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Research Article

Numerical Investigation of the Nonlinear Coupled Fractional Massive Thirring Equation Using Two-Scale Approach

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In this paper, we investigate the numerical solution of the coupled fractional massive Thirring equation with the aid of He's fractional complex transform (FCT). This study plays a significant aspect in the field of quantum physics, weakly nonlinear thrilling waves, and nonlinear optics. The main advantage of FCT is that it converts the fractional differential equation into its traditional parts and is also capable to handle the fractional order, whereas the homotopy perturbation method (HPM) is employed to tackle the nonlinear terms in the coupled fractional massive Thirring equation. An example is illustrated to present the efficiency and validity of the two-scale theory. The solutions are obtained in the form of series with simple and easy computations which confirm that the present approach is good in agreement and is easy to implement for such type of complex systems in science and engineering.

1. Introduction

In this modern era, fractional differential equations (FDEs) have gained importance greatly due to their applications in scientific and engineering disciplines, such as biochemical reaction, chemical engineering, nuclear physics, optical fibers, ecology, fluid dynamics mechanics, and complex systems [1–5]. Most of the time, it is not easy to find the analytical solution of some nonlinear partial differential equations (PDEs) in this modern era [6, 7]. Various numerical and computational approaches have been provided for such models to identify their dynamical behavior and applications. Wu and Baleanu [8] utilized variational iteration method coupled with Laplace transform for the approximate solution of initial and boundary value problems. Lin and Qu [9] proposed the Runge–Kutta spectral method for a nonlinear two-dimensional fractional differential equation with variable coefficients. Liu and Hou [10] obtained the numerical solutions of the space and time-

fractional-coupled burgers equations by a generalized differential transform method. Cheng and Chu [11] employed the Adomian decomposition method for the solution of fractional differential equations. Atangana and Baleanu [12] used the Sumudu transform for the approximate solution of nonlinear fractional Jaulent–Miodek and Whitham–Broer–Kaup equations. Ali et al. [13] obtained the approximate solution of fractional Volterra–Fredholm integrodifferential equations under mixed boundary conditions. A variety of approximate techniques available have been demonstrated for solving fractional order differential equations, such as the truncation method [14], monotone iterative technique [15], Galerkin method [16], homotopy perturbation method [17], Darboux transformation [18], and so on [19], but it is still challenging to find precise solutions to these nonlinear problems. Quantum field equations represent a general form of the Schrodinger wave equation, where the wave function is generalized to an infinite dimensional space of field configurations. The

nonlinear coupled fractional massive Thirring equation has been derived from the study of the Schrodinger equation [20, 21].

In 1958, Thirring [22] introduced a coupled fractional massive Thirring equation (FMTE) which appears in the quantum field theory such as

$$i\left(D_t^\alpha u + u_x\right) + v + u|v|^2 = 0,$$
$$i\left(D_t^\alpha v + v_x\right) + u + v|u|^2 = 0, \tag{1}$$

where α indicates the fractional order of the systems and $D_t^\alpha = \partial^\alpha / \partial t^\alpha \partial t^\alpha$ is He's fractional derivative [23].

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int (s-t)^{n-\alpha-1} \left[u_0(s) - u(s)\right] ds. \tag{2}$$

Here, (1) describes the physical nature of weakly dispersive water waves in nonlinear optics. This model represents a state of dynamic systems that exhibits the chaotic behavior and combines the nonlinear wave propagation with diffusion effects. Recently, Arafa and Hagag [24] used $q-$ HATM and Samdi et. al. [25] used the conformable residual power series method in order to obtain the approximate solution of such nonlinear fractional complex system and showed that numerical results are computationally attractive. Later, many authors [25–27] studied this model with some different phases.

In this work, we use a two-scale approach to investigate the numerical solution of the FMTE. The present approach generates the results in the form of series which converges quickly. The idea of converting the FDEs into their traditional partners reveals that a two-scale approach is very easy to implement and straightforward than those studied in literature. This paper is organized as follows. In Section 2, we present the idea of FCT with the study of scaling transformation. In Section 3, we study the basic idea of HPM which decomposes the nonlinear terms easily and reveals the findings in the sort of a series solution. In Section 4, we implement the present scheme to investigate the numerical solution of the FMTE model. We address some discussion on numerical results and conclusion in Sections 5 and 6, respectively.

2. Fractional Complex Transform

The fractional complex transform is [28, 29].

$$\Delta S = \frac{\Delta t^\alpha}{\Gamma(1+\alpha)}, \tag{3}$$

where Δt is on a small scale and ΔS is on a larger scale. In a small scale, the FMTE reacts as a discontinuous function, particularly at the top of the solitary wave. While on a larger scale, it acts like a simple solitary wave. So, (3) is also called the two-scale transform [30–32]. The same phenomenon, when observed by different scales, leads to different laws. For example, a flow is a continuum and its motion follows the fluid mechanics laws when it is observed on any observable scale, but when we observe the flow in a molecular size, the flow becomes discontinuous. A fractal space with a small scale is converted to an approximation smooth space in a larger scale using the two-scale theory.

3. Fundamental View of the HPM

In this segment, we illustrate a nonlinear functional equation to explain the basic view of the HPM [33, 34].

$$T(u) - g(h) = 0, \quad h \in \Omega, \tag{4}$$

with boundary conditions

$$S\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad h \in \Gamma, \tag{5}$$

where T and S are known as general functional operator and boundary operator, respectively. $g(h)$ is known as the function with Γ as a boundary of the domain Ω. The operator T can be separated into two parts such that T_1 represents a linear operator and T_2 represents a nonlinear operator. As a result, we can express (4) as follows:

$$T_1(u) + T_2(u) - g(h) = 0. \tag{6}$$

Assume a homotopy $v(h, q): \Omega \times (0, 1) \longrightarrow \mathbb{H}$ in such a way that it is appropriate for

$$H(v, q) = (1-q)\left[T_1(v) - T_1(u)_0\right] + q[T_1(v) - T_2(v) - g(h)], \tag{7}$$

or

$$H(v, q) = T_1(v) - T_1(u)_0 + qL(u_0) + q[T_2(v) - g(h)] = 0, \tag{8}$$

where $q \in [0, 1]$ is an embedding parameter and u_0 is an initial guess of (6), which is suitable for the boundary conditions. According to the HPM, we can use q as a slight variable, and suppose that the solution of (8) can be expressed as a power of q:

$$v = v_0 + q v_1 + q^2 v_2 + q^3 v_3 + \cdots = \sum_{i=0}^{\infty} p^i v_i. \tag{9}$$

Considering $p = 1$, the approximate solution of (4) will be obtained as follows:

$$u = \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \cdots = \sum_{i=0}^{\infty} v_i. \tag{10}$$

The series solution in (10) is mostly convergent due to the convergence rate of the series depending on the nonlinear operator T_2.

4. Numerical Application

In this segment, we test the capability of the two-scale approach for the investigation of the numerical solution of the FMTE model. We may rewrite (1) as follows:

$$i\left(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x}\right) + v + u|v|^2 = 0,$$
$$i\left(\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial v}{\partial x}\right) + u + v|u|^2 = 0, \tag{11}$$

with the following initial conditions

$$u(x, 0) = \mu e^{ix},$$
$$v(x, 0) = \eta e^{ix}. \tag{12}$$

The system of (11) can also be written as follows:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x}\right) - iv - iu|v|^2 = 0,$$

$$\left(\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial v}{\partial x}\right) - iu - iv|u|^2 = 0. \tag{13}$$

Now, using (3) in the abovementioned system, we obtain as follows:

$$\frac{\partial u}{\partial S} + \frac{\partial u}{\partial x} - iv - iu|v|^2 = 0,$$

$$\frac{\partial v}{\partial S} + \frac{\partial v}{\partial x} - iu - iv|u|^2 = 0. \tag{14}$$

Here, $|u|^2 = u\overline{u}$ and $|v|^2 = v\overline{v}$, whereas $\overline{u}$ and $\overline{v}$ are the conjugate of u and v in complex system, respectively. Thus,

the HPM for (14) together with initial guess (12) generates the following series.

For $i = 1$, the system of (14) becomes as follows:

$$\frac{\partial u_1}{\partial S} + \frac{\partial u_0}{\partial x} - iv_0 - iu_0 v_0 \overline{v}_0 = 0, \quad u_1(x, 0) = 0,$$

$$\frac{\partial v_1}{\partial S} + \frac{\partial v_0}{\partial x} - iu_0 - iv_0 u_0 \overline{u}_0 = 0, \quad v_1(x, 0) = 0. \tag{15}$$

For $i = 2$, the system of (14) becomes as follows:

$$\frac{\partial u_2}{\partial S} + \frac{\partial u_1}{\partial x} - iv_1 - i(u_0 v_0 \overline{v}_1 + u_0 v_1 \overline{v}_0 + u_1 v_0 \overline{v}_0) = 0,$$

$$u_2(x, 0) = 0,$$

$$\frac{\partial v_2}{\partial S} + \frac{\partial v_1}{\partial x} - iu_1 - i(v_0 u_0 \overline{u}_1 + v_0 u_1 \overline{u}_0 + v_1 u_0 \overline{u}_0) = 0,$$

$$v_2(x, 0) = 0. \tag{16}$$

For $i = 3$, the system of (14) becomes as follows:

$$\frac{\partial u_3}{\partial S} + \frac{\partial u_2}{\partial x} - iv_2 - i(u_0 v_0 \overline{v}_2 + u_0 v_1 \overline{v}_2 + u_0 v_2 \overline{v}_0 + u_1 v_0 \overline{v}_1 + u_1 v_1 \overline{v}_0 + u_2 v_0 \overline{v}_0) = 0, \quad u_3(x, 0) = 0,$$

$$\frac{\partial v_3}{\partial S} + \frac{\partial v_1}{\partial x} - iu_{12} - i(v_0 u_0 \overline{u}_2 + v_0 u_1 \overline{u}_1 + v_0 u_2 \overline{u}_0 + v_1 u_0 \overline{u}_1 + v_1 u_1 \overline{u}_0 + v_2 u_0 \overline{u}_0) = 0, \quad v_3(x, 0) = 0, \tag{17}$$

and so on.

Thus, the solution of these iterative components can be evaluated as follows:

$$u_0(x, 0) = \mu e^{ix},$$

$$v_0(x, 0) = \eta e^{ix},$$

$$u_1(x, S) = ie^{ix}\left[\eta - \mu + \eta^2\mu\right]S,$$

$$v_1(x, S) = ie^{ix}\left[\mu - \eta + \mu^2\eta\right]S,$$

$$u_2(x, S) = i^2 e^{ix}\left[\eta^3 + 2\mu + \eta^4\mu + 2\eta^2\mu(-2 + \mu^2) + \eta(-2 + 3\mu^2)\right]\frac{S^2}{2},$$

$$v_2(x, S) = i^2 e^{ix}\left[\mu^3 + 2\eta + \mu^4\eta + 2\mu^2\eta(-2 + \eta^2) + \mu(-2 + 3\eta^2)\right]\frac{S^2}{2}. \tag{18}$$

Proceeding it in similar practice, we can draw the following series:

$$u(x, S) = u_0(x, 0) + u_1(x, S) + u_2(x, S) + u_3(x, S) + \cdots,$$

$$v(x, S) = v_0(x, 0) + v_1(x, S) + v_2(x, S) + v_3(x, S) + \cdots, \tag{19}$$
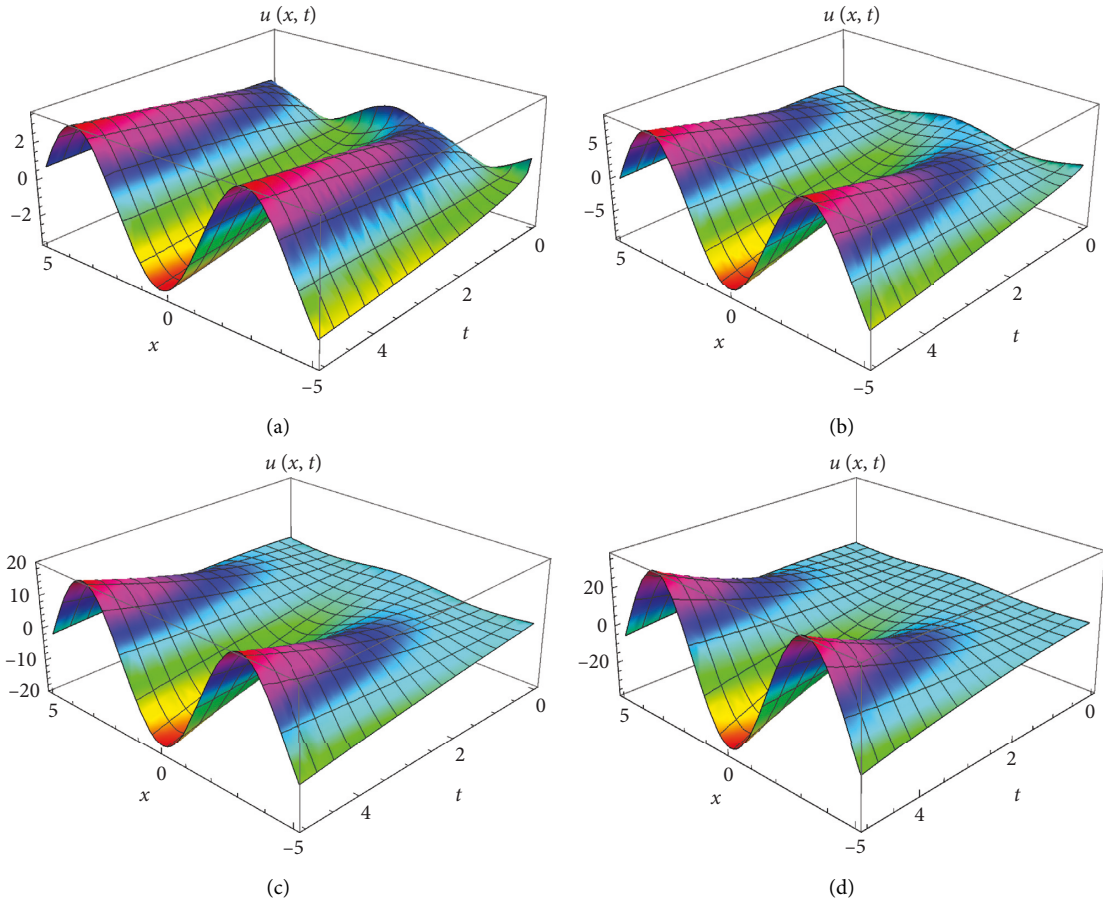
which implies that

(a)

(b)

(c)

(d)

Figure 1: Surfaces plots for real part of $u(x, t)$ for distinct values of α. (a) Numerical simulation of $u(x, t)$ when $\alpha = 0.25$. (b) Numerical simulation of $u(x, t)$ when $\alpha = 0.50$. (c) Numerical simulation of $u(x, t)$ when $\alpha = 0.75$. (d) Numerical simulation of $u(x, t)$ when $\alpha = 1$.

$$u(x, S) = \mu e^{ix} + ie^{ix}\left[\eta - \mu + \eta^2\mu\right]S + i^2 e^{ix}\left[\eta^3 + 2\mu + \eta^4\mu + 2\eta^2\mu\left(-2 + \mu^2\right) + \eta\left(-2 + 3\mu^2\right)\right]\frac{S^2}{2} + \cdots.$$

$$(20)$$

$$v(x, S) = \eta e^{ix} + ie^{ix}\left(\mu - \eta + \mu^2\eta\right)S + i^2 e^{ix}\left[\mu^3 + 2\eta + \mu^4\eta + 2\mu^2\eta\left(-2 + \eta^2\right) + \mu\left(-2 + 3\eta^2\right)\right]\frac{S^2}{2} + \cdots.$$

In other words,

$$u(x, t) = \mu e^{ix} + ie^{ix}\left[\eta - \mu + \eta^2\mu\right]t^\alpha + \frac{i^2}{2}e^{ix}\left[\eta^3 + 2\mu + \eta^4\mu + 2\eta^2\mu\left(-2 + \mu^2\right) + \eta\left(-2 + 3\mu^2\right)\right]\left(\frac{t^\alpha}{\Gamma(1 + \alpha)}\right)^2 + \cdots,$$

$$(21)$$

$$v(x, t) = \eta e^{ix} + ie^{ix}\left[\mu - \eta + \mu^2\eta\right]t^\alpha + \frac{i^2}{2}e^{ix}\left[\mu^3 + 2\eta + \mu^4\eta + 2\mu^2\eta\left(-2 + \eta^2\right) + \mu\left(-2 + 3\eta^2\right)\right]\left(\frac{t^\alpha}{\Gamma(1 + \alpha)}\right)^2 + \cdots.$$

The rest of the components can easily be obtained using the iterative formula.

5. Results and Discussion

In this section, we exhibit the surface behavior of weakly nonlinear dispersive water waves and nonlinear optic problems arising in the quantum field theory. Figure 1(a)–1(d) represents the surface behavior of the real part of $u(x, t)$ when $-5 \leq x \leq 5$ and $0 \leq t \leq 5$ with a step size of 0.5 at different fractional orders when $\alpha = 0.25, 0.50, 0.75$, and 1. In addition, Figures 2(a) and 2(b) show the two-dimensional configuration of the real part for $u(x, t)$ with multiple fractional orders. Similarly, Figure 3(a)–3(d) represents the surface behavior of the imaginary part of $v(x, t)$ when $-5 \leq x \leq 5$ and $0 \leq t \leq 5$ with a step size of 0.5 at different fractional orders when $\alpha = 0.25, 0.50, 0.75$, and 1. In addition, Figures 4(a) and 4(b) show the two-dimensional configuration of the imaginary part for $v(x, t)$ with multiple fractional orders. The comparison in Tables 1 and 2 shows
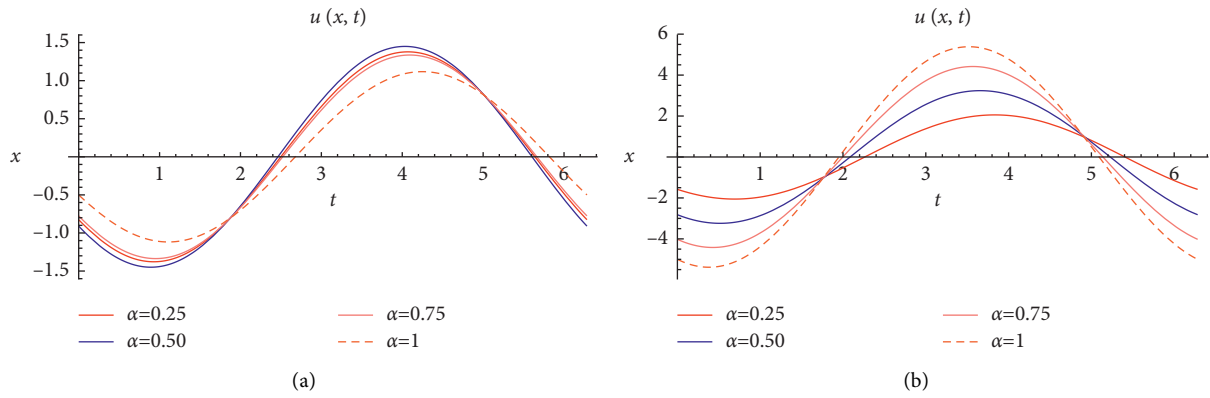
FIGURE 2: Plot of $u(x, t)$ for real part and imaginary part for different values. (a) Real part of $u(x, t)$ for equation (1) at $t = 1$. (b) Imaginary part of $u(x, t)$ for equation (1) at $t = 2$.

FIGURE 3: Surfaces plots for imaginary part of $v(x, t)$ for distinct values of α. (a) Numerical simulation of $v(x, t)$ when $\alpha = 0.25$. (b) Numerical simulation of $v(x, t)$ when $\alpha = 0.50$. (c) Numerical simulation of $v(x, t)$ when $\alpha = 0.75$. (d) Numerical simulation of $v(x, t)$ when $\alpha = 1$.

FIGURE 4: Plot of $v(x, t)$ for real part and imaginary part for different values. (a) Real part of $v(x, t)$ for equation (1) at $t = 1$. (b) Imaginary part o $v(x, t)$ for equation (1) at $t = 1$.
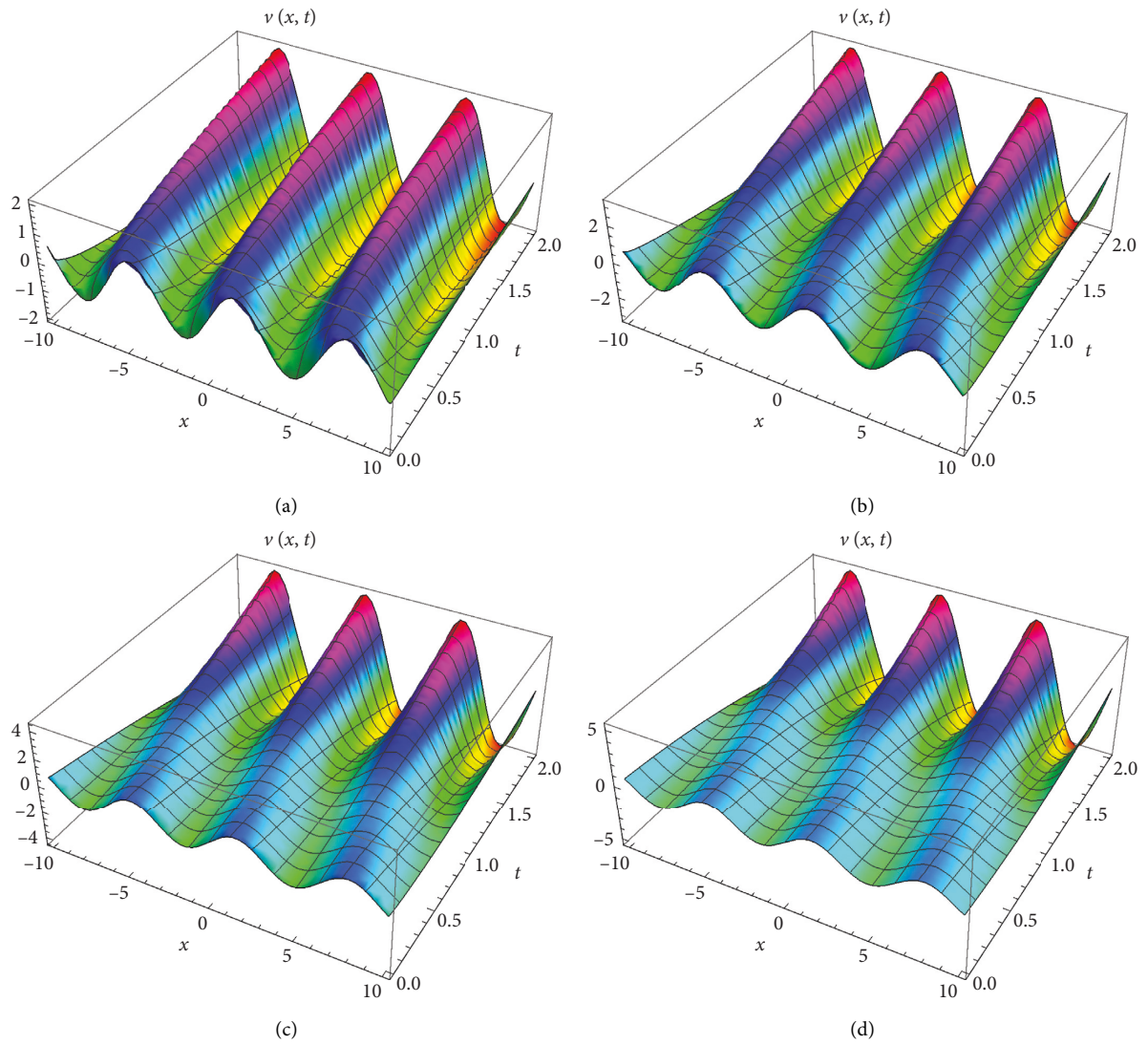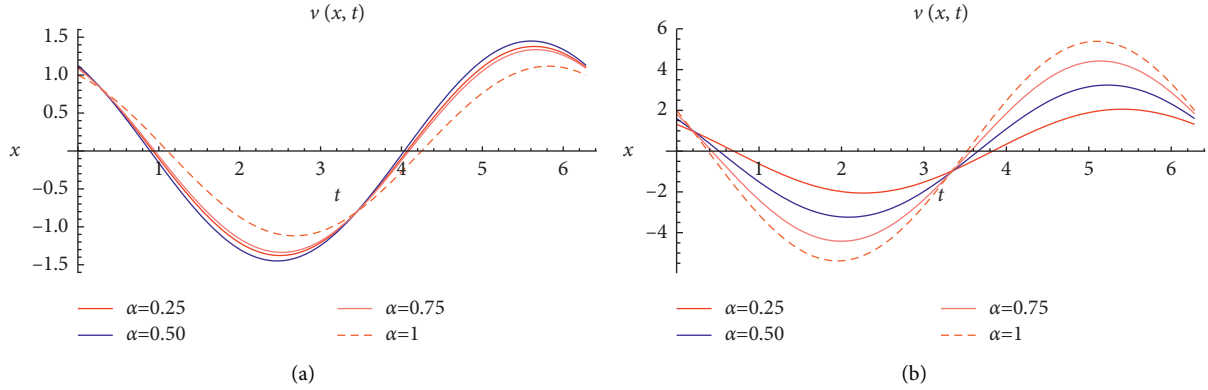
TABLE 1: The two-scale results for $u(x, t)$ and $v(x, t)$ with $\mu = \eta = 1$ and $t = 0.01$ when $\alpha = 0.95$.

Sr. No.	Real part of $u(x, t)$			Imaginary part of $v(x, t)$		
X	[24]	[25]	Present approach	[24]	[25]	Present approach
0.5	0.8768886	0.876879	0.871206	0.4806890	0.4807269	0.490582
1.0	0.5390875	0.5390512	0.529358	0.8422470	0.8422704	0.848204
1.5	0.0692560	0.0692990	0.0579041	0.9975937	0.9975967	0.998157
2.0	−0.417495	−0.417456	−0.427726	0.9086945	0.9086776	0.903726
2.5	−0.802029	−0.802004	−0.808634	0.59733153	0.5972807	0.588031
3.5	−0.990193	−0.99199	−0.99156	0.1396924	0.1396497	0.128366
3.5	−0.935948	−0.935933	−0.931718	−0.352132	−0.352172	−0.362728
4.5	−0.652551	−0.652518	−0.643759	−0.757742	−0.757770	−0.765013
4.5	−0.209386	−0.209344	−0.198185	−0.977831	−0.977840	−0.979996
5.0	0.2850436	0.2850849	0.295912	−0.958512	−0.958499	−0.955042

TABLE 2: Two-scale results for $u(x, t)$ and $v(x, t)$ with $\mu = \eta = 1$ and $t = 0.01$ when $\alpha = 1$.

Sr. No.	Real part of $u(x, t)$			Imaginary part of $v(x, t)$		
x	[24]	[25]	Present approach	[24]	[25]	Present approach
0.5	0.8771018	0.8771022	0.872657	0.4803024	0.4803015	0.488129
1.0	0.5394600	0.5394608	0.531807	0.8420100	0.8420095	0.846748
1.5	0.0697396	0.0697406	0.0607516	0.9975642	0.9975642	0.998053
2.0	−0.417056	−0.417055	−0.425177	0.9088799	0.9088803	0.9050
2.5	−0.801741	−0.801740	−0.807008	0.59767091	0.5976709	0.590371
3.0	0.990104	−0.936132	−0.991255	0.1401298	0.1401308	0.131199
3.5	−0.936104	−0.936105	−0.932808	−0.351719	−0.351718	−0.360095
4.0	−0.652886	−0.652887	−0.645978	−0.757455	−0.757454	−0.763225
4.5	−0.209818	−0.209819	−0.200989	−0.977739	−0.977739	−0.979491
5.0	0.2846207	0.209819	0.293209	−0.958639	−0.958639	−0.955944

that our findings are more efficient than those from the method applied in [24, 25]. The graphical representations of the real and imaginary parts describe the geometrical behavior of the solution in a bell shape. The obtained results seem to be more precise and demonstrate that the two-scale approach is very easy to implement and straightforward for finding the numerical solutions of the FTME model.

6. Conclusion

In this work, we present a two-scale approach for finding the numerical solution of the FMTE model in the field of quantum theory. The key benefit of FCT is that it converts

the fractional terms into their traditional partner to make it easy for the implementation of the HPM and provide the results in the form of series. A physical understanding of the fractional complex transform is elucidated by the two-scale fractal theory. The numerical results obtained by the two-scale approach have been depicted in the form of 2D and 3D figures for different fractional orders. All calculations are made with Wolfram Mathematica software 11.0.1. A comparison of solution graphs and tables demonstrates that this approach offers a straightforward means of obtaining the analytical or approximate solutions for a wide range of problems that might become a universal tool for fractional calculus. The procedure reveals that the present approach is

extremely effective and robust, which can be performed to other nonlinear evolution problems with fractal derivatives in the future demands.

Data Availability

All the data are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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