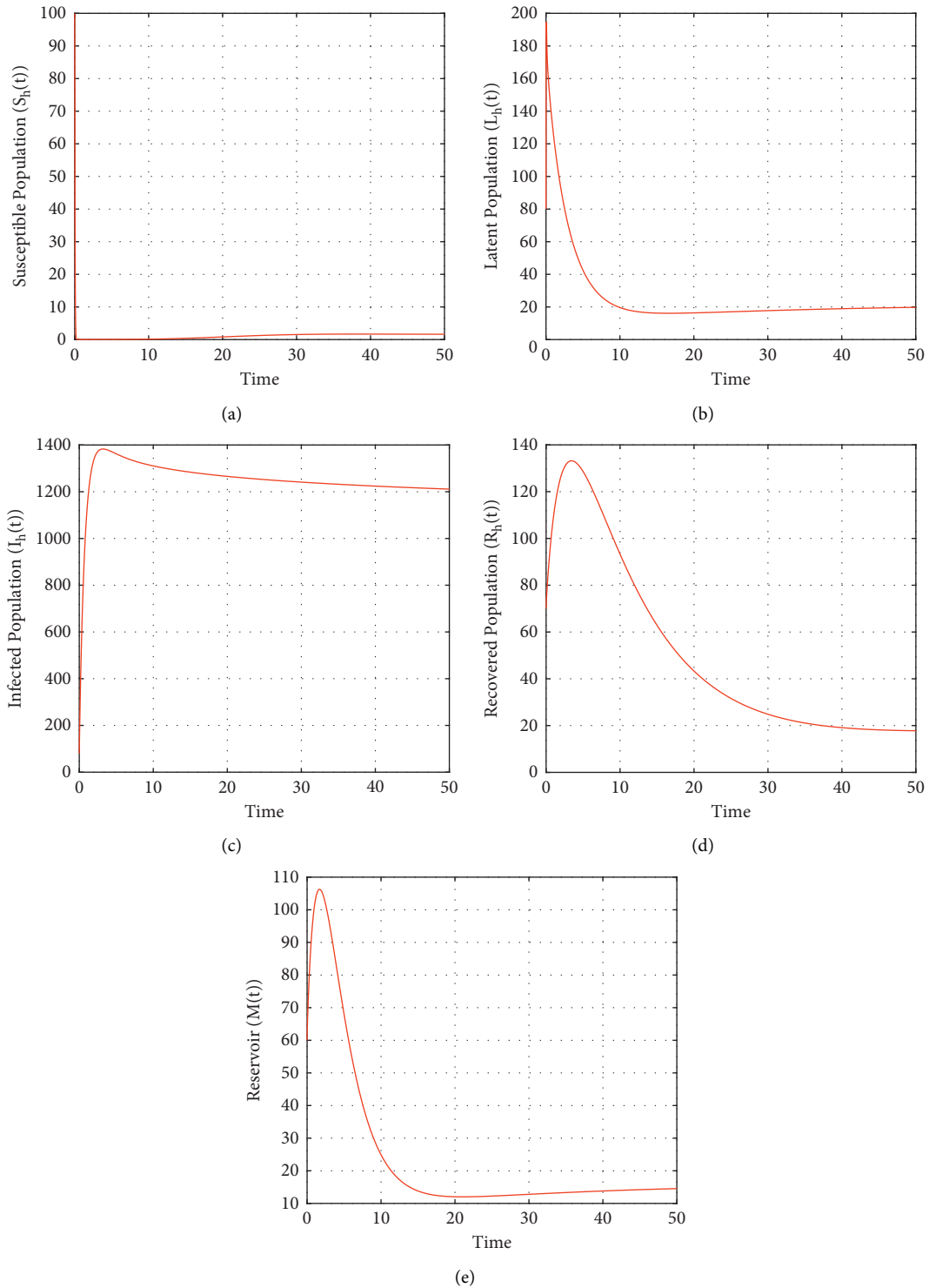


FIGURE 3: The results visualizes the dynamics of the proposed model at endemic state (X_2), where the value of the epidemic parameters are: $\Lambda = 0.5, \beta_1 = 0.11, \beta_2 = 0.00005, \beta_3 = 0.1, d = 0.1, \gamma_1 = 0.001, \gamma_2 = 0.2, \gamma_3 = 0.3, d_1 = 0.2, \eta_1 = 0.3, \eta_2 = 0.5,$ and $\alpha = 0.4$. In this case, the value of the reproductive number is greater than unity i.e., $R_0 > 1$. (a) Susceptible- $S_h(t)$. (b) Latent- $L_h(t)$. (c) Infected- $I_h(t)$. (d) Recovered- $R_h(t)$. (e) Reservoir- $M(t)$.

9.99999%, as shown in Figure 1(a). Similarly the collectively impact of other parameters of S_1 is approximately 9.89%, if their values increase or decrease by 10%, as shown in Figures 1(a) and 1(b). Moreover, the parameters of S_2 have a

negative relation and therefore its collectively impact is 9.94% whenever the value of the parameters given in S_2 increases or decreases by 10 % (see Figures 1(c) and Figures 1(d)). More precisely, if the value of the parameters of S_2 are

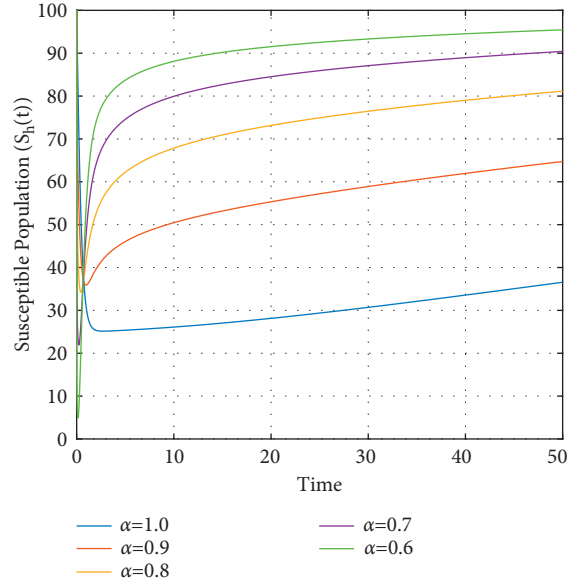


FIGURE 4: The graph visualizes the temporal dynamics of the susceptible for long run against various value of (σ) , while the value of epidemic parameters are: $\Lambda = 0.4$, $\beta_1 = 0.011$, $\beta_2 = 0.005$, $\psi = 0.016$, $\beta_3 = 0.01$, $d = 0.01$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\gamma_3 = 0.05$, $d_1 = 0.002$, $\eta_1 = 0.01$, $\eta_2 = 0.06$, and $\alpha = 0.06$. We also used the initial population sizes: (100, 90, 80, 70, 60).

increased by 10%, the basic reproductive number will be decreased by 9.94%, while if one decreases the value of the parameters, the threshold quantity will be increased by 9.94%.

4.2. Verification of Stability Results. We find out the numerical simulation to verify the theoretical work carried out for the fractional-order SARS-CoV-2 transmission epidemiological model (6). To show the validity of the analytical findings we present the large-scale simulation. There are not

many choices like the traditional numerical methods to choose various schemes for the numerical simulation of fractional-order models [43], therefore extensive attention is required to formulate new and convenient techniques for the simulation of fractional models. We follow a numerical scheme formulated in [18, 44]. We assume the time step $h = 10^{-3}$ for integration with the simulation interval $[0, t]$, $n = T/h$ and $n \in \mathbb{N}$. We also assume that $u = 0, 1, 2, \dots, n$, therefore the discretization for the proposed model looks like the following equation:

$$\begin{aligned}
CFS_{h(u+1)} &= S_h(0) + \{\Lambda^\sigma - (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) + d^\sigma) S_h(t)\} (1 - \alpha) \\
&\quad + \alpha h \sum_{k=0}^u \{\Lambda^\sigma - (\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) + d^\sigma) S_h(t)\}, \\
CFL_{h(u+1)} &= L_h(0) + \{(\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) S_h(t) - \varrho_1 L_h(t)\} (1 - \alpha) \\
&\quad + \alpha h \sum_{k=0}^u \{(\beta_1^\sigma L_h(t) + \gamma^\sigma \beta_2^\sigma I_h(t) + \psi^\sigma \beta_3^\sigma M(t) S_h(t) - \varrho_1 L_h(t)\}, \\
CFI_{h(u+1)} &= (1 - \alpha) \{\gamma_1^\sigma L_h(t) - \varrho_2 I_h(t)\} + \alpha h \sum_{k=0}^u \{\gamma_1^\sigma L_h(t) - \varrho_2 I_h(t)\}, \\
&\quad + I_h(0), \\
CFR_{h(u+1)} &= (1 - \alpha) \{\gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) - d^\sigma R_h(t)\} + \alpha h \sum_{k=0}^u \{\gamma_2^\sigma L_h(t) + \gamma_3^\sigma I_h(t) - d^\sigma R_h(t)\}, \\
&\quad + R_h(0), \\
CFM_{(u+1)} &= (1 - \alpha) \{\eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) - \alpha^\sigma M(t)\} + M(0) + \alpha h \sum_{k=0}^u \{\eta_1^\sigma L_h(t) + \eta_2^\sigma I_h(t) - \alpha^\sigma M(t)\}.
\end{aligned} \tag{33}$$

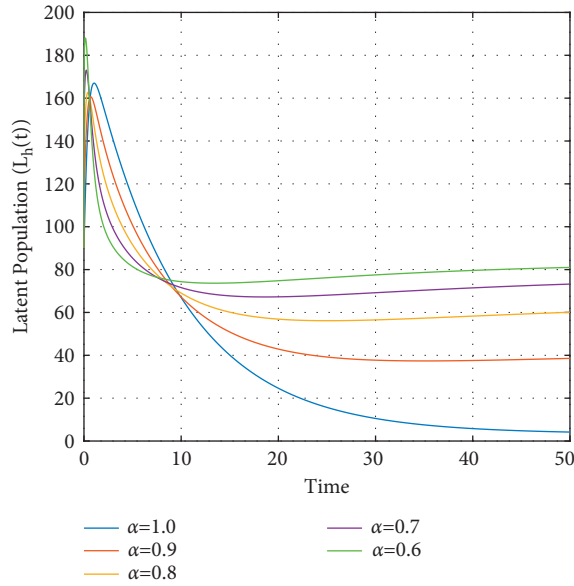


FIGURE 5: The plot demonstrate the dynamical behaviour of the latent individuals against the epidemic parameters having values: $\Lambda = 0.4$, $\beta_1 = 0.011$, $\beta_2 = 0.005$, $\psi = 0.016$, $\beta_3 = 0.01$, $d = 0.01$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\gamma_3 = 0.05$, $d_1 = 0.002$, $\eta_1 = 0.01$, $\eta_2 = 0.06$, and $\alpha = 0.06$, and different values of fractional parameter (σ), while the initial sizes for compartmental population are (100, 90, 80, 70, 60).

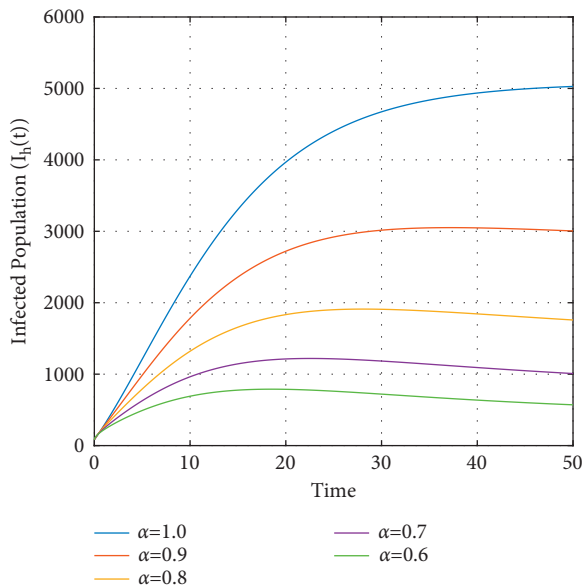


FIGURE 6: The graph represents the temporal dynamics of infected individuals against the fractional parameter (σ) and model parameters value are: $\Lambda = 0.4$, $\beta_1 = 0.011$, $\beta_2 = 0.005$, $\psi = 0.016$, $\beta_3 = 0.01$, $d = 0.01$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\gamma_3 = 0.05$, $d_1 = 0.002$, $\eta_1 = 0.01$, and $\eta_2 = 0.06$, while the sizes of compartmental population are (100, 90, 80, 70, 60).

Furthermore, we chose the value of epidemic parameters biologically while the initial population sizes are assumed to be non-negative values (100, 90, 80, 70, 60). We use the MATLAB software package to execute the model for

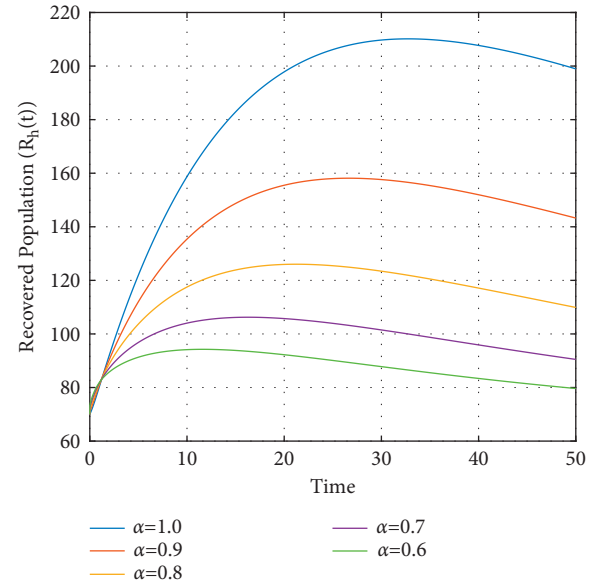


FIGURE 7: The graph describes the dynamics of the recovered individuals for different values of the fractional parameter (σ) and model parameters value are: $\Lambda = 0.4$, $\beta_1 = 0.011$, $\beta_2 = 0.005$, $\psi = 0.016$, $\beta_3 = 0.01$, $d = 0.01$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\gamma_3 = 0.05$, $d_1 = 0.002$, $\eta_1 = 0.01$, and $\eta_2 = 0.06$, while the initial guess are (100, 90, 80, 70, 60).

numerical simulations. We justify the stability results to show the dynamics of the disease-free and endemic states as given in Figures 2 and 3. This investigates the graphical verification of the dynamics of the considered problem around the disease-free state X_1 . Besides from a mathematical point of view, the biological interpretation reveals that whenever the value of the threshold parameter is less than unity, each solution curve of S_h will tend to its equilibrium position as shown in Figure 2(a). This shows that there will always be a susceptible population. Moreover, the dynamics of the other compartments around the disease-free state are depicted in Figures 2(b)–2(d), and which describe that the solution curves will tend to the associated equilibrium position and remain stable. So it could be noted that the elimination of the contagious disease of the novel coronavirus from the community depends on the value of R_0 , and the disease could be easily eliminated if $R_0 < 1$. Furthermore, the dynamics of the fractional-order model around the endemic equilibrium are shown in Figures 3(a)–3(e), which respectively show the temporal dynamics of susceptible, latent, infected, recovered, and reservoirs. From these results, we observed that if proper control measures are not adopted the disease will attain the endemic position. It is clear that the susceptible population decreases from the beginning and then has no effect after some time and so becomes stable as shown in Figure 3(a). The dynamics of the latent population state that there will be a sudden increase in the initial period of infection, while then decreases after some unit of time and become stable, as shown in Figure 3(b), which verifies that there will always be a latent population. Similarly, the dynamics of the infected population are shown in Figure 3(c). This reveals that the infected ratio increases day by day and

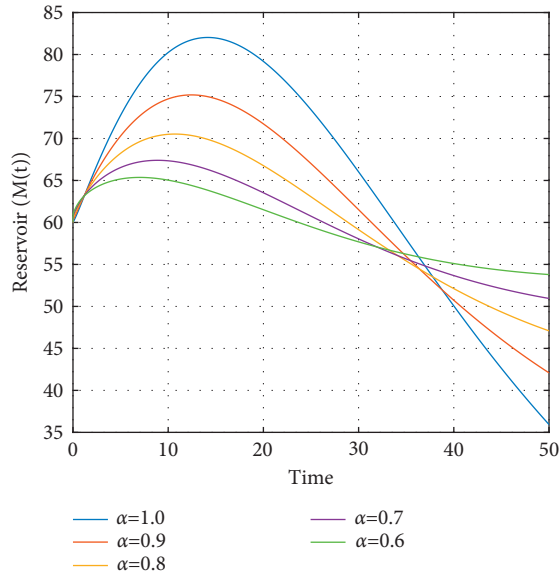


FIGURE 8: The graph describes the dynamics of the ratio of reservoir for different value of the fractional parameter (σ), and the model parameters value are: $\Lambda = 0.4$, $\beta_1 = 0.011$, $\beta_2 = 0.005$, $\psi = 0.016$, $\beta_3 = 0.01$, $d = 0.01$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\gamma_3 = 0.05$, $d_1 = 0.002$, $\eta_1 = 0.01$, and $\eta_2 = 0.06$, while the initial guess are (100, 90, 80, 70, 60).

reaches its endemic position in a few units of time. Furthermore, the simulation of the model for recovered population and reservoir is given in Figures 3(d)–3(e). All these results suggest that if no proper control measure is implemented, the disease will attain its endemic position whenever the value of the threshold quantity (R_0) is greater than unity.

We also show the significance of the fractional-order via disease transmission as shown in Figures 4–8, which respectively visualizes the temporal dynamics of S_h , A_h , I_h , R_h , and M . Particularly, the temporal dynamics of the susceptible are shown in Figure 4. We noted a significant impact of the fractional order on the transmission dynamics of susceptible individuals that if the fractional parameter σ increases than the number of susceptible individuals decreases as shown in Figure 4. The long run of the latent, infected, and the recovered population for various orders of fractional order are presented in Figures 5–7. We noted that there is a strong influence of (σ) on disease transmission. The temporal dynamics of the reservoir are presented in Figure 8. Thus, we investigate that the *CF* model gives more accurate dynamics of the disease and provides valuable outputs instead of classical models.

5. Conclusion

We investigated the dynamics of SARS-CoV-2 with asymptomatic, symptomatic, and quarantined individuals using an epidemic model. First, the formulation of the model is proposed, and then consequently fractionalized due to the increasing development in fractional calculus. Particularly, we used the well-known *Caputo-Fabrizio* operator for the

said purposes. Both the biological and mathematical feasibilities are discussed in detail for the proposed model and proved that the problem is well-posed. We also calculated the threshold parameter and performed stabilities of the fractional model. The detailed sensitivity is also discussed and quantified the role of every epidemic parameter and its relative impact on the disease transmission. We showed that the proposed model is stable in both local and global sense. Finally, we gave some graphical representations and showed the validations of the obtained results. We also presented the relative impact of the fractional parameter on the various groups of the compartmental populations graphically and proved that the major outcome of the reported work is that the fractional-order *CF* epidemic models are more appropriate and the best choice rather than the classical order. We believe that the findings of this work will be helpful for the audience working in the field of mathematical epidemiology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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