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Research Article

Numerical Approach for Solving the Fractional Pantograph Delay Differential Equations

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A new class of polynomials investigates the numerical solution of the fractional pantograph delay ordinary differential equations. These polynomials are equipped with an auxiliary unknown parameter a, which is obtained using the collocation and least-squares methods. In this study, the numerical solution of the fractional pantograph delay differential equation is displayed in the truncated series form. The upper bound of the solution as well as the error analysis and the rate of convergence theorem are also investigated in this study. In five examples, the numerical results of the present method have been compared with other methods. For the first time, a-polynomials are used in this study to numerically solve delay equations, and accurate approximations have been displayed.

1. Introduction

Since the 17th century, many problems in physics, mechanics, economics, biology, etc., have been investigated and solved by ordinary differential equations. Ordinary differential equations are suitable models for describing processes that occur at the moment, meaning that the process rate of change depends only on its current state, not in its past. However, time delays in many processes cannot be ignored. For example, a signal needs time to reach the desired point and a driver to react to a possible accident. Delays are also used to describe some aspects of infectious diseases: the initial contagion, drug treatment, and immunity to the disease. In modeling the control of blood carbon dioxide levels, the time interval of blood oxygen recovery in the lungs and the stimulation of the chemical receptor in the brainstem are considered as delays in the model [1]. Delayed differential equations have been studied for more than 200 years [2]. In the 19th century, Euler, Lagrange, and Laplace studied delayed differential equations. In the late 1930s and early 1940s, Voltaire introduced a number of delayed differential equations during his studies on the hunting-hunter model and was the first to study these equations in an organized manner.

In recent decades, fractional delay differential equations have had an important role in engineering and natural sciences. Applications of these equations include hydrology, signal processing, control theory, medical sciences, networks, cell biology, climate models, infectious diseases, navigation prediction, circulating blood, population dynamics, oncolytic virotherapy, delayed plant disease model, the body reaction to carbon dioxide, and many others [3–6].

Pantograph differential equations are one of the types of delay differential equations that were first created by studying on an electric locomotive [7]. It is difficult to find an accurate and analytical solution to this problem, so approximate and numerical methods should be used to solve these problems. Different numerical methods have been used to solve fractional pantograph differential equations such as Hermit wavelets method [8], spectral method [9, 10], fractional-order Bernoulli wavelet method [11], fractional-order Boubaker polynomials [12], Taylor collocation method [13, 14], Laguerre–Gauss collocation method [15], Legendre multiwavelet collocation method [16], two-step Runge–Kutta of order one method [17], one–Leg θ method [18], variational method [19], collocation method [20], generalized differential transform scheme [21], modified the predictor-corrector scheme [22], and Bernoulli wavelet method [23].

Various polynomials have been used to solve the pantograph differential equations. The authors in [24] have used a numerical method based on Hermit polynomials to obtain the approximate solution of the generalized pantograph differential equations with variable coefficients. In this study, the solution of the pantograph equation is approximated using the collocation method and creating matrix operators. Rabiei and Ordokhani [12] have used fractional-order Boubaker polynomials to approximate fractional pantograph differential equations in any arbitrary interval. The properties of these polynomials have been used to construct pantograph operational matrices and fractional integrals. Then, by the least-squares method, a nonlinear system is obtained, which is solved using Newton's iterative method. Sedaghat et al. [25] have used the shifted Chebyshev polynomials and Chebyshev pantograph operator matrix and transformed the pantograph differential equation into a system of equations. Also, using error analysis, the number of polynomials used in the approximation is obtained. Isah et al. [26] have used Genocchi polynomials to approximate the generalized fractional pantograph equations under boundary and initial conditions. With the help of the properties of these polynomials, the Genocchi delay operator matrix is produced.

Then, by converting this equation into a system of equations, the approximate solution is obtained. The authors in [27] chose Bernstein polynomials to solve the pantograph differential equations numerically and also studied the stability of the method. Using the spectral Galerkin method and shifted Legendre polynomial properties, Al-Suyuti et al. [28] transformed the multiorder fractional pantograph equations into a system of linear equations and solved this system with appropriate methods. Cakmak and Alkan [29] have used Fibonacci polynomials and collocation method to solve the system of nonlinear pantograph equations under initial conditions. In this method, solving the generated system of equations approximates the given differential equation. Yüzbaşı et al. [30] use the residual correction of Hermit polynomial solutions to solve the generalized pantograph differential equations. Hermit polynomials solutions are obtained by collocation method and transform the problem into a system of equations and obtain unknown coefficients. Javadi et al. [31] have first produced shifted orthonormal Bernstein polynomials and integral and delayed operator matrices of these polynomials. Using the properties of these polynomials, they have converted the generalized pantograph equations into a system of linear equations and proved the stability of this method. Wang et al. [32] has used shifted Chebyshev polynomials to solve generalized fractional pantograph equations with variable coefficients. By producing the generalized pantograph operational matrix using the properties of these polynomials and creating a system of equations and solving this system, the numerical solution of the pantograph equation is obtained. Akkaya et al. [33] have used first Boubaker polynomials to approximate the solution of the pantograph equations with variable coefficients. In this study, the authors have obtained the numerical solution of differential equations by using matrix operators and converting pantograph equations into a system of equations. Ezz–Eldien et al. [34] have approximated the numerical solution of multiorder fractional neutral pantograph equations with the help of Chebyshev polynomials and spectral tau and collocation methods. In this study, the linear and nonlinear pantograph equations and the system of fractional multipantograph equations have been investigated.

We consider a new method for the numerical solution of the following fractional pantograph delay differential equation problem:

$$\begin{aligned} &D^{\alpha}u(x) = c(x)u(x) + \sum_{r=1}^{l} d_r(x)D^{\alpha_r}\left(u(q_r x)\right), \quad m-1<\alpha\leq m,\\ &u^{(i)}(0) = \nu_i, \quad i = 0, 1, \ldots, m-1, \end{aligned} \tag{1}$$

where $0<q_r<1, 0\leq\alpha_r<\alpha\leq m$, u is the unknown function, and the functions c and d_r are the known functions defined in $x\in[0,T]$. Indeed, D^{α} is the Caputo fractional derivative of order α.

2. Basic Concepts

There are many types of fractional derivatives and integrals which are suggested by Riemann, Liouville, Riesz, Letnikov, Grünwald, Weyl, Marchaud, and Caputo. In this study, we will consider the fractional pantograph delay differential equation problem in the Caputo sense.

Definition 1. The Caputo fractional derivative of order αth, for a function $g\in C^{m-1}[a,x]$ and $g^{(m)}$ is integrable on $[a,x]$, is written as follows:

$$D^{\alpha}(g) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{g^{(m)}(\xi)}{(x-\xi)^{\alpha+1-m}}\mathrm{d}\xi, \tag{2}$$

where Γ is Euler's Gamma function (or Euler's integral of the second kind) and $m-1<\alpha\leq m, m\in\mathbb{Z}^{+}$.

Definition 2. The Riemann–Liouville fractional integral of order αth is defined as follows:

$$J_x^{\alpha}g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}g(\xi)\mathrm{d}\xi. \tag{3}$$

For $\alpha,\beta>0$, we know that

$$\begin{aligned} &J_x^{\alpha}\left(D^{\alpha}u(x)\right) = u(x) - \sum_{i=0}^{m-1}u^{(i)}(0)\frac{x^i}{i!}, m-1<\alpha\leq m, x>0,\\ &J_x^{\alpha}\left(D^{\beta}u(x)\right) = J_x^{\alpha-\beta}u(x)\\ &\qquad - \sum_{i=0}^{\lceil\beta\rceil-1}u^{(i)}(0)\frac{x^{i+\alpha-\beta}}{\Gamma(i+\alpha-\beta+1)}, 0<\beta<\alpha,\\ &D^{\alpha}(u(qx)) = q^{\alpha}D^{\alpha}u(qx), \alpha>0, q\neq 0. \end{aligned} \tag{4}$$

As we know, using the polynomials is very useful for finding the solutions to the differential equation problems, especially in engineering applications. It is due to the simple application of them. Recently, a new class of polynomials equipped with an auxiliary parameter has been introduced by the first author in [35] and some applications of it have been shown in [36–39]. This class is introduced as follows.

Definition 3 (see [35]). The a-polynomial functions is defined as follows:

$$\begin{aligned} A_0(x) &= 1, \\ A_n(x) &= axU_{n-1}(x) + U_n(x), \\ &\quad n \geq 1, \end{aligned} \tag{5}$$

where $U_n(.)$ is the second kind of Chebyshev polynomial and a is an auxiliary real parameter.

The following equations are also established as follows:

$$\begin{aligned} A_{n+1}(x) &= 2xA_n(x) - A_{n-1}(x), \quad n \geq 1, \\ A_n(x) &= \left(1 + \frac{a}{2}\right)U_n(x) + \frac{a}{2}U_{n-2}(x), \quad n \geq 2. \end{aligned} \tag{6}$$

See [35–37], for more properties.

3. The Implementation and Error Analysis

In this method, the solution of (1) is approximated by the following truncated series form:

$$u(x) \approx \tilde{u}_N(x) = \sum_{k=0}^{N} c_k A_k(x), \tag{7}$$

where c_k are the unknown coefficients. The collocation points of the present method on the interval $[0, T]$ are defined as $x_j = jh$ such that $h = (T/N)$. Therefore, the unknown coefficients c_k and the unknown auxiliary parameter, a, are obtained by using the collocation method on (1) and hence solving the following nonlinear system of equations:

$$\begin{aligned} \operatorname{Res}(x_j) &= D^{\alpha}u(x_j) - c(x_j)u(x_j) \\ &\quad - \sum_{r=1}^{l} d_r(x_j)D^{\alpha_r}u(q_r x_j) = 0, \end{aligned} \tag{8}$$

for $j = 0, 1, \ldots, N$. Above residual and initial conditions produce a nonlinear system of equations. To solve this nonlinear system of equations, we used the Mathematica software version 12.0 and the **FindMinimum** command.

Now, we want to specify a bound for norm solution $u(x)$ using Gronwall's inequality. Consider the fractional pantograph delay differential (1); by integration by J_x^{α} of the two sides of (1) and using properties (4), (5), and (6), we will have

$$\begin{aligned} u(x) &= J_x^{\alpha}(c(x)u(x)) + \sum_{r=1}^{l}\frac{1}{q_r^{\alpha_r}}J_x^{\alpha-\alpha_r}\left(d_r(x)u(q_r x)\right) \\ &\quad + J_x^{\alpha}f(x) - F(x), \end{aligned} \tag{9}$$

where

$$\begin{aligned} F(x) &= \sum_{i=0}^{m-1} u^{(i)}(0)\frac{x^i}{i!} + \sum_{r=1}^{l}\frac{1}{q_r^{\alpha_r}}\sum_{i=0}^{\lceil\alpha_r\rceil-1} \\ &\quad (d_r(x)u(q_r x))^{(i)}(0)\frac{(q_r x)^{i+\alpha-\alpha_r}}{\Gamma(i+\alpha-\alpha_r+1)}. \end{aligned} \tag{10}$$

Now, we calculate the absolute value (10):

$$|u(x)| \leq |C|\int_a^x |u(s)|ds + \sum_{r=1}^{l}\frac{|D_r|}{q_r^{\alpha_r}}\int_a^x |u(q_r s)|ds + |G(x)|, \tag{11}$$

where

$$\begin{aligned} |C| &= \operatorname*{Max}_{0\leq x\leq T}\left|\frac{c(x)(b-x)^{\alpha-1}}{\Gamma(\alpha)}\right|, \\ |D_r| &= \operatorname*{Max}_{0\leq x\leq T}\left|\frac{d_r(x)(b-x)^{\alpha-\alpha_r-1}}{\Gamma(\alpha-\alpha_r)}\right|, \\ |G(x)| &= \left|J_x^{\alpha}f(x) - F(x)\right|. \end{aligned} \tag{12}$$

With a simple change of variables, we have

$$|u(x)| \leq |C|\int_a^x |u(s)|ds + \sum_{r=1}^{l}\frac{|D_r|}{q_r^{1+\alpha_r}}\int_{q_r a}^{q_r x} |u(s)|ds + |G(x)|. \tag{13}$$

According to $0 < q_r < 1$, we have

$$|u(x)| \leq |C|\int_a^x |u(s)|ds + \sum_{r=1}^{l}\frac{|D_r|}{q_r^{1+\alpha_r}}\int_a^x |u(s)|ds + |G(x)|. \tag{14}$$

So, from the above inequality, we have

$$\|u(x)\| \leq \|G(x)\| + |C|\int_a^x \|u(s)\|ds + \sum_{r=1}^{l}\frac{|D_r|}{q_r^{1+\alpha_r}}\int_a^x \|u(s)\|ds, \tag{15}$$

and hence, Gronwall's inequality [40] concludes that

$$\|u(x)\| \leq \|G(x)\|e^{\left(|C|+\sum_{r=1}^{l}\left(|D_r|/q_r^{1+\alpha_r}\right)\right)x}. \tag{16}$$

3.1. The Rate of Convergence Theorem. In this section, we want to analyze the convergence rate of the present method. For this purpose, first, we express the rate of convergence of Chebyshev polynomials of the second kind.

Theorem 1 (see [41]). *Suppose $g(x) = \sum_{k=0}^{\infty} d_k U_k(x)$ is a continuous function of bounded variation on $[-1, 1]$, that $\{U_k(x)\}_{k=0}^{\infty}$ are Chebyshev polynomials of the second kind, and $V_{-1}^{x}(g)$ is total variation of g on $[-1, 1]$ that satisfies the following Lipschitz condition:*

$$\left|V_{-1}^{x}(g) - V_{-1}^{y}(g)\right| \leq \sigma|x - y|, \quad \sigma \in (0, 1). \tag{17}$$

Then, we have, for $x \in (-1, 1)$ and $n \geq 3$,

$$\left|g(x) - g_n(x)\right| = O\left(\frac{1}{n^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right), \tag{18}$$

where $g_n(x) = \sum_{k=0}^{n} d_k U_k(x)$.

Theorem 2. *If $g(x) = \sum_{k=0}^{\infty} d_k A_k(x)$ satisfies the conditions of Theorem 1, then we have, for $x \in (-1, 1)$ and $n \geq 3$,*

$$\left|g(x) - g_n(x)\right| \leq 2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right), \tag{19}$$

for some real constant $\tilde{a}$.

Proof. According to (8) and Theorem 1, we have

$$\begin{aligned}\left|g(x) - g_n(x)\right| &= \left|\sum_{k=n+1}^{\infty} d_k A_k(x)\right| \\ &= \left|\left(1 + \frac{a}{2}\right)\sum_{k=n+1}^{\infty} d_k U_k(x) + \frac{a}{2}\sum_{k=n+1}^{\infty} d_k U_{k-2}(x)\right| \\ &\leq 2\tilde{a}\left|\sum_{k=n+1}^{\infty} d_k U_{k-2}(x)\right| \\ &= 2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right),\end{aligned} \tag{20}$$

where $\tilde{a} = \max\{|1 + (a/2)|, |(a/2)|\}$. □

Theorem 3. *If $u(x)$ is the exact solution to the fractional pantograph delay differential equation (1) and satisfies the conditions of Theorem 2, then we have, for $x \in (-1, 1)$ and $n \geq 3$,*

$$E_n(x) = \left|u(x) - u_n(x)\right| \leq 2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right), \tag{21}$$

for some real constant $\tilde{a}$.

Proof. According to (10), for the numerical solution of equation (1), we have

$$\begin{aligned}u_n(x) &= J_x^{\alpha}\left(c(x)u_n(x)\right) + \sum_{r=1}^{l}\frac{1}{q_r^{\alpha_r}}J_x^{\alpha-\alpha_r}\left(d_r(x)u_n(q_r x)\right) \\ &\quad + J_x^{\alpha} f(x) - F(x).\end{aligned} \tag{22}$$

Subtracting (12) from (10) gives

$$\begin{aligned}\left|u(x) - u_n(x)\right| &\leq \left|J_x^{\alpha}\left(c(x)u(x)\right) - J_x^{\alpha}\left(c(x)u_n(x)\right)\right| \\ &\quad + \sum_{r=1}^{l}\frac{1}{q_r^{\alpha_r}}\left|J_x^{\alpha-\alpha_r}\left(d_r(x)u(q_r x)\right) - J_x^{\alpha-\alpha_r}\left(d_r(x)u_n(q_r x)\right)\right|.\end{aligned} \tag{23}$$

Using Theorem 2, we have

$$\begin{aligned}&\left|J_x^{\alpha}\left(c(x)u(x)\right) - J_x^{\alpha}\left(c(x)u_n(x)\right)\right| \leq 2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right), \\ &\sum_{r=1}^{l}\frac{1}{q_r^{\alpha_r}}\left|J_x^{\alpha-\alpha_r}\left(d_r(x)u(q_r x)\right) - J_x^{\alpha-\alpha_r}\left(d_r(x)u_n(q_r x)\right)\right| \leq \\ &2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1 - x^2\right)^{(3/2)-\sigma}}\right).\end{aligned} \tag{24}$$

As a result,

$$E_n(x) = |u(x) - u_n(x)| \le 2\tilde{a}O\left(\frac{1}{(n-2)^{\sigma}\left(1-x^2\right)^{(3/2)-\sigma}}\right). \quad (25)$$

Since the exact solution of (1) is often unknown for noninteger values α, the following error can help us to get a more accurate numerical solution:

$$E(x) = \left| D^{\alpha}u(x) - c(x)u(x) - \sum_{r=1}^{l} d_r(x) D^{\alpha_r}\left(u(q_r x)\right)\right|. \quad (26)$$

□

4. Numerical Examples

In this section, we show the advantage and high accuracy of the present method by presenting and analyzing five examples. Let N be the number of collocation points, we use maximum absolute error $\varepsilon_N = \mathrm{Max}_{x\in[0,T]}|\tilde{u}_N(x) - u(x)|$ to verify and validate the results where the following example codes are written by Mathematics software version 12.0.

4.1. Example 1. Assume the linear fractional pantograph delay differential equation problem (1) is as follows:

$$D^{\alpha}u(x) = \frac{3}{4}u(x) + u\left(\frac{x}{2}\right) + f(x), \quad 1 < \alpha \le 2, \qquad u(0) = u'(0) = 0, \quad (27)$$

where $z(x) = x^2$ is the exact solution and the function f is obtained by the exact solution at $\alpha = 2$.

The error obtained from the present method is compared with the fractional-order Boubaker polynomials method [12] in Table 1. In continuation, the fractional-order Bernoulli wavelet method [11] and the present method are compared in Table 2. By observing these tables, the high accuracy of the present method is confirmed in comparison with the other two methods. In Figures 1 and 2, graphs of numerical and exact solutions are plotted for different values α and at $T = 1$, $N = 15$, $a = -3.23987 \times 10^{-8}$, and at $T = 5$, $N = 20$, $a = 0.13461$, respectively. In these figures, it can be seen that as the value of α to 2 approaches, the approximate solutions converge to the exact solution.

4.2. Example 2. Assume the neutral fractional pantograph delay differential equation problem (1) is as follows:

$$D^{\alpha}u(x) = -u(x) + 0.1u\left(\frac{4x}{5}\right) + 0.5D^{\alpha}u\left(\frac{4x}{5}\right) + f(x), \quad 0 < \alpha \le 1, \qquad u(0) = 0, \quad (28)$$

where $u(x) = xe^{-x}$ is the exact solution and the function f is obtained as before at $\alpha = 1$.

Table 1: The absolute errors in example 1 at $T = 1$, $\alpha = 2$, and $a = -3.23987 \times 10^{-8}$.

x	[12]	Our method ($N = 15$)
0	7.20e-18	5.81699e-18
0.2	1.60e-17	2.34981e-18
0.4	4.27e-17	4.31820e-19
0.6	8.71e-17	5.32169e-17
0.8	1.49e-16	4.07432e-17
1	2.29e-16	8.26400e-17

Table 2: The absolute errors in example 1 at $T = 2$, $\alpha = 2$, and $a = 1.11350 \times 10^{-9}$.

x	[11]	Our method ($N = 10$)
0	2.71e-17	5.80358e-19
0.4	2.78e-17	2.40380e-17
0.8	0	1.78719e-16
1.2	2.22e-16	3.69657e-16
1.6	3.10e-15	1.58153e-16
2	5.33e-15	3.64813e-16

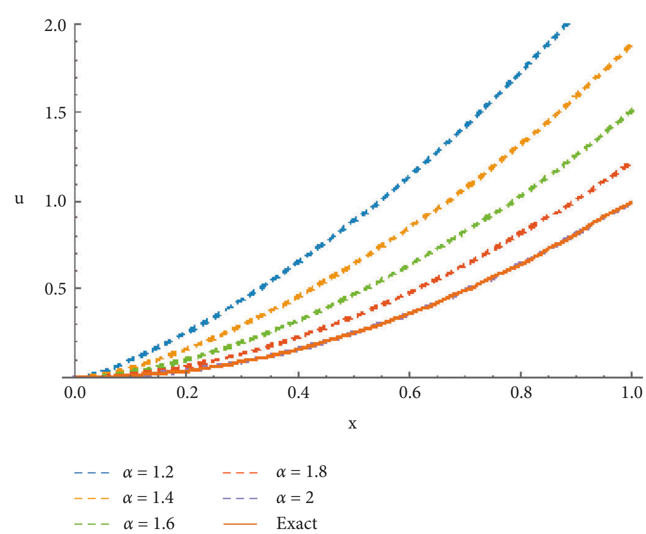


Figure 1: Graphs of numerical and exact solution at different values of α for example 1 at $T = 1$, $N = 15$, and $a = -3.23987 \times 10^{-8}$.

In Table 3, the results of this method are compared with the following methods. Two-step Runge–Kutta of order one method [17], one-leg θ method ($h = 0.01, \theta = 0.8$) [18], variational iteration method ($m = 6$) [19], fractional-order Boubaker polynomials method ($m = 6$) [12], Bernoulli wavelets method ($k = 2, M = 6$) [23], spectral method based on modification of the hat functions ($n = 64$) [10], Taylor wavelets method ($k = 2, M = 6$) [14], and fractional-order Bernoulli wavelet method ($k = 2, M = 6$) [11].

The results of this table show the tangible advantage of the proposed method in comparison with other methods. In Table 4, by increasing the number of collocation points, the maximum error decreases, which also confirms the convergence theorem. In Figure 3, numerical solutions converge

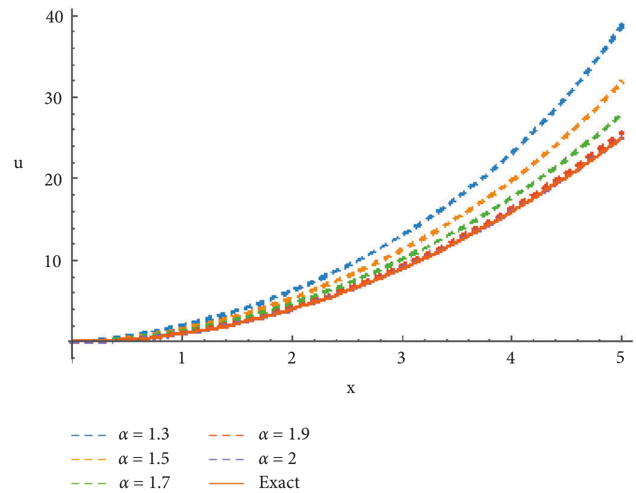


FIGURE 2: Graphs of numerical and exact solution at different values of α for example 1 at $T = 5$, $N = 20$, and $a = 0.13461$.

TABLE 3: The absolute errors in example 2 at $T = 1$, $\alpha = 1$, and $a = 3.8274 \times 10^{-8}$.

x	[17]	[18]	[19]	[12]	[23]	[10]	[14]	[11]	Our method ($N = 15$)
0.1	8.68e-04	4.65e-03	1.30e-03	3.80e-05	1.98e-08	4.69e-07	9.76e-09	1.98e-08	1.47764e-16
0.3	1.90e-03	2.57e-02	2.63e-03	2.81e-05	7.78e-09	5.39e-07	5.67e-09	7.78e-09	1.07469e-16
0.5	2.28e-03	4.43e-02	2.83e-03	2.79e-06	6.34e-05	1.15e-07	7.75e-09	6.34e-05	5.12793e-17
0.7	2.27e-03	5.37e-02	2.39e-03	2.39e-05	4.36e-05	2.27e-07	6.91e-09	4.36e-05	6.41964e-17
0.9	2.03e-03	6.35e-02	1.64e-03	3.52e-05	2.80e-05	3.37e-07	5.57e-09	2.80e-05	1.12318e-16

TABLE 4: Results of ε_N error of $u(x)$ in example 2 at different values N at $T = 1$, $\alpha = 1$, and $a = 3.8274 \times 10^{-8}$.

N	ε_N
5	1.65537e-05
10	1.60136e-12
15	4.44089e-16

to the exact solution of the problem as the value of α to 1 approaches.

4.3. Example 3. Assume the nonlinear fractional pantograph delay differential equation problem (1) is as follows:

$$D^{\alpha}u(x) = 1 - 2u^2\left(\frac{x}{2}\right), \quad 1 < \alpha \leq 2,$$
$$u(0) = 1, \quad u'(0) = 0, \tag{29}$$

where $u(x) = \cos(x)$ is the exact solution at $\alpha = 2$. In Table 5, the results of the proposed method are compared to the following methods: fractional-order Boubaker polynomials method ($m = 6$) [12], Bernoulli wavelet method ($k = 2, M = 3$) [11], spectral method ($n = 40$) [10], Bernoulli wavelets method ($k = 2, M = 9$) [23], and Taylor wavelets method ($k = 2, M = 9$) [14].

This table displays the high accuracy and efficiency of the present method compared to other methods. In Table 6, by increasing the number of collocation points, the maximum error decreases, which confirms the convergence theorem. In Figure 4, the graphs of numerical and exact solutions of the problem are plotted in different values of α at $T = 2$, $N = 15$, and $a = 0.63078$. It can be observed that, as the value of α to 2 approaches, numerical solutions converge to the exact solution. Also, in this figure, the absolute error graph is plotted in $\alpha = 2$.

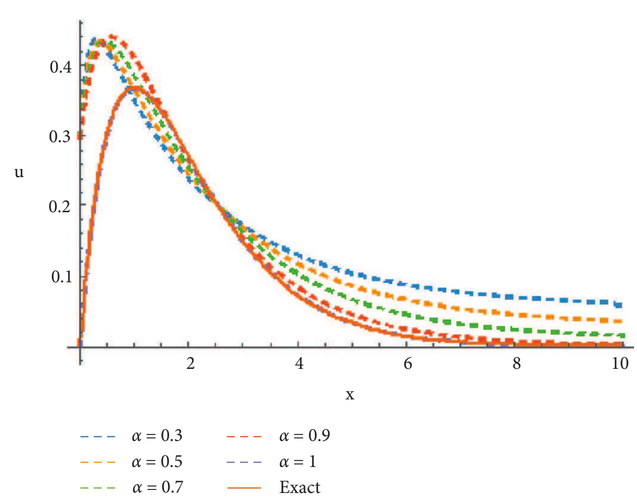


FIGURE 3: Graphs of numerical and exact solution at different values of α for example 2 at $T = 10$, $N = 30$, and $a = 1694.22$.

Table 5: The absolute errors in example 3 at $T = 1$, $\alpha = 2$, and $a = 0.53837$.

x	[12]	[23]	[10]	[14]	[11]	Our method $(N = 15)$
0	2.50e-08	1.62e-11	0	0	5.78e-11	0
0.2	8.45e-09	3.30e-13	6.13e-10	3.12e-16	6.25e-11	0
0.4	7.60e-09	4.17e-12	2.41e-09	5.69e-16	4.90e-11	3.33067e-16
0.6	8.20e-09	1.08e-08	5.29e-09	1.11e-15	3.01e-11	1.11022e-16
0.8	9.72e-09	1.62e-08	9.05e-09	1.89e-15	2.62e-07	1.11022e-16
1	3.28e-08	—	—	—	3.46e-07	2.22045e-16

Table 6: Results of ε_N error of $u(x)$ in example 3 at different values N at $T = 2$, $\alpha = 2$, and $a = 0.63078$.

N	ε_N
5	2.70962e-03
10	3.36929e-08
15	2.52741e-15

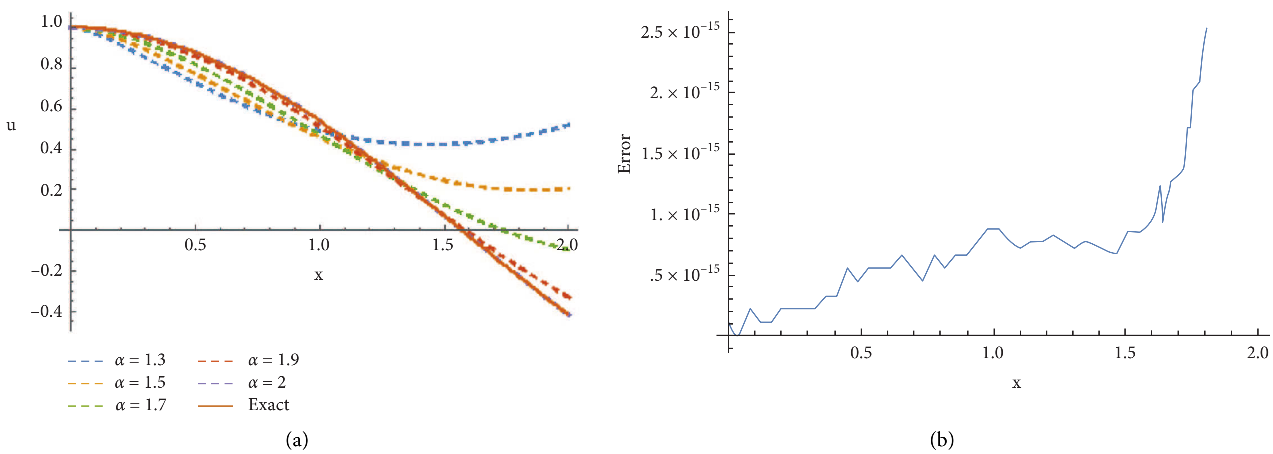


Figure 4: Graphs of numerical and exact solution at different values of α (a) and graphs of absolute error at $\alpha = 2$ (b), for example 3, at $T = 2$, $N = 15$, and $a = 0.63078$.

4.4. Example 4. Assume the neutral fractional delay differential equation problem (1) is as follows:

$$D^{\alpha}u(x) = cu(x) + du(\tau x) + f(x), \quad 0 < \alpha \leq 1, \ 0 \leq x \leq 1,$$
$$u(0) = 0, \tag{30}$$

where $0 < \tau < 1$, arbitrary $c, d \in \mathbb{R}$, $u(x) = \sin(x)$ is the exact solution, and the function f is obtained as before at $\alpha = 1$. In Table 7, the results of the proposed method are compared to the following methods: fractional-order Boubaker polynomials method $(m = 7)$ [12], spectral method $(n = 40)$ [10], Bernoulli wavelets method $(k = 2, M = 6)$ [23], and Taylor wavelets method $(k = 2, M = 8)$ [14].

In Table 8, the maximum error results of the proposed method are compared with discontinuous Galerkin method [42], Taylor wavelets method [14], and Bernoulli wavelets method [23]. In this table, the results of maximum error decrease by increasing the number of collocation points. Observing these two tables shows the high accuracy of the present method compared to other methods. In Figure 5, the graphs of numerical and exact solutions of the problem are plotted in different values of α at $c = -1$, $d = \tau = 0.5$, $T = 1$, $N = 15$, and $a = 4.77550$. It can be observed that, as the value of α to 1 approaches, numerical solutions converge to the exact solution. Also, in this figure, the absolute error graph is plotted in $\alpha = 1$.

4.5. Example 5. Assume the fractional pantograph delay differential equation problem (1) is as follows:

$$D^{\alpha}u(x) = -u(x) + 0.1u\left(\frac{x}{5}\right) + f(x), \quad 0 < \alpha \leq 1, \tag{31}$$
$$u(0) = 1,$$

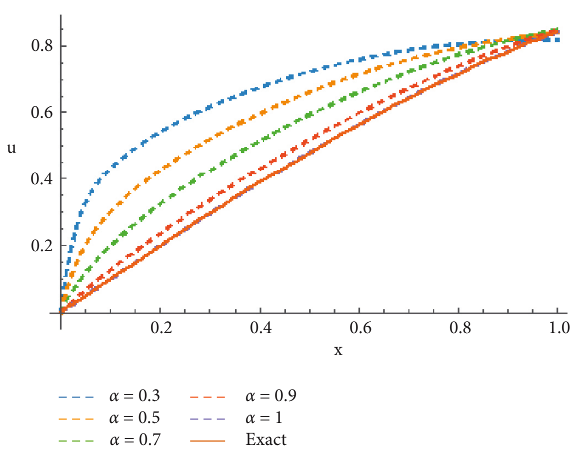
where $u(x) = e^{-x}$ is the exact solution and the function f is obtained by the exact solution at $\alpha = 1$. In Table 9, the results of this method are reported in comparison with the following methods: collocation method [20], fractional-order Boubaker polynomials method $(m = 5)$ [12], Bernoulli wavelets method $(k = 2, M = 5)$ [23], and spectral method

Table 7: The absolute errors in example 4 at $c = -1$, $d = \tau = 0.5$, $T = 1$, $\alpha = 1$, and $a = 4.77550$.

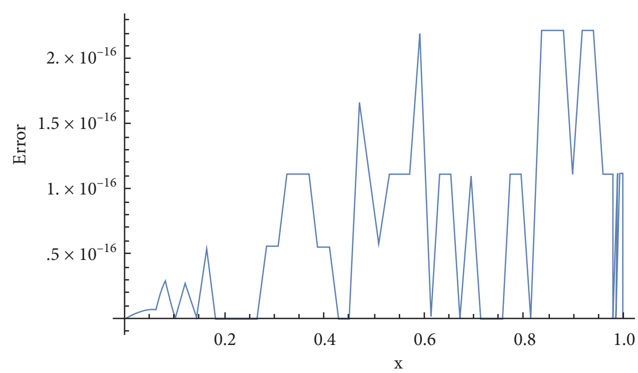
x	[12]	[23]	[10]	[14]	Our method ($N = 15$)
0	7.03e-08	5.93e-09	0	0	2.65201e-21
0.2	1.07e-08	2.27e-10	1.97e-10	3.42e-13	2.77550e-17
0.4	5.49e-09	1.22e-09	5.36e-11	3.41e-13	0
0.6	6.00e-09	5.57e-06	3.29e-10	9.17e-13	0
0.8	1.45e-08	4.56e-06	9.01e-10	8.07e-13	1.11022e-16
1	—	—	—	—	0

Table 8: The absolute errors in example 4 at $c = -1$, $d = \tau = 0.5$, $T = 1$, $\alpha = 1$, and $a = 4.77550$.

	[42]	[23]	[14]		Our method	
	$N = 64$	$k = 2, M = 6$	$k = 2, M = 6$	$N = 5$	$N = 10$	$N = 15$
The maximum error	2.1429e-05	6.1725e-06	2.3022e-09	1.41173e-06	1.39125e-13	2.22045e-16



(a)



(b)

Figure 5: Graphs of numerical and exact solution at different values of α (a) and graphs of absolute error at $\alpha = 1$ (b), for example 4, at $c = -1$, $d = \tau = 0.5$, $T = 1$, $N = 15$, and $a = 4.77550$.

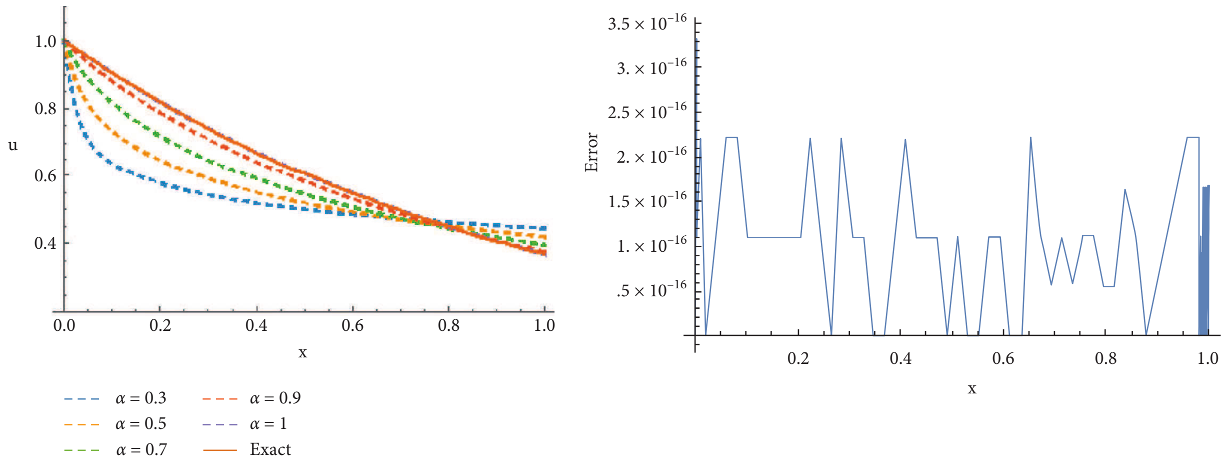
Table 9: The absolute error in example 5 at $c = -1$, $d = \tau = 0.5 T = 1$, $\alpha = 1$, and $a = 3.69727 \times 10^{-8}$.

x	[20]	[12]	[23]	[10]	Our method ($N = 15$)
2^{-2}	1.08e-05	1.11e-08	1.19e-08	1.18e-09	2.22045e-16
2^{-3}	3.81e-05	2.38e-08	8.70e-08	5.39e-10	1.11022e-16
2^{-4}	1.26e-05	1.07e-08	3.32e-07	1.17e-09	2.22045e-16
2^{-5}	4.09e-05	2.56e-08	1.52e-07	5.34e-10	2.22045e-16
2^{-6}	1.20e-05	2.72e-08	1.97e-07	2.27e-09	0

Table 10: Results of ε_N error of $u(x)$ in example 5 at different values N at $T = 1$, $\alpha = 1$, and $a = 3.69727 \times 10^{-8}$.

N	ε_N
5	1.05718e-06
10	1.11910e-13
15	4.44089e-16

$(n = 64)$ [10]. This table displays the high accuracy and efficiency of the present method compared to other methods. In Table 10, the maximum error decreases by increasing the number of collocation points. In Figure 6, the graphs of numerical and exact solutions of the problem are plotted in different values of α at $T = 1$, $N = 15$, and $a = 3.69727 \times 10^{-8}$. It can be observed that, as the value of α



(a) (b)

Figure 6: Graphs of numerical and exact solution at different values of α (a) and graphs of absolute error at $\alpha = 1$ (b), for example 5, at $T = 1$, $N = 15$, and $a = 3.69727 \times 10^{-8}$.

to 1 approaches, numerical solutions converge to the exact solution. Also, in this figure, the absolute error graph is plotted in $\alpha = 1$.

5. Conclusion

As you can see in this study, the present method has more accurate results than the other method. A much smaller number of collocation points are used in the present method. The maximum absolute error in this method is much less than the other method. As shown in this study, the convergence of the present method is guaranteed. This method is easily implemented to solve the linear and nonlinear fractional pantograph delay differential equations. The simplicity of using a-polynomials in fractional derivatives can be one of the advantage points of the present method, which creates less complexity to solve. In future studies, the use of these a-polynomials to numerically solve the fractional nonlinear multipantograph and partial delay differential equations will be analyzed. According to the accurate results obtained from these polynomials in this study, we hope to observe the same accuracy in our future studies [43].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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