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Research Article

Application of Transcendental Bernstein Polynomials for Solving Two-Dimensional Fractional Optimal Control Problems

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The aim of this study is to introduce a novel method to solve a class of two-dimensional fractional optimal control problems. Since there are some difficulties solving these problems using analytical methods, thus finding numerical methods to approximate their solution is a challenging topic. In this study, we use transcendental Bernstein series. In fact, for solving the problem, we generalize the Bernstein polynomials to a larger class of functions which can provide more accurate approximate solutions. The convergence theorem is proved. Some examples are solved to demonstrate the validity and applicability of this technique. Comparing the results with other methods, we can find the efficiency and applicability of the scheme.

1. Introduction

Fractional differential equations (FDEs) provide some advantages in the simulation of problems arising in system biology [1], physics [2], hydrology [3], chemistry and biochemistry [1] and finance [4], economic growth models with memory effect [5], and many more. Constructing analytical and numerical methods for solving various types of FDEs has become an ongoing research topic. We mention here the finite element method, the wavelet method [6], the spectral tau method [7], the Gegenbauer spectral method [8], the iterative method [9], the fractional-order wavelet method [6], etc. Also, there are many mathematical and engineering problems which can be modelled in the form of fractional-order differential equations [10–16].

Generally, two approaches exist for solving fractional-order optimal control problems (FOCPs) such as integer order optimal control problems. First, the use of Pontryagin's maximum principle leads to a two-point boundary value problem, while the second approach involves solving the problem directly by discretizing and approximating the state and control functions [17]. To obtain the optimal solution, one can solve the Hamiltonian system of FOCPs. When constructing the Hamiltonian system using integration by parts, the left and right fractional derivatives appear simultaneously, which makes it very difficult to find the exact solution. Hence, some researchers have focused on the numerical solution of FOCPs.

A central difference numerical scheme for FOCPs was given in [18]. In [19], the given optimization problem is reduced to a system of algebraic equations utilizing polynomial basis functions. For the fractional variational problems, an approximate solution is achieved by solving the system. In [20], FOCPs and their solutions were analyzed by means of rational approximation. In [21], the shifted Legendre-tau method was presented to solve a class of initial-boundary value problems for the fractional diffusion equations with variable coefficients on a finite domain.

The Bezier curves method (BCM) is discussed in some papers. In [22, 23], the BCM was utilized for solving delay differential equation (DDE) and optimal control of switched systems (OCSSs). In [24], the BCM was proposed for some linear optimal control systems (LOCPs) with pantograph delays. Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, the BCM is utilized (see [25]). Other uses of the BCM are found in [26].

In the present work, FDEs are utilized as the dynamic constraints, leading to the fractional optimal control problem (FOCP). However, very little work has been done in the area of FOCP, although FOCPs have gained much attention for some of their applications in engineering and physics [27]. In this paper, we solve this problem using the transcendental Bernstein series (TBS). Numerical examples demonstrate the efficiency of the stated technique in solving FOCP.

In what follows, we want to obtain a numerical technique for solving two-dimensional fractional optimal control problems (TFOCPs) of the form

$$\begin{gathered}\min I_1(\zeta,u)=\int_0^{a'}\int_0^{b'}G\left(x,s,\zeta(x,s),u(x,s),\frac{\partial\zeta}{\partial x}(x,s),\frac{\partial\zeta}{\partial s}(x,s)\right)\mathrm{d}x\mathrm{d}s,\\ \text{s.t.}\quad \frac{\partial^2\zeta}{\partial x\,\partial s}(x,s)=c_1\frac{\partial^\nu\zeta}{\partial x^\nu}(x,s)+c_2\frac{\partial^\beta\zeta}{\partial s^\beta}(x,s)+c_3\zeta(x,s)+c_4u(x,s),\\ \zeta(0,s)=g_1(s),\quad \zeta(x,0)=f_1(x),\quad 0<\nu,\beta\le 1,\quad c_1,c_2,c_3,c_4\in R,\quad c_4\neq 0,\quad a',b'\in R,\end{gathered}\tag{1}$$

where G, ζ, and u are smooth functions, and $f_1(x)$ and $g_1(s)$ are the given functions.

The paper is organized as follows. In Section 2, the TBS are defined. Also, the convergence theorem is proved in this section. In Section 3, we discuss several numerical examples and present some comparative results in some tables. Finally, conclusions and future work ideas are given in Section 4.

2. The TBS

Definition 1. The Bezier polynomial of degree n is defined over the interval $[t_{j-1},t_j]$ as follows:

$$x_j(t)=\sum_{r=0}^{n}a_r^jB_{r,n}\left(\frac{t-t_{j-1}}{h}\right),\tag{2}$$

where $h=t_j-t_{j-1}$, and

$$B_{r,n}\left(\frac{t-t_{j-1}}{h}\right):=\binom{n}{r}\frac{1}{h^n}\left(t_j-t\right)^{n-r}\left(t-t_{j-1}\right)^r,\tag{3}$$

is the Bernstein polynomial of degree n over the interval $[t_{j-1},t_j]$, and a_r^j, $r=0,1,\ldots,n$, and they are unknown control points.

BCMs can be used to approximate some problems (see [22, 23]). One may note that the BCM is similar to the TBS.

Definition 2. The Caputo fractional derivatives of order $\nu>0$ of the function $u(x,s)$ are defined as (see [28])

$$\begin{aligned}\frac{\partial^\nu u}{\partial x^\nu}(x,s)&=\frac{1}{\Gamma(n-\nu)}\int_0^s(x-\eta)^{n-\nu-1}\frac{\partial^n u}{\partial\eta^n}(\eta,s)\mathrm{d}\eta,\\ \frac{\partial^\nu u}{\partial s^\nu}(x,s)&=\frac{1}{\Gamma(n-\nu)}\int_0^s(s-\theta)^{n-\nu-1}\frac{\partial^n u}{\partial\theta^n}(x,\theta)\mathrm{d}\theta,\quad n-1<\nu\le n.\end{aligned}\tag{4}$$

Now, the TBS of degree m is defined as

$$B_{i,m}(s)=\binom{m}{i+\gamma_i}s^i(1-s)^{m-i},\quad 0\le i\le m,\quad s\in[0,1].\tag{5}$$

Notice that $\gamma_i=0$, then the TBS coincides with the Bernstein polynomial of degree m.

The expansions of the function $\zeta(x,s)$ and $u(x,s)$ in terms of TBS can be written in the following forms:

$$\begin{aligned}\zeta(x,s)&\simeq\phi_{m_1}(x)^TP'\psi_{m_2}(s),\\ u(x,s)&\simeq\theta_{n_1}(x)^TQ'\Omega_{n_2}(s),\end{aligned}\tag{6}$$

where $P' = [p_{ij}]$, $i = 0, 1, \ldots, m_1$, $j = 0, 1, \ldots, m_2$ and $Q' = [q_{ij}]$, $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$ are unknown matrices, and $m_1, m_2, n_1, n_2 \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. We have

$$\begin{aligned}
\phi_{m_1}(x) &= \begin{bmatrix} 1 & 1 & B_{2,m_1}(x) & \ldots B_{m_1,m_1}(x) \end{bmatrix}^T = A' T_{m_1}(x), \\
\psi_{m_2}(s) &= \begin{bmatrix} 1 & B_{1,m_2}(s) & B_{2,m_1}(s) & \ldots B_{m_2,m_2}(s) \end{bmatrix}^T = B' T_{m_2}(s), \\
\theta_{n_1}(x) &= \begin{bmatrix} 1 & B_{1,n_1}(x) & B_{2,n_1}(x) & \ldots B_{n_1,n_1}(x) \end{bmatrix}^T = C' T_{n_1}(x), \\
\Omega_{n_2}(s) &= \begin{bmatrix} 1 & B_{1,n_2}(s) & B_{2,n_2}(s) & \ldots B_{n_1,n_1}(s) \end{bmatrix}^T = D' T_{n_2}(s),
\end{aligned} \tag{7}$$

with

$$\begin{gathered}
A' = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ a_{20} & a_{21} & a_{22} & \ldots & a_{2m_1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m_1 0} & a_{m_1 1} & a_{m_1 3} & \ldots & a_{m_1 m_1} \end{bmatrix}, \\
B' = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ b_{10} & b_{11} & \ldots & b_{1m_2} \\ b_{20} & b_{21} & \ldots & a_{2m_2} \\ \vdots & \vdots & \ldots & \vdots \\ b_{m_2 0} & b_{m_2 1} & \ldots & b_{m_2 m_2} \end{bmatrix}, \\
C' = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ c_{10} & c_{11} & \ldots & c_{1n_1} \\ c_{20} & c_{21} & \ldots & c_{2n_1} \\ \vdots & \vdots & \ldots & \vdots \\ c_{m_2 0} & c_{m_2 1} & \ldots & c_{n_1 n_1} \end{bmatrix}, \\
D' = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \mathrm{d}_{10} & \mathrm{d}_{11} & \ldots & \mathrm{d}_{1n_2} \\ \mathrm{d}_{20} & \mathrm{d}_{21} & \ldots & \mathrm{d}_{2n_2} \\ \vdots & \vdots & \ldots & \vdots \\ \mathrm{d}_{n_2 0} & \mathrm{d}_{n_2 1} & \ldots & \mathrm{d}_{n_2 n_2} \end{bmatrix}, \\
a_{ij} = \begin{cases} (-1)^{j-i} \binom{m_1}{i} \binom{m_1 - i}{j - i}, & i \le j \\ 0, & i > j, \end{cases} \quad b_{ij} = \begin{cases} (-1)^{j-i} \binom{m_2}{i} \binom{m_2 - i}{j - i}, & i \le j \\ 0, & i > j, \end{cases} \\
c_{ij} = \begin{cases} (-1)^{j-i} \binom{n_1}{i} \binom{n_1 - i}{j - i}, & i \le j \\ 0, & i > j, \end{cases} \quad \mathrm{d}_{ij} = \begin{cases} (-1)^{j-i} \binom{n_2}{i} \binom{n_2 - i}{j - i}, & i \le j \\ 0, & i > j, \end{cases} \\
T_{m_1}(x) = \begin{bmatrix} \phi_0(x) & \phi_1(x) \ldots \phi_{m_1}(x) \end{bmatrix}^T, \quad T_{m_2}(x) = \begin{bmatrix} \psi_0(x) & \psi_1(x) \ldots \psi_{m_2}(x) \end{bmatrix}^T, \\
T_{n_1}(x) = \begin{bmatrix} \theta_0(x) & \theta_1(x) \ldots \theta_{n_1}(x) \end{bmatrix}^T, \quad T_{n_2}(x) = \begin{bmatrix} w_0(x) & w_1(x) \ldots w_{n_2}(x) \end{bmatrix}^T, \\
\phi_i(x) = \begin{cases} x^i & i = 0, 1, \\ x^{i+k_i} & i = 2, 3, \ldots, m_1, \end{cases} \quad \psi_i(s) = \begin{cases} 1 & j = 0, \\ s^{i+q_j} & j = 1, 2, 3, \ldots, m_2, \end{cases} \\
\theta_i(x) = \begin{cases} 1 & i = 0, \\ x^{i+r_i} & i = 1, 2, 3, \ldots, n_1, \end{cases} \quad w_j(s) = \begin{cases} 1 & j = 0, \\ s^{j+l_j} & j = 1, 2, 3, \ldots, n_2, \end{cases}
\end{gathered} \tag{8}$$

where the symbols k_i, q_j, r_i, and l_j represent the control parameters.

2.1. Convergence Analysis

Theorem 1. *Suppose that* $V\colon [0,1]\times[0,1]\leftrightarrow R$ *is* n_1+n_2+1 *times continuously differentiable,* $V\in C^{n_+n_2+1}([0,1]\times[0,1])$, *and* $Y=\langle x^{\beta_i}t^{\gamma_j}, 0\le i\le n_1, 0\le j\le n_2\rangle$. *If* $\theta_{n_1}(x)^T Q\Omega_{n_2}(t)$ *is the good approximation, then we have*

$$\left\|V-\theta_{n_1}^T Q\Omega_{n_2}\right\|_2\le\frac{N(n_1+n_2+2)}{k!(n_1+n_2+1-k)!}, k\in\{0,1,2,\ldots,n_1+n_2+1\}, \tag{9}$$

where

$$N:=\sup\left\{\left|\frac{\partial^{n_1+n_2+1}}{\partial s^{n_1+n_2+1-i}}V(s,t)\right|: s,t\in[0,1],\; i=0,1,\ldots,n_1+n_2+1\right\}. \tag{10}$$

Proof. One may write

$$V(s,t)=q(s,t)+\frac{1}{(n_1+n_2+1)!}\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)^{n_1+n_2+1} V(\lambda_0 s,\lambda_0 t),\quad \lambda_0\in(0,1), \tag{11}$$

$$q(s,t)=\sum_{i=0}^{n_1+n_2}\frac{1}{i!}\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)^{i}V(0,0).$$

Hence,

$$|V(s,t)-q(s,t)|=\left|\frac{1}{(n_1+n_2+1)!}\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)^{n_1+n_2+1}, V(\lambda_0 s,\lambda_0 t)\right|,\quad \lambda_0\in(0,1), \tag{12}$$

which leads to

$$\left\|V-\theta_{n_1}^T Q\Omega_{n_2}\right\|_2=\|V-q\|_2. \tag{13}$$

Therefore,

$$\begin{aligned}\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)^{n_1+n_2+1}, \\ V(\lambda_0 s,\lambda_0 t) &= s^{n_1+n_2+1}\frac{\partial^{n_1+n_2+1}}{\partial s^{n_1+n_2+1}}V(\lambda_0 s,\lambda_0 t)\\ &+ s^{n_1+n_2}t\binom{n_1+n_2+1}{1}\frac{\partial^{n_1+n_2+1}}{\partial s^{n_1+n_2}\partial t}V(\lambda_0 s,\lambda_0 t)\\ &+\vdots\\ &+ st^{n_1+n_2}\binom{n_1+n_2+1}{n_1+n_2}\frac{\partial^{n_1+n_2+1}}{\partial s\,\partial t^{n_1+n_2}}V(\lambda_0 s,\lambda_0 t)\\ &+ t^{n_1+n_2+1}\frac{\partial^{n_1+n_2+1}}{\partial t^{n_1+n_2+1}}V(\lambda_0 s,\lambda_0 t).\end{aligned} \tag{14}$$

Since

$$\binom{n_1+n_2+1}{i}=\binom{n_1+n_2+1}{n_1+n_2+1-i},\quad i=0,1,\ldots,n_1+n_2+1, \tag{15}$$

there exists a value $k\in\{0,1,2,\ldots,n_1+n_2+1\}$ such that

$$\max\left\{\binom{n_1+n_2+1}{i},\quad i=0,1,\ldots,n_1+n_2+1\right\}=\binom{n_1+n_2+1}{k}=\frac{(n_1+n_2+1)!}{k!(n_1+n_2+1-k)!}. \tag{16}$$

Now, one may define

$$N:=\sup\left\{\left|\frac{\partial^{n_1+n_2+1}}{\partial s^{n_1+n_2+1-i}}V(s,t)\right|: s,t\in[0,1],\quad i=0,1,\ldots,n_1+n_2+1\right\}. \tag{17}$$

For finding an upper bound in equation (13), we have

$$
\begin{aligned}
&\int_0^1\int_0^1 s^{2n_1+2n_2+2-2i}t^{2i}\mathrm{d}s\mathrm{d}t = \frac{1}{(2i+1)(2n_1+2n_2+3-2i)}, \quad i = 0,1,\ldots,n_1+n_2+1,\\
&\int_0^1\int_0^1 s^{2n_1+2n_2+1-i}t^{i+1}\mathrm{d}s\mathrm{d}t = \frac{1}{(i+2)(2n_1+2n_2+2-i)}, \quad i = 0,1,\ldots,n_1+n_2,\\
&\int_0^1\int_0^1 s^{2n_1+2n_2-1-i}t^{i+3}\mathrm{d}s\mathrm{d}t = \frac{1}{(i+4)(2n_1+2n_2-i)}, \quad i = 0,1,\ldots,n_1+n_2-1,\\
&\int_0^1\int_0^1 s^{2n_1+2n_2-3-i}t^{i+5}\mathrm{d}s\mathrm{d}t = \frac{1}{(i+6)(2n_1+2n_2-2-i)}, \quad i = 0,1,\ldots,n_1+n_2-2,\\
&\int_0^1\int_0^1 s^{i+1}t^{2n_1+2n_2-i+1}\mathrm{d}s\mathrm{d}t = \frac{1}{(i+2)(2n_1+2n_2+2-i)}, \quad i = 0.
\end{aligned} \tag{18}
$$

Then, we get

$$
\begin{aligned}
&\|V-q\|_2\left(\int_0^1\int_0^1\left|\frac{1}{(n_1+n_2+1)!}\left(s\frac{\partial}{\partial s}+t\frac{\partial}{\partial t}\right)^{n_1+n_2+1}V(\lambda_0 s,\lambda_0 t)\right|^2\mathrm{d}s\mathrm{d}t\right)^{1/2},\\
&= \leq\frac{(n_1+n_2+1)!}{k!(n_1+n_2+1-k)!}\times\frac{M}{(n_1+n_2+1)!}\left(\int_0^1\int_0^1\left|\sum_{i=0}^{n_1+n_2+1}s^{n_1+n_2+1-i}t^i\right|^2\mathrm{d}s\mathrm{d}t\right)^{1/2}\\
&= \psi\left(\sum_{i=0}^{n_1+n_2+1}\frac{1}{(2i+1)(2n_1+2n_2+3-2i)}+\ldots+\sum_{i=0}^{1}\frac{1}{(i+3)(2n_1+2n_2-i+1)}+\frac{1}{4n_1+4n_2+4}\right)^{1/2}\\
&\leq\psi\left((n_1+n_2+2)+(n_1+n_2+1)+\cdots+2+1\right)^{1/2} = \psi\left((n_1+n_2+2)^2\right)^{1/2}\\
&= \psi(n_1+n_2+2), \quad \text{where} \quad \psi = \frac{M}{k(n_1+n_2+1-k))!}.
\end{aligned} \tag{19}
$$

□

3. Numerical Examples

In this section, some examples are approximated using the mentioned method. The results are presented in tables and are compared with the results obtained in [29–31]. Also, the graphs of the approximate solutions are plotted for different values of ν and β.

Example 1. Consider the following problem [29],

$$
\begin{aligned}
&I_1 = \min\int_0^3\int_0^3\left[\left(\frac{\partial\zeta}{\partial s}(x,s)+\zeta(x,s)\right)^2+\zeta^2(x,s)+u^2(x,s)\right]\mathrm{d}x\,\mathrm{d}s,\\
&\text{s.t.}\quad \frac{\partial^2\zeta}{\partial x\,\partial s}(x,s) = -\frac{\partial^\nu\zeta}{\partial x^\nu}(x,s)-3\frac{\partial^\beta\zeta}{\partial s^\beta}(x,s)+0.2\zeta(x,s)+0.3u(x,s),\\
&\zeta(0,s) = e^{-2s}, \quad \zeta(x,0) = e^{-3x}\cos(2\pi x).
\end{aligned} \tag{20}
$$

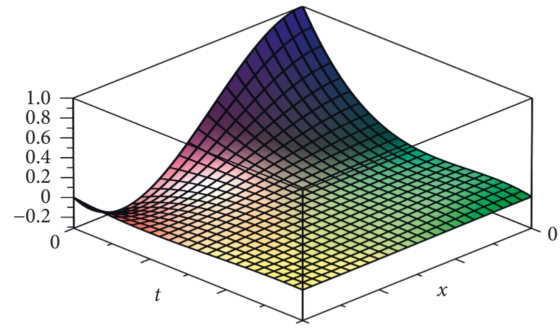

Figure 1: The approximate solution of $\zeta(x, s)$ with $n_1 = n_2 = m_1 = m_2 = 3$ for Example 1.

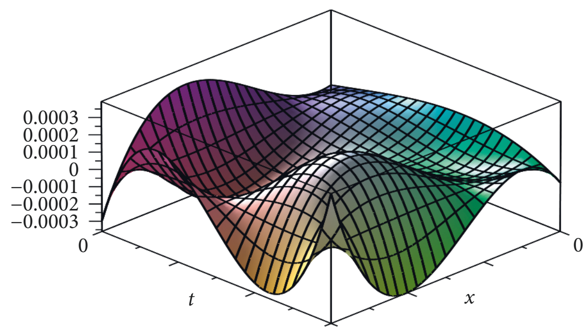

Figure 2: The approximate solution of $u(x, s)$ with $n_1 = n_2 = m_1 = m_2 = 3$ for Example 1.

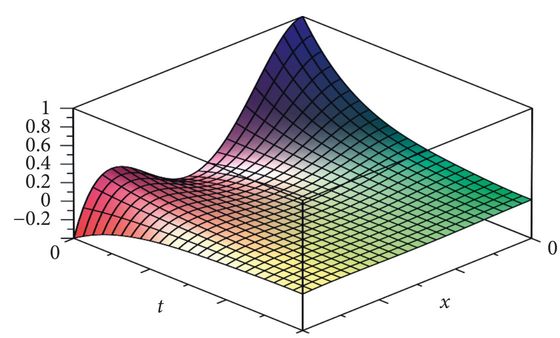

Figure 3: The approximate solution of $\zeta(x, s)$ with $n_1 = n_2 = m_1 = m_2 = 4$ for Example 1.

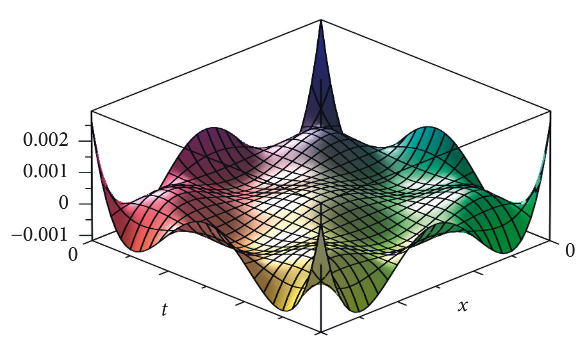

Figure 4: The approximate solution of $u(x, s)$ with $n_1 = n_2 = m_1 = m_2 = 4$ for Example 1.

Table 1: The value of I_1 in some references.

Method in [29]	Value of I_1 [29]
$M = 1, M' = 1$	6.3883
$M = 2, M' = 3$	5.0251
$M = 6, M' = 4$	2.2775
$M = 7, M' = 5$	0.5997
$M = 8, M' = 5$	0.1906
$M = 9, M' = 5$	0.1770
$M = 10, M' = 5$	0.0951
$M = 10, M' = 6$	0.0947
Method in [30]	Value of I_1 [30]
$M = 6, M' = 1$	4.0203
$M = 6, M' = 8$	2.2905
$M = 9, M' = 2$	0.8024
$M = 7, M' = 8$	0.6202
$M = 8, M' = 3$	0.2792
$M = 8, M' = 8$	0.2026
Method in [31]	Value of I_1 [31]
$X = 0.3, T = 0.3$	1.4979
$X = 0.2, T = 0.2$	1.0953
$X = 0.1, T = 0.1$	0.7348
$X = 0.05, T = 0.05$	0.5510
$X = 0.03, T = 0.03$	0.4760
Value of I_1 in method with $n_1 = n_2 = m_1 = m_2 = 3$	0.0355047374520341296
Value of I_1 in method with $n_1 = n_2 = m_1 = m_2 = 4$	0.00199061404713575028

This example is solved using our method with $\gamma_i = 1$ (see Figures 1–4). Applying the method, we have

$$\begin{aligned} I_{1,\text{approx}} &= 0.0355047374520341296, \; n_1 = n_2 = m_1 = m_2 = 3, \nu = \beta = 0.5, \\ I_{1,\text{approx}} &= 0.00199061404713575028, \; n_1 = n_2 = m_1 = m_2 = 4, \nu = \beta = 0.5, \\ I_{1,\text{approx}} &= 0.0219108300379033950, \; n_1 = n_2 = m_1 = m_2 = 4, \nu = \beta = 1. \end{aligned} \tag{21}$$

Note that these results are better than the obtained in [29]. The comparative study can be found in Table 1. We note that in Table 1, parameters M, M' are from the method used in [29]. The CPU time for solving the problem is 4.13 s using a Core i3 laptop.

$$\begin{aligned} \zeta_{\text{approx}}(x, s) = {} & 1 - 1.45240394376420t + 0.194460761429751x^3 \\ & + 0.694697572298107t^2 - 0.106958345297291t^3 \\ & - 0.694479683208798x^2 + 1.04133400405558x^2 t \\ & - 0.505832311967745x^2 t^2 + 0.0784741699699019x^2 t^3 \\ & - 0.297681404809978x^3 t + 0.146001746220140x^3 t^2 \\ & - 0.0227563106212003x^3 t^3, \quad \text{for } n_1 = n_2 = m_1 = m_2 = 3, \nu = 0.5, \beta = 0.5. \end{aligned} \tag{22}$$

Example 2. Next, let us consider the following problem [30]:

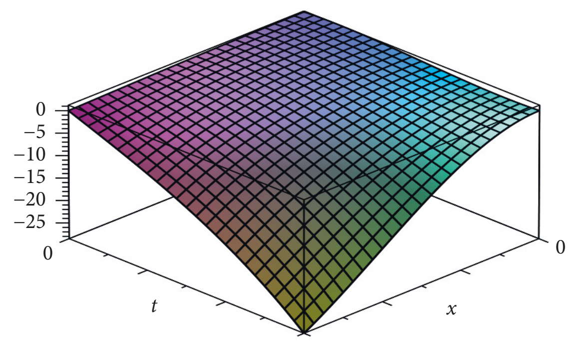

Figure 5: The approximate solution of $\zeta(x,s)$ with $n_1 = n_2 = m_1 = m_2 = 3$ for Example 2.

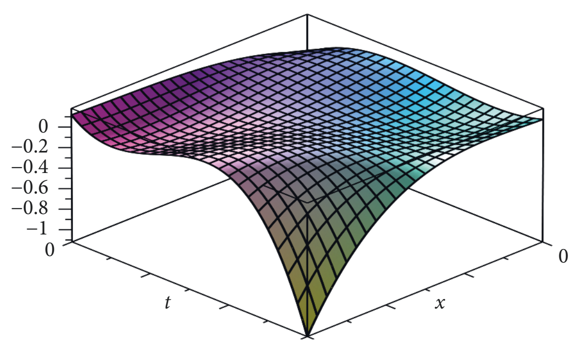

Figure 6: The approximate solution of $u(x,s)$ with $n_1 = n_2 = m_1 = m_2 = 3$ for Example 2.

Table 2: The value of I_1 in some references.

Method in [30]	Value of I_1[30]
$M = 7, M' = 6$	2.03080×10^6
$M = 6, M' = 8$	1.82721×10^6
$M = 7, M' = 7$	1.60773×10^6
$M = 6, M' = 9$	1.54056×10^6
$M = 7, M' = 8$	1.45170×10^6
$M = 7, M' = 9$	1.33534×10^6
$M = 8, M' = 8$	1.30907×10^6
Value of I_1 in method with $n_1 = n_2 = m_1 = m_2 = 3$	$1.00374989684 \times 10^6$

$$\begin{aligned} I_1 &= \min \frac{1}{2}\int_0^5\int_0^5 \left[10^7 (\zeta(x,s) - \sin(x+s))^2 + u^2(x,s)\right] \mathrm{d}x\, \mathrm{d}s, \\ \text{s.t.} \quad & \frac{\partial^2 \zeta}{\partial x\, \partial s}(x,s) = -\frac{\partial^\nu \zeta}{\partial x^\nu}(x,s) - 3\frac{\partial^\beta \zeta}{\partial s^\beta}(x,s) + 0.2\zeta(x,s) + 0.3u(x,s), \\ & \zeta(0,s) = e^{-2s}, \quad \zeta(x,0) = e^{-3x}\cos(2\pi x). \end{aligned} \tag{23}$$

This example is solved using our method with $\gamma_i = 1$. The graphs of the approximate solutions are shown in Figures 5 and 6. Also, we get

$$I_{1,\text{approx}} = 1.00374989684 \times 10^6,\ n_1 = n_2 = m_1 = m_2 = 3, \nu = 0.5, \beta = 0.5. \tag{24}$$

A comparative study is presented in Table 2.

4. Conclusion

In this paper, an efficient algorithm based on TBS was presented to solve 2-D FOCPs. The main idea of the method is to use the TBS as a new approximation instrument. The validity of the stated method based on TBS was verified in Section 3. Furthermore, one may note the 2-D FOCP can be reduced to a system of algebraic equations. Finding the control parameters provides the approximate solution of the problem. The efficiency of the method was confirmed by several numerical examples. Solving the problem based on the Caputo-Hadamard fractional derivative and considering infinite horizon optimal control for nonlinear interconnected large-scale dynamical systems with an application to optimal attitude control will be studied in future.

Abbreviations

FDEs:	Fractional differential equations
BCM:	Bezier curves method
DDE:	Delay differential equation
OCSSs:	Optimal control of switched systems
LOCPs:	Linear optimal control systems
FOCP:	Fractional optimal control problem
TBS:	Transcendental Bernstein series
TFOCPs:	Two-dimensional fractional optimal control problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

F.G., S.M., and S.N. conceptualized the study; F.G. and S.N. curated the data; F.G., S.M., and S.N. did formal analysis; S.N. and S.M. acquired the funding; F.G. and S.N. investigated the study; F.G., S.M., and S.N. developed the methodology; S.N. did project administration; F.G. and S.N. provided the resources; F.G. and S.N. provided software; S.N. supervised the study; F.G. and S.N. validated the study; F.G. and S.N. wrote the original draft; F.G., S.M., and S.N reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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