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Research Article

The Extended Exponential Weibull Distribution: Properties, Inference, and Applications to Real-Life Data

Adam Braima S. Mastor,[1] **Oscar Ngesa**,[2] **Joseph Mung'atu**,[3] **Nada M. Alfaer**,[4] **and Ahmed Z. Afify**[5]

[1]*Department of Mathematics (Statistics Option) Programme, Pan African University, Institute for Basic Sciences, Technology and Innovation (PAUSTI), Nairobi 62000-00200, Kenya*
[2]*Mathematics, Statistics and Physical Sciences Dept, Taita Taveta University, Voi, Kenya*
[3]*Department of Mathematics, Jomo Kenyatta University of Agriculture and Technology, Juja, Kenya*
[4]*Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*
[5]*Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt*

Correspondence should be addressed to Adam Braima S. Mastor; soliman.adam1@students.jkuat.ac.ke

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A novel version of the exponential Weibull distribution known as the extended exponential Weibull (ExEW) distribution is developed and examined using the Lehmann alternative II (LAII) generating technique. The new distributions basic mathematical properties are derived. The maximum likelihood estimation (MLE) technique is used to estimate the unknown parameters of the proposed distribution. The estimators' performance is further assessed using the Monte Carlo simulation technique. Eventually, two real-world data sets are utilized to show the applicability of the new distribution.

1. Introduction

Analysis of lifetime data has seen applications in various fields, including health, business, engineering, finance, etc. [1, 2]. The main objective of such analysis is usually to model the distribution time to an event and/or the determinants of time to event of interest.

For modeling lifetime data, a variety of probability models are available, including log-logistic, beta, gamma, Weibull, exponential, and others. Furthermore, in many cases, these traditional models are inappropriate for modeling lifetime data [3, 4] necessitating the use of updated versions of current distributions [5]. New models bring up new possibilities for theoretical and practical researchers to solve real-world issues since they suit asymmetric and complicated random occurrences so well [6].

The Weibull distribution has been used to cope with several challenges in a wide range of survival data and to model lifespan data. The Weibull distribution, with its negatively and positively skewed density forms, is the primary option when modeling monotone hazard rates [7].

The parameters of this distribution's tremendous flexibility allow for a range of techniques, all of which have the same key property.

The hazard rate is a monotone function that can be decreasing, constant, or growing [8]. The Weibull distribution is inappropriate for survival data with a non-monotone failure rate function.

As a result, scientists explored for extensions and modifications of the Weibull distribution. Mudholkar and Srivastava [9] developed a three-parameter model by exponentiating the Weibull distribution, known as the exponentiated Weibull distribution. Silva et al. [10] proposed the beta modified Weibull distribution. Bourguignon et al. [11] introduced the Weibull-G family. Pinho et al. [12] developed the gamma-exponentiated Weibull distributions (GEW). Xie et al. [13] developed a three-parameter modified Weibull extension.

Lee et al. [14] introduced the beta Weibull model, which can be incorporated into data sets with nonmonotone and monotone hazard rate functions (hrf) and has the exponential, exponentiated Weibull, and exponentiated exponential models as submodels, among others. Carrasco et al. [15] defined and investigated a novel four-parameter modification of the Weibull distribution that can simulate bathtub-shaped failure rate forms.

Cordeiro et al. [16] developed the exponential Weibull distribution. Bander et al. [17] proposed the (P-A-L) extended Weibull distribution, which is a novel generalization of the three-parameter extended Weibull distribution. Almheidat et al. [18] suggested a generalization of the Weibull distribution, described by four parameters that specify the shape and scale properties. Famoye et al. [19] developed the Weibull-normal distribution and found that the Weibull-normal distribution can be unimodal or bimodal. Aldahlan [20] introduced a novel model called the inverse Weibull inverse exponential (IWIE) distribution. Hassan and Abd-Allah [21] proposed the exponentiated Weibull Lomax, a novel five-parameter model derived from the exponentiated Weibull-generated family.

Additional important generalized forms of the Weibull model are introduced by Korkmaz et al. [22–24], Abouelmagd et al. [25–27], Cordeiro et al. [28], Bhatti et al. [29], Nasir et al. [30], Alizadeh et al. [31], Afify et al. [32, 33], Hussein et al. [34], Mead et al. [35] and Nassar et al. [36].

In recent years, the technique of parameter induction has attracted a lot of attention, for example, Tahir and Nadarajah [37] reviewed the most common G families introduced in the last decade using the parameter induction technique. Tahir and Cordeiro [38] presented a survey for probability families formulated by parameter induction techniques, and they developed some new G families. Ahmad et al. [39] reviewed and presented a brief survey of recent advances in distribution theory with a focus on the parameter induction technique. Recently, Muse et al. [40] presented a survey of the log-logistic (LL) distribution and its generalizations by focusing on the new LL distributions formulated from the parameter induction technique.

The addition of one or more extra shape parameters to the parent distribution makes it more flexible, which is especially useful when analyzing tail features.

Based on the above discussion, this study proposes a new probability distribution called the extended exponential Weibull (ExEW) distribution, as a modification of the exponential Weibull distribution using the Lehmann-type II approach.

There are two main methods for developing the exponentiated family (EF) of distributions in the literature. These techniques are the Lehmann alternative I (LAI) technique, which has gotten a lot of attention, and the Lehmann alternative II (LAII) technique, which has received less attention.

The method of LAII aids in the derivation and understanding of its many features. According to Nadarajah and Tahir [37], the LAII technique is defined as follows:

If $G(x)$ is the cumulative distribution function (cdf) and $\overline{G}(x) = 1 - G(x)$ is the survival function (sf) of the existing distribution, then, by taking one minus the α^{th} power of $\overline{G}(x)$, the cdf of the LAII family or the exponential family (EF) follows as

$$F(x) = 1 - [1 - G(x)]^{\alpha}. \quad (1)$$

According to (1), the probability density function (pdf) reduces to

$$f(x) = [1 - G(x)]^{\alpha-1} G(x)(\alpha). \quad (2)$$

The sf is defined as follows:

$$S(t) = [1 - G(t)]^{\alpha}. \quad (3)$$

The hazard rate function (HRF) is

$$h(t) = \frac{g(t)(\alpha)}{1 - G(t)}. \quad (4)$$

The reverse hazard function is

$$r(t) = \frac{pdf}{cdf} = \frac{g(t)(\alpha)}{G(t)}. \quad (5)$$

The cumulative hazard function $H(t)$ is

$$H(t) = -\log[1 - G(t)](\alpha). \quad (6)$$

The following can be used to briefly outline the article's motivations: (i) using the LAII parameter induction approach, a tractable extension of the Weibull distribution will be introduced, which provides increasing, decreasing, and constant hazard rate forms. (ii) Createing a modified Weibull distribution with a more adaptable kurtosis compared to the standard Weibull model. (iii) Extending the parent Weibull distribution to one that is extended so that its density function can display symmetrical, asymmetrical, unimodal, J, and reversed-J shapes. (iv) Creating a more comprehensive model that can be used to represent different types of data in the fields of engineering, medicine, actuarial science, and other applied fields. This fact is demonstrated by modeling two real-life data sets from the engineering and medical disciplines, demonstrating its superiority as compared to other competing distributions. (v) lastly, our motivation stems from the desire to demonstrate how the inclusion of a single parameter may increase the application and tractability of the parent distribution. The remainder of the research develops and talks about each of those issues.

The remainder of the article is structured as follows: Section 2 discusses the basic lifetime functions of the proposed distribution and its submodels. Section 3 provides some of the mathematical properties of the ExEW distribution. The estimation of the ExEW parameters is investigated in Section 4. An extensive Monte Carlo simulation study is presented in Section 5. Section 6 presents two real-life data applications of the proposed distribution. In Section 7, concluding remarks are presented.

2. The ExEW Distribution

The ExEW distribution generalizes the exponential Weibull distribution. It is formulated by using the LAII approach.

Let $X \sim ExEW(a, b, c, \alpha)$, then the cdf of the ExEW distribution can be defined by applying (1) as follows:

$$F(x) = 1 - \left[e^{-(ax+bx^c)}\right]^{\alpha}, x > 0, \tag{7}$$

where $\alpha > 0$, $a > 0$, and $c > 0$ are shape parameters and $b > 0$ is the scale parameter.

2.1. Submodels. The proposed ExEW distribution has some submodels that are often utilized in parametric survival modeling. Its submodels include the exponential (E), Weibull (W), exponentiated exponential (EE) [41], and the exponential Weibull (EW) distributions [16]. These submodels are listed in Table 1.

Table 1: Submodels of the ExEW(a, b, c, α) distribution.

Model	a	b	c	α
E	a	0	0	1
W	0	b	c	1
EE	a	0	0	α
EW	a	b	c	1

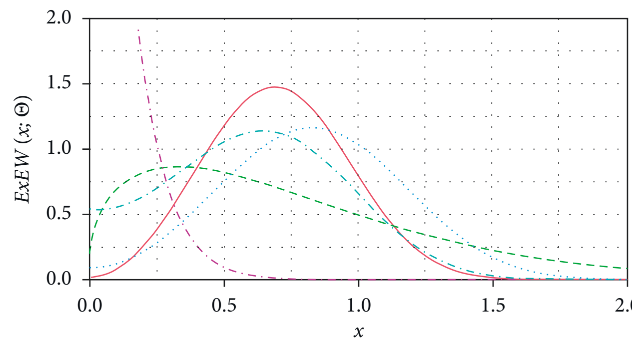

Figure 1: Shapes of the pdf of the ExEW distribution for various choices of the parameters.

2.2. Probabilistic Functions for the ExEW Distribution. In this section, the pdf, hrf, and sf of the ExEW are presented. In addition to the above probabilistic functions, cumulative hrf (chrf) and reverse hrf (rhrf) are also formulated.

(1) The pdf corresponding to (7) takes the form

$$f(x) = \alpha\left(\left(a + bcx^{c-1}\right)e^{-(ax+bx^c)}\right)\left[e^{-(ax+bx^c)}\right]^{\alpha-1}, x > 0. \tag{8}$$

Figure 1 illustrates pdf shapes of the ExEW distribution for various choices of the parameters. The pdf of the ExEW distribution can be symmetrical, asymmetrical, unimodal, J, and reversed-J shapes.

(2) The sf corresponding to (7) is as follows:

$$S(x) = \left[e^{-(ax+bx^c)}\right]^{\alpha}. \tag{9}$$

(3) The hrf of the proposed distribution is expressed as

$$h(x) = \frac{\alpha\left(\left(a + bcx^{c-1}\right)e^{-(ax+bx^c)}\right)}{\left[e^{-(ax+bx^c)}\right]^{\alpha}}. \tag{10}$$

Figure 2 shows the hrf which is clearly decreasing, increasing, constant, J-shaped, and reverse J-shaped.

(4) The rhrf is written as follows:

$$r(x) = \frac{\alpha\left(\left(a + bcx^{c-1}\right)\right)\left[e^{-(ax+bx^c)}\right]^{\alpha}}{1 - \left[\exp\left(-\left(ax + bx^c\right)\right)\right]^{\alpha}}. \tag{11}$$

(5) The chrf is obtained as

$$H(x) = -\alpha \log\left[1 - \left[\exp\left(-\left(ax + bx^c\right)\right)\right]^{\alpha}\right]. \tag{12}$$

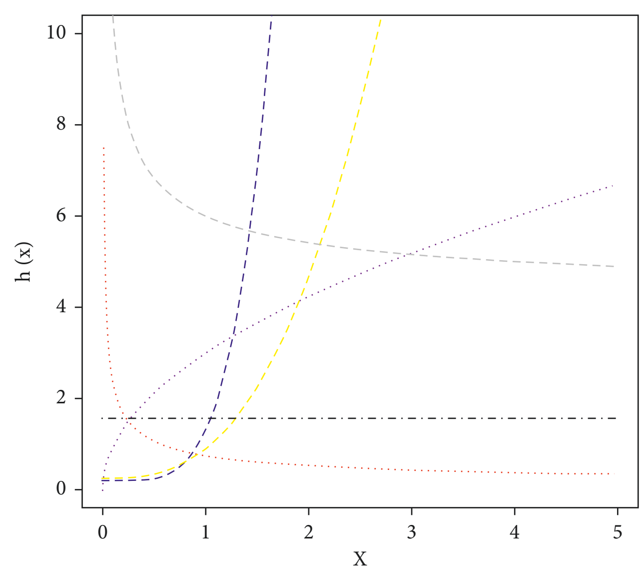

Figure 2: Shapes of the HRF of the ExEW distribution for different values of the parameters.

3. Mathematical Properties

In this part, the ExEW distribution's mathematical properties are discussed.

3.1. The Quantile Function. The quantile function (qf) is the reverse of the cdf and is significant in quantitative and statistical data analysis. A probability distribution can be described using either the qf or the cdf [42].

Let $x = Q(p) = F^{-1}(p)$, for $0 < p < 1$, then the qf of the ExEW distribution is given by reversing (7), thus $F(x) = 1 - [e^{-(ax+bx^c)}]^{\alpha} = p$ for x, then the qf will be the solution of the equation,

$$ax + bx^c = -\ln\{1-p\}^{1/\alpha}. \quad (13)$$

3.2. Residual and Reverse Residual Life Functions. In reliability analysis and risk management, residual life has a wide range of applications. The residual life of the ExEW r.v. is

$$R_{(t)}(x) = \frac{S(x+t)}{S(t)},$$
$$R_{(t)}(x) = \frac{\left[e^{-(a(x+t)+b(x+t)^c)}\right]^{\alpha}}{\left[e^{-(at+bt^c)}\right]^{\alpha}}. \quad (14)$$

Furthermore, the reverse residual life of the ExEW distribution can be calculated as follows:

$$\widehat{R}_{(t)}(x) = \frac{S(x-t)}{S(t)},$$
$$\widehat{R}_{(t)}(x) = \frac{\left[e^{-(a(x-t)+b(x-t)^c)}\right]^{\alpha}}{\left[e^{-(at+bt^c)}\right]^{\alpha}}. \quad (15)$$

3.3. Moments. Moments can be used to analyze some of a distribution's most important characteristics and properties, such as dispersion, tendency, kurtosis, and skewness.

The r^{th} moment of an r.v. $X \sim \mathrm{ExEW}(a,b,c,\alpha)$ is

$$E(X^r) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k+m}\Gamma(\alpha)}{j!k!m!\Gamma(\alpha-j)} a^m \left\{\frac{\alpha a\Gamma(r+k+m+-2c/c)}{c[b(1+j)]r+k+m+1/c} + \frac{b\Gamma(r+m-c/c)}{[b(1+j)]^{r+c+m/c}}\right\}. \quad (16)$$

Proof. An r.v. X with pdf f(x), the r^{th} moment, is written as follows:

$$\mu_r' = \int_0^{\infty} x^r f(x)dx. \quad (17)$$

(8) is substituted for equation (17) and the result is

$$E(X^r) = \alpha a \int_0^{\infty} x^r e^{-(ax+bx^c)}\left[e^{-(ax+bx^c)}\right]^{\alpha-1}dx + bc\int_0^{\infty} x^r x^{c-1} e^{-(ax+bx^c)}\left[e^{-(ax+bx^c)}\right]^{\alpha-1}dx. \quad (18)$$

Applying binomial expansion and further simplification, (18) becomes

$$E(X^r) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\alpha)(aj)}{j!k!\Gamma(\alpha-j)} \alpha a \int_0^{\infty} x^{r+k} e^{-(ax+bx^c)} e^{-jbx^c}dx + \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\alpha)(aj)}{j!k!\Gamma(\alpha-j)} bc\int_0^{\infty} x^{r+c-1} e^{-(ax+bx^c)} - jbx^c dx. \quad (19)$$

But,

$$e^{-(ax+bx^c)} = e^{-ax} \times e^{-bx^c}. \quad (20)$$

By using McLaurin's series expansion

$$e^{-ax} = \sum_{m=0}^{\infty} (-1)^m \times \frac{a^m x^m}{m!}. \quad (21)$$

Substituting (21) in (19), we have

$$E(X^r) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k+m}\Gamma(\alpha)}{j!k!m!\Gamma(\alpha-j)}\left\{\alpha a^{m+1}\int_0^{\infty} x^{r+k+m} e^{-bx^c(1+j)}dx\right\} + \left\{bca^m \int_0^{\infty} x^{r+c+m-1} e^{-bx^c(1+j)}dx\right\}. \quad (22)$$

Let

$$w = bx^c(1+j). \quad (23)$$

We get

$$x^c = \frac{w}{b(1+j)} \Rightarrow \frac{w^{1/c}}{[b(1+j)]^{1/c}}. \quad (24)$$

Hence,

$$x = \frac{w^{1/c}}{[b(1+j)]^{1/c}}. \quad (25)$$

Then,

$$\frac{\mathrm{d}x}{\mathrm{d}w} = \frac{1/cw^{1-c/c}}{[b(1+j)]^{1/c}}, \quad (26)$$

which is as follows:

$$dx = \frac{w^{1-c/c}}{c[b(1+j)]^{1/c}}dw. \quad (27)$$

Substituting (27) in equation (22), we obtain

$$E(X^r) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k+m}\Gamma(\alpha)}{j!k!m!\Gamma(\alpha-j)}\left\{\alpha a^{m+1}\int_0^{\infty} \frac{w^{r+k+m/c}}{[b(1+j)]^{r+k+m/c}} e^{-w}\right\} \times \left\{\frac{w^{1-c/c}}{c[b(1+j)]^{1/c}}dw + bca^m \int_0^{\infty} \frac{w+c+m-1/c}{[b(1+j)]^{r+c+m-1/c}} e^{-w} \times \frac{w^{1-c/c}}{c[b(1+j)]^{1/c}}dw\right\}. \quad (28)$$

After simplification, (28) becomes

$$E\left(X^r\right) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k+m}\Gamma(\alpha)}{j!k!m!\Gamma(\alpha-j)} a^m \left\{\frac{\alpha a\Gamma(r+k+m+-2c/c)}{c[b(1+j)]^{r+k+m+1/c}} + \frac{b\Gamma(r+m-c/c)}{[b(1+j)]^{r+c+m/c}}\right\}. \tag{29}$$

Table 2 reports the values of the first five moments, standard deviation (SD), coefficient of variation (CV), skewness (CS), and kurtosis (CK) of the ExEW model for various parameter values.

From Table 2, the ExEW distribution is quantitatively versatile in terms of mean and variance. As evidenced by its values, CS can be right skewed, almost symmetrical, or somewhat left-skewed. The CK values indicate whether the ExEW distribution is leptokurtic, platykurtic, or mesokurtic. All of these features point to the ExEW distribution's versatility, which makes it an ideal choice for modeling.

3.4. Moment Generating Function (mgf). The mgf of the ExEW distribution is written as follows:

$$M_t(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f_{ExEW}(x)\mathrm{d}x. \tag{30}$$

Using the results from Subsection 3.3, we can get the closed-form for the mgf. Then, the mgf of ExEW distribution reduces to

$$E\left(e^{tx}\right) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^{j+k+m}\Gamma(\alpha)}{k!m!j!\Gamma(-j+\alpha)} \frac{(t)^r}{r!} a^m \left\{\alpha a \frac{\Gamma(((r+k+m+-2c)/c))}{c[b(1+j)]^{((r+k+m+1)/c)}} + b\frac{\Gamma(((r+m-c)/c))}{[b(1+j)]^{((r+c+m)/c)}}\right\}. \tag{31}$$

4. Maximum Likelihood Estimation

Maximum likelihood (ML) is used to estimate the unknown parameters of the ExEW distribution using a full sample.

If $X_1, X_2, \ldots, X_n$ denote a random sample from the ExEW distribution with an unknown parameter vector $\phi = (a, b, c, \alpha)$, then the ML function follows as

$$L(\phi) = \prod_{i=1}^{n} \alpha\left(a + bcx_i^{c-1}\right) e^{-\left(ax_i + bx_i^c\right)} \left[e^{-\left(ax_i + bx_i^c\right)}\right]^{\alpha-1}. \tag{32}$$

Then, the log-likelihood function reduces to

$$\ell(\phi) = n\ln a + n\ln b + n\ln c + n\ln\alpha + n\ln(c-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(ax_i + bx_i^c\right) + (\alpha-1)\sum_{i=1}^{n}\ln\left(e^{-\left(ax_i+bx_i^c\right)}\right). \tag{33}$$

The parameter estimates are produced by performing a partial derivative of $\ell(\phi)$ with respect to each parameter, as follows:

$$\begin{aligned}
\frac{\partial\ell}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n} x_i - (\alpha-1)\sum_{i=1}^{n} x_i. \\
\frac{\partial\ell}{\partial b} &= \frac{n}{b} - \sum_{i=1}^{n} x_i^c - (\alpha-1)\sum_{i=1}^{n} x_i^c, \\
\frac{\partial\ell}{\partial c} &= \frac{n}{c} + \frac{n}{(c-1)}\sum_{i=1}^{n}\ln x_i - b\sum_{i=1}^{n} x_i^c \ln x_i, \\
\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^c.
\end{aligned} \tag{34}$$

The unknown parameters can be calculated via resetting the aforementioned equations to zero and calculating them all at once. These equations can also be numerically solved using statistical software (for example, the adequacy model package in R software) or an iterative technique such as the Newton–Raphson algorithm.

Because of the predicted information matrix is too complicated to set confidence intervals for the parameters, the observed information matrix $I_{(\phi)}$ is used.

The following is how the information matrix is obtained:

$$I_{(\phi)} = -\begin{bmatrix} \frac{\partial\ell^2}{\partial a^2} & \frac{\partial\ell^2}{\partial a\partial b} & \frac{\partial\ell^2}{\partial a\partial c} & \frac{\partial\ell^2}{\partial a\partial\alpha} \\ & \frac{\partial\ell^2}{\partial b^2} & \frac{\partial\ell^2}{\partial b\partial c} & \frac{\partial\ell^2}{\partial b\partial\alpha} \\ & & \frac{\partial\ell^2}{\partial c^2} & \frac{\partial\ell^2}{\partial\alpha\partial c} \\ & & & \frac{\partial\ell^2}{\partial\alpha^2} \end{bmatrix}, \tag{35}$$

whereas the regularity criteria are satisfied and the parameters are within the interior of the parameter space but not on the boundary; $\sqrt{n}(\cong \widehat{\phi} - \phi)$ converges in distribution to $N_4(0, I^{-1}(\phi))$, where $I_{(\phi)}$ is the predicted. When $I_{(\phi)}$ is substituted by the observed information matrix assessed at $J_{(\widehat{\phi})}$, the asymptotic behavior remains true. To construct $100(1-\tau)\%$ two-sided 95% confidence interval for model parameters, we use the asymptotic multivariate normal distribution $N_4(0, I^{-1}(\widehat{\phi}))$, where τ is the significance level.

5. Simulation Study

In this part, a comprehensive numerical inspection using Monte Carlo simulations is achieved to evaluate the capability of the ML estimates (MLEs) for the ExEW model. The absolute biases (AB), root mean square errors (RMSEs), and coverage probability (CP) are calculated for different small

Table 2: Numerical values of the first five moments, SD, CV, CS, and CK for various parametric values.

	a, b, c, α				
Moments	(1, 10, 1.9, 0.05)	(0.1, 1, 1.1, 1.05)	(0.5, 7, 0.5, 1.1)	(0.1, 1, 1.05, 1.1)	(1, 7, 1.1, 2.1)
μ_1'	1.197	0.584	0.208	0.557	0.035
μ_2'	3.0132	0.433	0.119	0.414	0.012
μ_3'	1.007	0.344	0.079	0.329	0.009
μ_4'	4.173	0.286	0.058	0.273	0.003
μ_5'	2.047	0.244	0.045	0.233	0.002
SD	5.357	0.305	0.275	0.322	0.106
CV	4.474	0.522	1.323	0.577	3.068
CS	5.869	−0.594	1.109	−0.508	3.843
CK	4.513	2.239	3.036	2.008	19.828

and large samples and parameter settings to evaluate the performance of MLEs. To produce random samples from the ExEW, the qf (13) is employed. With $n = 25, 50, 75, 100, 150$, and 200, the simulation experiments are repeated $N = 1000$ times. For set I, $a = 1, b = 1, c = 1.5$, $\alpha = 0.12$, set II: $a = 1, b = 1, c = 1.5$, $\alpha = 0.14$, set III: $a = 1, b = 1, c = 1.5$, $\alpha = 0.25$, and set IV: $a = 1, b = 1, c = 1.5$, $\alpha = 0.50$.

The following equations are used to calculate the AB, RMSE, and CP for the estimates.

$$\mathrm{AB} = \frac{\sum_{i=1}^{n} (\hat{\phi} - \phi)}{N},$$
$$\mathrm{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^{n} (\hat{\phi} - \phi)^2}, \tag{36}$$

where $\phi = a, b, c, \alpha$.

The CP is the proportional of times the $100(1 - \tau)\%$ confidence interval, which is given by

$$CP = \hat{\phi} \mp Z_{\tau} \times SE(\hat{\phi}). \tag{37}$$

The AB and RMSE values of the parameters a, b, c, and α for various sample sizes are shown in Tables 3 and 4. The visual comparisons of these results are shown in Figures 3–10. The findings show that the RMSE decreases as the sample size grows until it hits zero. Furthermore, the AB decreases as sample size grows. As a result, the MLEs and their asymptotic features can be used to build confidence ranges even for tiny sample numbers. Additionally, the confidence intervals' CPs are quite close to the nominal 95 percent level.

Table 3: The results of AB_s, RMSEs, and CP for the MLEs of ExEW distribution for I and II.

		I			II		
Parameters	n	AB_s	RMSEs	CP	AB_s	RMSEs	CP
a	25	2.327	3.646	0.995	2.390	3.643	1.000
	50	1.784	2.897	0.998	1.804	2.927	0.999
	75	1.603	2.675	1.0	1.571	2.698	1.000
	100	1.247	2.223	0.999	1.122	2.121	1.000
	150	0.818	1.555	1.000	0.793	1.583	1.000
	200	0.631	1.273	1.00	0.539	1.113	1.000
b	25	0.695	0.818	0.872	0.710	0.829	0.859
	50	0.570	0.736	0.926	0.578	0.749	0.914
	75	0.518	0.708	0.93	0.514	0.709	0.927
	100	0.451	0.651	0.952	0.403	0.642	0.959
	150	0.334	0.571	0.982	0.324	0.576	0.975
	200	0.281	0.535	0.99	0.259	0.528	0.993
c	25	1.332	1.910	0.97	1.509	2.139	0.957
	50	0.884	1.362	0.969	1.035	1.562	0.957
	75	0.785	1.263	0.97	0.857	1.376	0.964
	100	0.609	1.047	0.971	0.595	1.046	0.976
	150	0.387	0.705	0.971	0.404	0.764	0.971
	200	0.292	0.518	0.98	0.285	0.546	0.975
α	25	0.033	0.057	0.999	0.041	0.067	0.999
	50	0.028	0.044	1.000	0.036	0.051	1.000
	75	0.027	0.040	1.000	0.032	0.045	1.000
	100	0.021	0.034	1.000	0.023	0.038	1.000
	150	0.015	0.027	1.000	0.017	0.031	1.000
	200	0.012	0.023	1.000	0.013	0.025	1.000

6. Applications to Real-Life Data

To illustrate the applicability of the ExEW distribution, we analyze and compare the fitting of the ExEW distribution with other competing models by using two real-life data sets. The ExEW distribution is compared to submodels such as the W, EE [41], and EW distributions [16], and other common lifetime distributions including the log-logistic (LL), beta Weibull (BW) [14], beta extended Weibull (BEW) [43], modified beta Weibull (MBW) [44], and tan-log-logistic (TanLL) distributions [45].

The competing models' pdfs are as follows:

(1) The pdf of the W distribution is

$$f(x) = acx^{a-1} \exp(-cx^{a}). \tag{38}$$

(2) The pdf of EE distribution is

$$f(x) = ac \exp(-cx)\{1 - \exp(-cx)\}^{a-1}. \tag{39}$$

(3) The pdf of EW distribution is

$$f(x) = \left(a + bcx^{c-1}\right)\exp - \left(ax + bx^{c}\right). \tag{40}$$

(4) The pdf of the LL distribution is

$$f(x) = \frac{ac((cx))^{a-1}}{\left[1 + (cx)^{a}\right]^{2}}. \tag{41}$$

Table 4: The results of AB_s, RMSEs, and CP for the MLEs of ExEW distribution for III and IV.

Parameters	n	III AB_s	III RMSEs	III CP	IV AB_s	IV RMSEs	IV CP
a	25	1.750	2.816	0.996	1.304	2.243	0.999
	50	1.362	2.330	1.000	1.111	2.070	0.997
	75	1.029	1.965	0.999	0.798	1.543	1.000
	100	0.871	1.617	0.999	0.742	1.372	0.999
	150	0.578	1.222	1.000	0.463	1.045	1.000
	200	0.455	01.036	1.00	0.361	0.998	1.000
b	25	0.592	0.725	0.87	0.518	0.685	0.914
	50	0.494	0.658	0.912	0.449	0.647	0.925
	75	0.386	0.617	0.944	0.370	0.593	0.943
	100	0.367	0.595	0.954	0.361	0.577	0.964
	150	0.285	0.536	0.976	0.250	0.508	0.988
	200	0.240	0.489	0.99	0.187	0.476	0.986
c	25	1.764	2.671	0.946	2.417	3.911	0.938
	50	1.179	1.892	0.969	1.705	2.917	0.955
	75	0.844	1.480	0.977	1.174	2.153	0.965
	100	0.745	1.314	0.976	0.984	1.726	0.966
	150	0.487	0.933	0.981	0.569	1.12	0.974
	200	0.352	0.711	0.973	0.466	1.054	0.983
α	25	0.063	0.111	1.000	0.092	0.197	1.000
	50	0.054	0.085	1.000	0.082	0.160	1.000
	75	0.040	0.071	1.000	0.062	0.136	1.000
	100	0.035	0.065	1.000	0.061	0.121	1.000
	150	0.025	0.049	1.000	0.035	0.093	1.000
	200	0.017	0.041	1.000	0.028	0.083	1.000

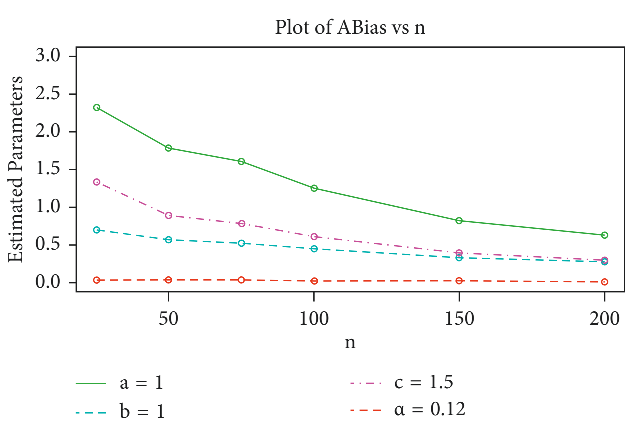

Figure 3: The plots of ABs for the ExEW parameters in set I.

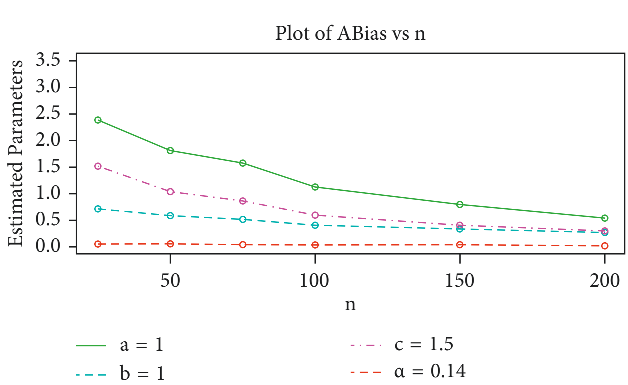

Figure 4: The plots of ABs for the ExEW parameters in set II.

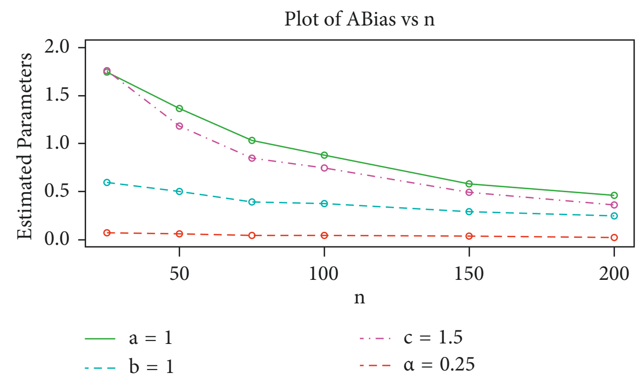

Figure 5: The plots of ABs for the ExEW parameters in set III.

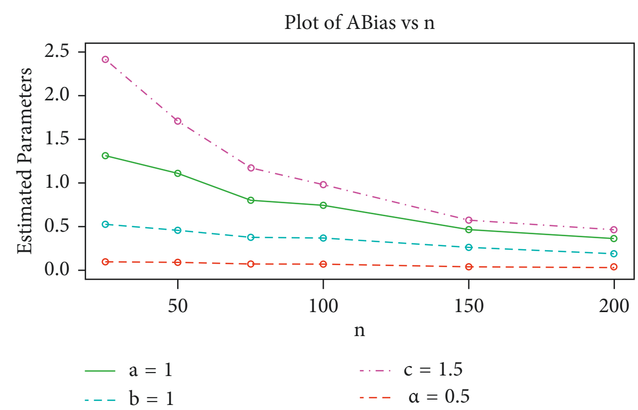

Figure 6: The plots of ABs for the ExEW parameters in set IV.

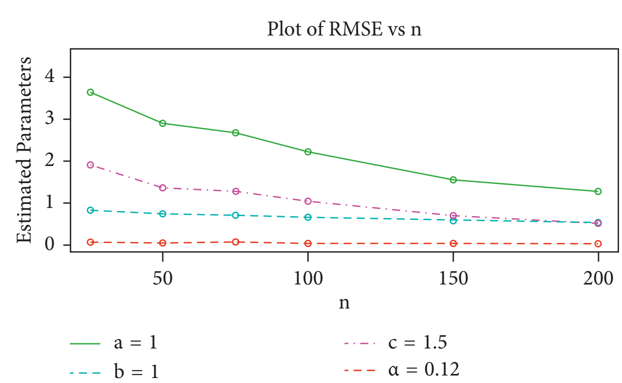

Figure 7: The plots of RMSEs for the ExEW parameters in set I.

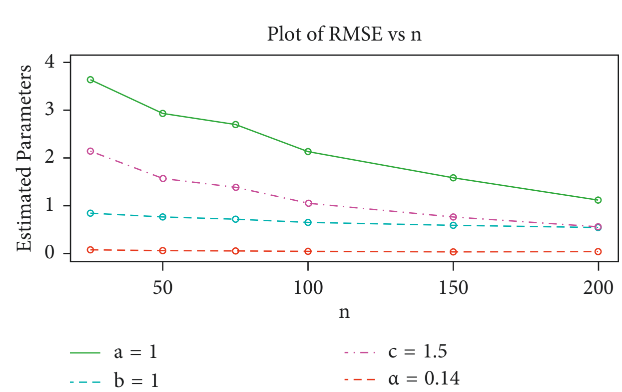

Figure 8: The plots of RMSEs for the ExEW parameters in set II.

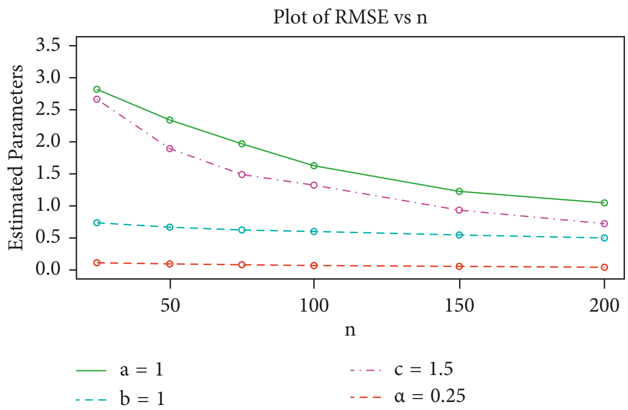

FIGURE 9: The plots of RMSEs for the ExEW parameters in set III.

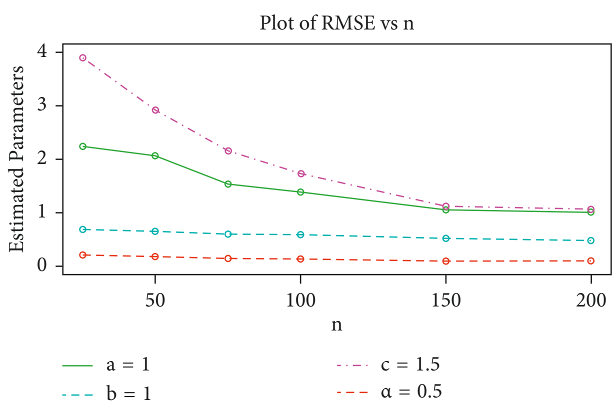

FIGURE 10: The plots of RMSEs for the ExEW parameters in set IV.

(5) The pdf of BW distribution is

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{c}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left[1-e^{-\left(\frac{x}{\alpha}\right)^{c}}\right]^{a-1} e^{-b\left(\frac{x}{\alpha}\right)^{c}}. \tag{42}$$

(6) The pdf of BEW distribution is

$$f(x) = \frac{\alpha}{\beta(c,d)} b a^{b} x^{b-1} e^{-ax^{b}} \left(1-e^{-\alpha\left(1-e^{-ax^{b}}\right)}\right)^{c-1} e^{-\alpha d\left(1-e^{-ax^{b}}\right)}. \tag{43}$$

(7) The pdf of MBW is

$$f(x) = \frac{\alpha^{c}}{\beta(c,d)}\frac{b a^{b} x^{b-1} e^{1-ax^{b}}\left[1-e^{-ax^{b}}\right]^{c-1}\left[e^{-ax^{b}}\right]^{d-1}}{\left(1-(1-\alpha)\left(1-e^{-ax^{b}}\right)\right)^{c+d}}. \tag{44}$$

(8) The pdf of TanLL distribution is

$$f(x) = \frac{\pi}{4}\left\{\frac{(b/a)(x/a)^{b-1}}{\left(1+(x/a)^{b}\right)^{2}}\right\}\sec^{2}\left[\frac{\pi}{4}\left\{\frac{(x/a)^{b}}{\left(1+(x/a)^{b}\right)}\right\}\right]. \tag{45}$$

To specify which statistical distribution best fits the two data, a variety of analytical measures are applied, such as the Bayesian information criterion (BIC), the Akaike information criterion (AIC), consistent AIC (CAIC), and the Hannan–Quinn information criterion (HQIC). Moreover, goodness-of-fit measures like the log-likelihood are also adopted.

6.1. Likelihood Ratio Tests. The ExEW distribution has some submodels, including the W, EE, and EW distributions. As a result, the likelihood ratio test (LRT) is used to evaluate the following hypotheses:

(1) H_0: $a = 0$, and $\alpha = 1$, this means that the sample comes from the W distribution.

H_1: $a \neq 0$, and $\alpha \neq 1$, this means that the sample comes from the ExEW distribution.

(2) H_0: $b = 0$, this indicates that the sample size comes from the EE distribution.

H_1: $b \neq 0$, this indicates that the sample size comes from the ExEW distribution.

(3) H_0: $\alpha = 1$, this means that the sample comes from the EW distribution.

H_1: $\alpha \neq 0$, this means that the sample comes from the ExEW distribution.

The LRT is written as

$$LRT = -2\ln\frac{\left(L\left(\hat{\phi}^{*};x\right)\right)}{\left(L(\hat{\phi},x)\right)}, \tag{47}$$

where $\hat{\phi}^{*}$ denotes the constrained MLEs for the null hypothesis H_0, whereas $\hat{\phi}$ denotes the unconstrained MLEs for the alternative hypothesis H_1. The LRT follows the chi-square distribution with degrees of freedom $(df)(df_{alt} - df_{null})$ when the null hypothesis is true.

The null hypothesis is rejected if the p value is less than 5%.

6.2. Application to Airplane Windshield Data. The airplane windshield data consists of 84 observations. Ramos et al. [46] recently studied the data set.. The data set is reported in Table 5, and its descriptive statistical analysis is shown in Table 6 which indicates that the skewness coefficient has a positive value, the data is right skewed. Due to the kurtosis having negative value, the data are platykurtic.

Figure 11 illustrates the TTT transform plot with a concavity pattern, indicating that the data has an increasing hazard rate shape. This confirms that the hrf in Figure 2 is appropriate for analyzing this data.

Table 5: The airplane windshield failure times data (in thousand hours).

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

Table 6: Descriptive analysis of airplane windshield data.

μ	*Mid*	Mode	Var	*CS*	*CK*	Min	Max
2.55745	2.3545	2.25	1.25177	0.09949	−0.65232	0.04	4.663

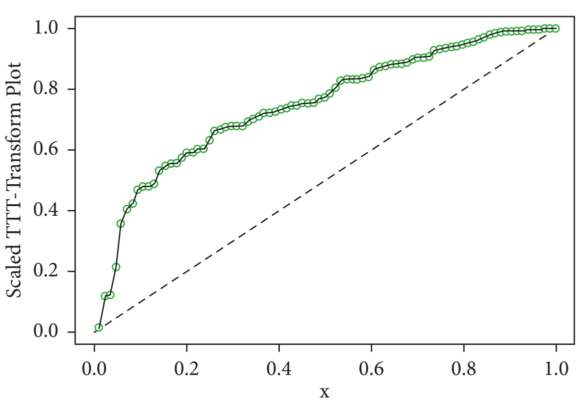

Figure 11: The TTT Plot of the airplane windshield data.

The MLEs of the parameters of the fitted models, as well as the corresponding standard errors, are shown in Table 7. At the 5% significance level, all of the ExEW parameters are significant. The ExEW model fits the airplane windshield data better than its submodels and other rival distributions. Table 8 shows that the ExEW has the highest log-likelihood and the lowest CAIC, HQIC, BIC, and AIC values as compared to the other models. Although the ExEW model provides the greatest fit to the data, the W distribution is a suitable option since its fit values are more similar to the ExEW model. Table 9.

Figure 12 illustrates the fitted density shapes for competitive models, demonstrating that the ExEW distribution fits aircraft windshields better.

Table 7: The MLEs of the competing models with standard errors for windshield data.

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	d
ExEW	0.712	0.504	2.931	0.0715	—
	(6.338)	(5.510)	(0.354)	(0.637)	—
W	2.375	2.863	—	—	—
	(0.210)	(0.138)	—	—	—
EE	3.560	0.758	—	—	—
	(0.611)	(0.077)	—	—	—
EW	−0.017	0.159	1.701	—	—
	(0.023)	(0.041)	(0.183)	—	—
LL	2.391	0.224	—	—	—
	(0.137)	(0.297)	—	—	—
BW	0.271	0.783	5.911	3.770	—
	(0.258)	(0.784)	(4.604)	(1.117)	—
	(0.754)	(401)	(1.013)	(0.419)	—
BEW	0.262	5.182	5.208	6.212	7.063
	(0.032)	(1.203)	(0.053)	(0.054)	(0.053)
MBW	0.348	11.855	0.129	4.630	4.663
	(0.414)	(23.226)	(0.266)	(4.660)	(0.888)
TanLL	2.139	3.340	—	—	—
	(0.126)	(0.304)	—	—	—

Table 8: The analytical performance measures for comparing distributions for windshield data.

Model	AIC	BIC	CAIC	HQIC	(ℓ)
ExEW	261.310	271.033	261.816	265.218	−126.655
W	264.107	268.968	264.255	266.061	−130.0533
EE	283.681	288.543	283.829	285.635	−139.841
EW	280.794	288.087	281.094	283.726	−137.397
LL	283.163	288.024	283.311	285.117	−139.581
BW	263.167	272.890	263.673	267.075	−127.583
BEW	265.641	277.795	266.410	270.526	−127.820
MBW	264.563	276.717	265.332	269.449	−127.281
TanLL	283.89	288.752	284.038	285.844	−139.945

6.3. Application to COVID-19 Fatality Rate Data. The second set of data about the COVID-19 fatality rate from Mexico contains 108 days, and it collected between March 4 and July 20, 2020. It is available at https://covid19.who.int. The data are recently studied by Almongy et al. [47]. Table 10 lists the data observations and Table 11 shows the descriptive statistical analysis of the data. Because the skewness coefficient has a positive value, the data are right skewed. The data are platykurtic since the kurtosis is smaller than three. Figure 13 displays the TTT plot with a concavity shape, indicating that the data have an increasing failure rate. This demonstrates that the ExEW distribution is appropriate for analyzing this type of data.

Table 9: The LRT statistic for the windshield data.

Dist.	Hypothesis	LRT	p value
W	H_0: $a = 0, \&\alpha = 1$ vice versa H_1: H_0 is false	260	< 0.001
EE	H_0: $b = 0$ vice versa H_1: H_0 is false	253	< 0.001
EW	H_0: $\alpha = 1$ vice versa H_1: H_0 is false	280	< 0.001

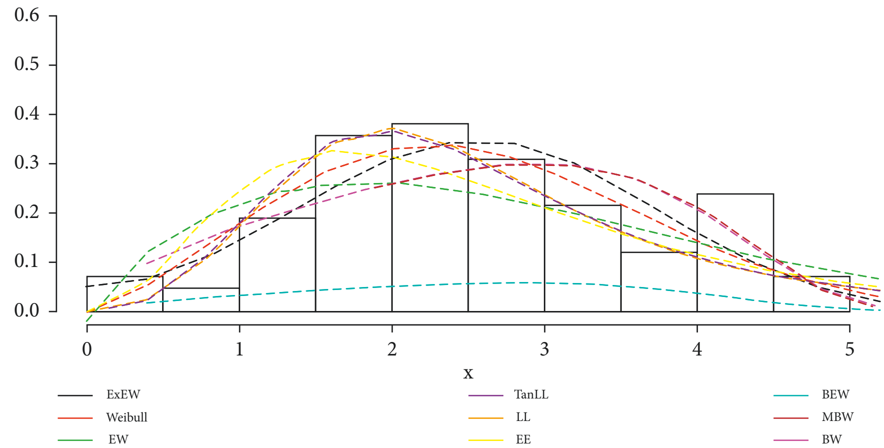

Figure 12: The fitted density shapes of the ExEW distribution and other distributions for windshield data.

Table 10: COVID-19 fatality rate data set.

8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.871, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923

Table 11: Descriptive statistics of COVID-19 fatality rate data.

μ	*Mid*	Mode	Var	*CS*	*CK*	Min	Max
5.758	5.193	3	10.5893	0.98668	0.68134	1.041	16.498

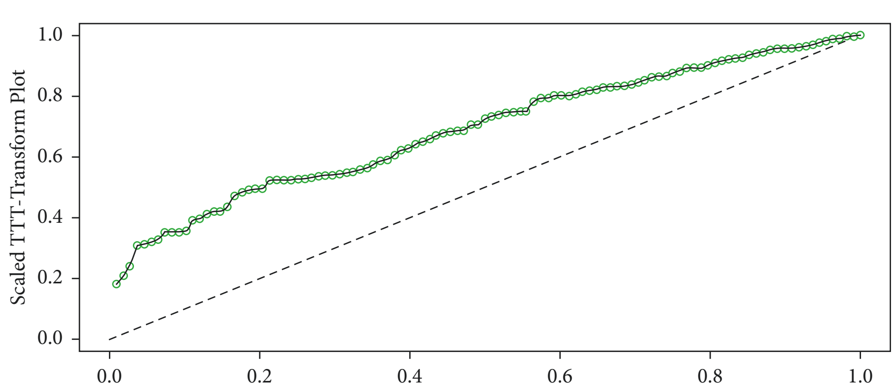

Figure 13: The TTT Plot of the COVID-19 fatality rate data.

Table 12: MLEs of the competing models with standard errors for COVID-19 fatality rate data.

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	d
ExEW	−2.422 (0.005)	2.285 (0.015)	1.059 (0.001)	1.227 (0.020)	— —
W	1.897 (0.138)	6.521 (0.350)	— —	— —	— —
EE	3.998 (0.674)	0.362 (0.035)	— —	— —	— —
EW	−0.410 (0.232)	0.395 (0.206)	1.272 (0.092)	— —	— —
LL	4.973 (0.288)	2.935 (0.231)	— —	— —	— —
BW	2.531 (0.567) (0.162)	0.158 (0.017) (2.297)	1.544 (0.009) (0.126)	1.698 (0.012) (0.122)	— — —
BEW	4.345 (6.153)	8.915 (14.228)	2.495 (3.654)	0.949 (0.686)	33.450 (47.771)
MBW	3.323 (5.260)	0.451 (0.255)	9.698 (10.930)	2.038 (0.950)	5.103 (3.619)
TanLL	4.418 (0.258)	3.054 (0.237)	— —	— —	— —

Table 13: The analytical performance measures for comparing distributions for COVID-19 fatality rate data.

Model	AIC	BIC	CAIC	HQIC	(ℓ)
ExEW	506.444	517.172	506.832	510.794	−249.222
W	541.911	547.275	542.025	544.086	−268.955
EE	536.352	541.716	536.466	538.527	−266.176
EW	516.850	524.897	517.081	520.113	−255.425
LL	542.164	547.527	542.279	544.339	−269.082
BW	263.167	549.962	539.622	2543.584	−265.617
BEW	542.686	556.096	543.274	548.123	−266.343
MBW	539.714	553.124	540.302	545.151	−264.857
TanLL	541.295	546.659	541.409	543.470	−268.647

Table 14: The LRT statistic for COVID-19 fatality rate data.

Dist	Hypothesis	LRT	p value
W	H_0: $a = 0, \&\alpha\ = 1$ vs H_1: H_0 is false	0.920	0.631
EE	H_0: $b = 0$ vs H_1: H_0 is false	40.055	< 0.001
EW	H_0: $\alpha = 1$ vs H_1: H_0 is false	6.479	< 0.039

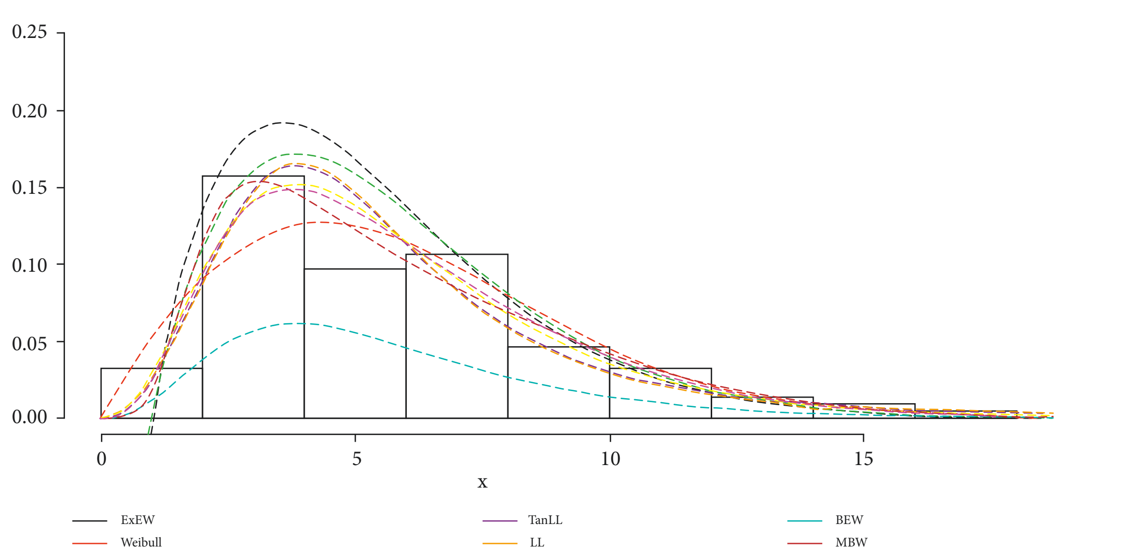

Figure 14: The fitted density shapes of the ExEW distribution and other distributions for COVID-19 fatality rate data.

Table 12 shows the MLEs of the parameters of the fitted models, together with the standard errors in brackets. At the 5% significance level, all of the ExEW parameters are significant. Table 13 provides the analytical measures of competing distributions and shows that the ExEW provides the best fit to the data. Table 14) shows the LRT tests for the proposed model and its submodels.

The fitted densities for competing models are depicted in Figure 14, demonstrating that the ExEW distribution fits the COVID-19 mortality rate data better.

7. Conclusion

The mathematical and statistical properties of the ExEW are proposed and described in this study. The ExEW distribution includes several known sub models as special cases. Some mathematical features of the new model are derived. The ExEW parameters are estimated via the maximum likelihood method, and the estimators' behaviour is evaluated via Monte Carlo simulations. Based on goodness-of-fit statistics and analytical performance measurements, the ExEW model fits two real-world data sets better than its submodels and other typical parametric survival models. As a consequence, we conclude that the ExEW distribution is the most fitting model among the distributions studied and is a good contender for modeling lifetime events.

There are various possible future extensions to this study. The presence of explanatory factors and long-term survival, for example, is common in practice. In addition, a survival regression model that works for both whole and incomplete (truncation, censored) data might be useful. Hence, our approach can be investigated more in various settings.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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