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Research Article

Dynamics of Multimodal Families of m-Modal Maps

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In this work, we introduce families of multimodal maps based on logistic map, i.e., families of m-modal maps are defined on an interval $I \subset \mathbb{R}$, which is partitioned into non-uniform subdomains, with $m \in \mathbb{N}$. Because the subdomains of the partition are not uniform, each subdomain contains a unimodal map, given by the logistic map, that can have different heights. Therefore, we give the necessary and sufficient conditions for these modal maps present a multimodal family of m-modal maps, i.e., a bifurcation parameter can set a unimodal map, a bimodal map, up to a m-modal map. Some numerical examples are given according to the developed theory. Some numerical examples are given in accordance with the developed theory.

1. Introduction

Many interesting results have been given in discrete dynamical systems, for example, in [1] the authors showed that period three implies whatever period so as a consequence chaos. Results about chaotic properties of non-autonomous discrete dynamical systems have been extensively studied in [2, 3]. Due to chaotic behavior presents ergodicity, sensitivity to initial conditions, transitivity, and not predictable evolution behaviors, chaotic dynamical systems have been considered with great potential in engineering applications. For example, to development pseudo-random number generators [4–6]; some applications of these generators are in video [7, 8], secure communications [9, 10], and cryptographic systems [11–15] due to the close relationship between chaos and cryptography. In [16, 17], a comparison between the properties of chaos and cryptosystems is given and showed that the ergodicity, sensitivity to initial conditions and the bifurcation parameter, deterministic dynamics and complex structure are analogous to confusion, diffusion, pseudo-randomness and algorithm complexity, respectively.

Many proposals of bit generators have been developed with new discrete dynamical systems called maps which are capable of generating chaotic behavior via unidimensional systems, so there is a great interest in developing new chaotic maps for the purpose mentioned above or with the intention to understand the chaotic behavior [18]. Different maps have been used to tackle the aforementioned aims, like unimodal chaotic maps [19, 20], piecewise linear chaotic maps [21], and chaotic maps based on combining more that one chaotic map [10, 22]. To ensure the boundedness of chaotic trajectories, the systems are usually restricted to maps that are mapping from a compact interval into itself, usually the compact interval is $I = [0, 1]$. However, there is no constraint to this interval, for example, in [23] the authors derived analytical expressions for the autocorrelation sequence and power spectral density of chaotic signals generated by one-dimensional continuous piecewise linear maps with three slopes $f\colon [-1, 1] \longrightarrow [-1, 1]$.

Multimodal maps have been studied by Smania [24], who studied the dynamics of the renormalization operator for this kind of maps. Particularly, he developed a combinatory theory for a certain kind of multimodal maps. On the other hand, it is possible to generate multimodal maps based on unimodal maps like logistic map or tent map. Campos-Cantón et al. [25, 26] introduced multimodal maps based on the logistic map, but restricted to a regular partition of the space, i.e., the interval I is divided into uniform subintervals and each subinterval has a critical point that presents the same modal in all subintervals. The consequence of using a regular partition of the space is that the map shape in each interval is the same. In the same spirit that the previous work, mutimodal maps have been introduced based on the tent map [27], where analytical expressions have been derived for the autocorrelation function and the auto-spectral density function of chaotic signals generated by a multimodal skew tent map. In this work we present results on irregular partitions of space, this fact allows us to define different modals in all subintervals and obtain different maps. However, there are some map configurations that do not allow multimodal maps. So in this work we give the necessary and sufficient conditions for these modal maps present a multimodal family of m-modal maps, i.e., a bifurcation parameter can set a unimodal map, a bimodal map, up to a m-modal map.

The useful insights for the study of chaos can be known by analyzing one-dimensional (1D) maps, e.g., the logistic and tent maps. These systems have been extensively studied [28] and implemented experimentally [29–34]. A unimodal map is a continuous 1D function $\mathbb{R} \longrightarrow \mathbb{R}$ with a single critical point c_0, monotonically increasing on one side of c_0 and decreasing on the other. Here, we introduce a class of multimodal maps which is obtained by translating and scaling the logistic maps based on the bi-parametric equation proposed by Verhulst in 1838 [35].

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha_g\left(1 - \frac{N}{\gamma_c}\right)N, \tag{1}$$

where $N(t)$ is the state of the system at time t, α_g is the intrinsic growth rate, and γ_c is the carrying capacity.

Motivate by a large number of applications of the chaotic maps, we study a multimodal maps family based on the logistic map. The importance of having new unidimentional chaotic maps helps to understand the properties of chaotic behavior in discrete dynamical systems and allows applications such as new proposal of bit generators. Now, in this work we study multimodal maps considering irregular partitions of the space. Also the critical points present different local modals giving extra degrees of freedom for sequences generation. The development of this new class of families of chaotic map are based on piecewise continuous function. The map is defined such that only one parameter is considered to takes different values, the others are fixed, thus we will be working with monoparametric families formed with multiple unimodal mappings.

The article is organized as follows: In Section 2, we set the basis of m-modal maps and give its definition. In Section 3, we define a multimodal family of m-modal maps and give the necessary and sufficient conditions for these families to behave as a unimodal map, a bimodal map, up to a m modal map. In Section 4, the conditions under a multimodal family of maps can show transitivity are given and the analysis of the fixed points is presented, as well. In Section 5, numerical examples are given through a trimodal map. Finally, in the last Section 6 conclusions are given.

2. m-Modal Maps

Let (I, d) be the compact metric space $I = [0, 1]$ endowed with the Euclidean metric. It is given the next definition:

Definition 1. Let S and f be an interval and a function, respectively, such that $S \subset I$ and $f \in C(S, I)$. If there exists an $x_c \in S$ such that for all $x \in S$ we have that $f|_{x<x_c}$ is strictly increasing and $f|_{x_c < x}$ is strictly decreasing then f is called a unimodal map on S.

Logistic map and tent map are examples of unimodal maps $(I = S)$. A bimodal map can be defined by considering $I = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$, such that the map f is a unimodal map in each interval $S_1 \subset I$ and $S_2 \subset I$. Therefore, there is a map with two modals on I, called a bimodal map on I. In this way, it is possible to generalize to an arbitrary number of modals to get a m-modal map based on a unimodal map. The interest of this work is to build piecewise functions to generate m-modal maps based on a unimodal map. Particularly in this work, the logistic map is used, which is given as follows:

$$f_\gamma(x) = \gamma(\beta - x)(x - \alpha), \text{ with } x \in S = [\alpha, \beta] \subset I, \tag{2}$$

where x is the state variable; $\gamma \in \mathbb{R}$ is a bifurcation parameter, and α, β are fixed arbitrary parameters with restricted values $0 \le \alpha < \beta \le 1$. The first and second derivatives of f_γ are $f'_\gamma = \gamma(-2x + \alpha + \beta)$ and $f''_\gamma = -2\gamma$, respectively, then f_γ is continuous for all $x \in \mathbb{R}$ and has a local maximum or a local minimum at $x_c = (\alpha + \beta)/2$, $(f'_\gamma(x_c) = 0)$, for $\gamma \neq 0$. Therefore, to get a unimodal map on the interval (α, β) based on the logistic map, it is necessary to consider $\gamma > 0\,(\gamma \in \mathbb{R}^+)$.

In the case of m-modal maps based on a unimodal map, the interval I is divided into multiple subintervals, where each subinterval contains its own maximum, and every subinterval can have the same or different length. Let us consider a partition of the interval I as follows:

Definition 2. Let Π be a partition of I which is determined by a finite sequence $\{\zeta_i\}_{i=0}^{m}$, with $m \in \mathbb{N}\backslash\{1\}$,such that $\zeta_0 = 0 < \zeta_1 < \cdots < \zeta_m = 1$:

$$\Pi = \{S_1 = [\zeta_0, \zeta_1], S_2 = (\zeta_1, \zeta_2], \ldots, S_{m-1} = (\zeta_{m-2}, \zeta_{m-1}], S_m = (\zeta_{m-1}, \zeta_m]\}. \tag{3}$$

If there exists at least a pair of subintervals S_i and S_j, with $i \neq j$, such that $\sigma_i \neq \sigma_j$, where $\sigma_i = d(\zeta_{i-1}, \zeta_i)$ is the diameter of S_i, with $i, j \in \{1, 2, \ldots, m\}$, then the partition on I is called nonuniform. In the contrary case, the partition on I is called uniform.

To illustrate the above definition, it is given the following example.

Example 1. Consider the following sequence of points on I, $\{\zeta_i\}_{i=0}^{3} = \{0, 1/2, 2/3, 1\}$, we have $\zeta_0 = 0 < \zeta_1 = 1/2 < \zeta_2 = 2/3 < \zeta_3 = 1$ which determine the partition of I, $\Pi = \left\{[0, 1/2]t, n(q1/2, 2/3h]_{,}x(72/3, 1]\right\}$, where $\sigma_1 = 1/2 \neq \sigma_2 = 1/6 \neq \sigma_3 = 1/3$, therefore, the partition is nonuniform.

Now, a m-modal map is defined based on m unimodal maps as follows.

Definition 3. Let $g \in C(I, R)$ and $\Pi = \{S_1, \ldots, S_m\}$ be a map and a partition on I with $m \in \mathbb{N}\{1\}$,respectively. If the map g is unimodal on each $S_i \in \Pi$, with $i = 1, \ldots, m$, then it is called m-modal map.

Now, we construct a m-modal map based on the unimodal maps given by Equation (2), which are defined on all subintervals S_i of the partition Π of I. To determine the piecewise function that defines the m-modal map, we start by giving the sequence of points $\{\zeta_i\}_{i=0}^{m} = \{\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_m\}$, which determine a partition Π on I. Therefore, the continuous piecewise function is defined as follows:

$$h_\gamma(x) = \gamma \begin{cases} (\zeta_1 - x)(x - \zeta_0), & \text{for} \quad \zeta_0 \leq x \leq \zeta_1; \\ (\zeta_2 - x)(x - \zeta_1), & \text{for} \quad \zeta_1 < x \leq \zeta_2; \\ \vdots & \vdots \\ (\zeta_m - x)(x - \zeta_{m-1}), & \text{for} \quad \zeta_{m-1} < x \leq \zeta_m. \end{cases} \tag{4}$$

Equation (4) defines a map with m modals, where γ is the bifurcation parameter. Recall that the modal of each subinterval S_i is given by $x_c^{(i)} = (\zeta_{i-1} + \zeta_i)/2$, for $i = 1, \ldots, m$. Then, the local maximum of the map (4) is given by $h(x_c^{(i)}) = (\gamma/4)\sigma_i^2$,in each S_i. In the case that the partition Π on I is uniform, each $\sigma_i = 1/m$ with $i \in \{1, 2, \ldots, m\}$ and the local maximum of each unimodal map in Equation (4) is the same given by $\gamma/(4m^2)$.But if Π is nonuniform, then there exists at least a pair of subintervals S_i with different diameter $\sigma_i \neq \sigma_j$; therefore, for a given value of γ, the m-modal map (4) presents multiple local maximums $h(x_c^{(i)}) \neq h(x_c^{(j)})$. Notice that the local maximums are determined by the parameter γ and the diameter σ_i of each subinterval S_i. The interest is to control the local maximums independently of each σ_i, so a new parameter ρ_i is considered in Equation (4). To avoid the effect of the parameter σ_i to determine the local maximum, then we consider ρ_i/σ_i^2 in each piece of the m-modal map (4). A m-modal map is given as follows:

$$g_\gamma(x) = \gamma \begin{cases} \dfrac{\rho_1}{\sigma_1^2}(\zeta_1 - x)(x - \zeta_0), & \text{for} \quad \zeta_0 \leq x \leq \zeta_1; \\ \dfrac{\rho_2}{\sigma_2^2}(\zeta_2 - x)(x - \zeta_1), & \text{for} \quad \zeta_1 < x \leq \zeta_2; \\ \vdots & \vdots \\ \dfrac{\rho_m}{\sigma_m^2}(\zeta_m - x)(x - \zeta_{m-1}), & \text{for} \quad \zeta_{m-1} < x \leq \zeta_m; \end{cases}, \tag{5}$$

where $\rho_i > 0$, with $i = 1, \ldots, m$, are arbitrary fixed parameters but restricted to $g_\gamma\colon \cup_{i=1}^{k} S_i \longrightarrow \cup_{i=1}^{k} S_i$, for $1 \leq k \leq m$, for all $\gamma \in (0, 4]$.

Lemma 1. ***Maximum ρ -value:*** *Consider a m-modal map given by (5). If $g_4\colon I \longrightarrow I$, then at least one $\rho_i = 1$ and the others $\rho_i \in (0, 1]$, with $i = 1, \ldots, m$.*

Proof. For $\gamma = 4$, the local maximums are given by $\rho_1, \rho_2, \ldots, \rho_m$. Because $g_4\colon I \longrightarrow I$ then at least one $\rho_i = 1$ and none of them is greater than 1. If the ρ's are different $\rho_1 \neq \rho_2 \neq \cdots \neq \rho_m$, then only one $\rho_i = 1$ and the others $\rho_i \in (0, 1)$, with $i = 1, \ldots, m$. If all ρ's are not different then this allows the possibility of having more $\rho's = 1$. However, they can be always ordered as $\rho_{j_1} \leq \rho_{j_2} \leq \cdots \leq \rho_{j_{m-1}} \leq \rho_{j_m}$, with $j_1, j_2, \ldots, j_m = 1, 2, \ldots, m$. This configuration allows to have at least one $\rho_i = 1$, and the others $\rho_i \in (0, 1]$. The proof is completed. □

3. Multimodal Family of m-Modal Maps

One of the main contributions of this work is the definition of the multimodal family of m-modal maps. However, we are also interested in those families of m-modal maps that are able to behave like a unimodal map, a bimodal map, etc., up to a m-modal map according to the bifurcation parameter γ. In other words, these monoparametric families g_γ behave like a k-modal map, with $k = 1, \ldots, m$, if it is possible to choose the appropriate values $\gamma_1, \gamma_2, \ldots, \gamma_m$ for the bifurcation parameter γ which makes g_{γ_k} behaves like a unimodal map, a bimodal map, up to a m-modal map, respectively. The property is that $\cup_{i=1}^{k} S_i \subset I$ will be invariant under each g_{γ_k} such that arbitrary length orbits could be calculated from iterating any initial value $x_0 \in I$, i.e., for each γ_k, the interval $[0, \zeta_k]$ would be invariant under g_{γ_k}.

Definition 4. Let $g_{\gamma_k} \in C(I, R)$ be a continuous m-modal map given by (5), where $m \in N\backslash\{1\}$,if there exist m different values for the bifurcation parameter $\gamma \in \{\gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \gamma_m\}$ such that they fulfill

(i) $g_{\gamma_k}\colon [0, \zeta_k] \longrightarrow [0, \zeta_k]$ is surjective, with $k = 1, \ldots, m$.

The monoparametric family $\left\{g_{\gamma_k}\right\}_{k=1}^{m}$ is called multimodal family of m-modal maps.

The above definition allows us to see that to obtain a multimodal family, parameter values of $\{\zeta_i\}_{i=0}^{m}$, and $\{\rho_i\}_{i=1}^{m}$ should determine the existence of parameters γ_k. So, in the next theorem, necessary and sufficient conditions are given to construct multimodal families by using m-modal maps. Considering that for $\gamma = 4$, g_γ is surjective on I.

Theorem 1 (configurations). *Let $g_\gamma(x)$ be a m-modal map given by equation (5) on I with $\gamma = 4$. There exist control parameter values $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ such that the family $\left\{g_{\gamma_k}\right\}_{k=1}^{m}$ is a multimodal family of m-modal map if and only if:*

$$\frac{\zeta_1}{\rho_1} \neq \frac{\zeta_2}{\max_{i\leq 2}\rho_i} \neq \cdots \neq \frac{\zeta_{m-1}}{\max_{i\leq m-1}\rho_i}, \tag{6}$$

and for every $j \in \{1, 2, \ldots, m-1\}$:

$$\frac{\zeta_j}{\max_{i\leq j}\rho_i} < 1. \tag{7}$$

Proof. ($\Leftarrow$) We have that g_4 is a m-modal map and is sujerctive on I and suppose that conditions given by equations (6) and (7) are fulfilled; it must be proved that there exist m different control parameter values $\gamma_1, \gamma_2, \ldots, \gamma_m$, such that $g_{\gamma_i}: [0, \zeta_i] \longrightarrow [0, \zeta_i]$ are surjective, i.e., $g_{\gamma_i}(I) \subseteq I$, where $i \in \{1, 2, \ldots, m\}$.

As g_4 is a m-modal map function given by (5), then each subinterval of the partition $\Pi = \{S_1, S_2, \ldots, S_m\}$ on I satisfies that $g_4(\zeta_i) = 0$, $g_4(x_c^{(i)}) = \rho_i$ and $\rho_i > 0$, due to they are local maximums. According to Corollary 1, as g_4 is surjective on I there exists at least a local maximum $\rho_i = 1$ and it is also satisfied that every $\rho_i \leq 1$.

The local maximum under g_γ on each subinterval of Π is given by

$$g_\gamma\left(x_c^{(i)}\right) = \rho_i \frac{\gamma}{4}. \tag{8}$$

Thus, when g_γ is restricted to the interval $J_k = [0, \zeta_k]$, then there are k local maximums $\rho_1\gamma/4$, $\rho_2\gamma/4, \cdots, \rho_k\gamma/4$, with $k = 1, \ldots, m$. Since for each local maximum, the parameter value of γ is the same, i.e., it is a constant, then the maximum value of g_γ restricted to the interval $J_k = [0, \zeta_k]$ is

$$\max_{x\in J_k} g_\gamma(x) = \frac{\gamma}{4}\max_{j\leq k}\rho_j. \tag{9}$$

We are finding a parameter value $\gamma_k \in (0, 4)$ such that $\max_{x\in J_k} g_{\gamma_k}(x) = \zeta_k$. Doing some algebra, it results that

$$\gamma_k = 4\frac{\zeta_k}{\max_{j\leq k}\rho_j}. \tag{10}$$

Due to (7), $\gamma_k < 4$, for $k \in \{1, 2, \ldots, m-1\}$, and the local maximum of g_{γ_k} on $J_k = [0, \zeta_k]$ is

$$\max_{x\in J_k} g_{\gamma_k}(x) = \frac{4}{\max_{j\leq k}\rho_j}\frac{\max_{j\leq k}\rho_j}{4}\zeta_k = \zeta_k. \tag{11}$$

As g_{γ_k} is continuous on J_k then it is satisfied g_{γ_k} is surjective on $J_k = [0, \zeta_k]$; besides, as k is arbitrary then there exist m control parameters γ_k such that g_{γ_k}: $[0, \zeta_k] \longrightarrow [0, \zeta_k]$ are surjective, with $k = 1, \ldots, m$. By equations (10) and (6) we have:

$$\gamma_1 = 4\frac{\zeta_1}{\rho_1} \neq \gamma_2 = 4\frac{\zeta_2}{\max_{j\leq 2}\rho_j} \neq \cdots \neq \gamma_{m-1} = 4\frac{\zeta_{m-1}}{\max_{j\leq m-1}\rho_j} \neq \gamma_m = 4, \tag{12}$$

so $\gamma_i \neq \gamma_j$ for all $i \neq j$. Then there are m different values for the bifurcation parameter γ: $\gamma_1 \neq \gamma_2 \neq \cdots \neq \gamma_{m-1} \neq \gamma_m$ and I is invariant under each g_{γ_k} where $(k = 1, 2, \ldots, m)$; therefore the family $\left\{g_{\gamma_k}\right\}$ is a multimodal family of a m-modal maps. ($\Rightarrow$)

Now, we have that $\left\{g_\gamma\right\}_{k=1}^{m}$, for $\gamma \in (0, 4]$, is a multimodal family of m-modal maps given by (5), so g_4 is surjective on I and there exists γ_k such that g_{γ_k}: $[0, \zeta_k] \longrightarrow [0, \zeta_k]$, for $k = 1, \ldots, m$. We need to prove that the existence of γ_k implies that equations (6) and (7) are fulfilled.

We know that for each γ_k there are k local maximums at $g_{\gamma_k}(x_c^{(1)}) = \rho_1\gamma_k/4$, $g_{\gamma_k}(x_c^{(2)}) = \rho_2\gamma_k/4$, $\cdots$, $g_{\gamma_k}(x_c^{(k)}) = \rho_k\gamma_k/4$. Because g_{γ_k} is surjective on J_k, then it is satisfied that $\max_{x\in J_i} g_{\gamma_k}(x) = \zeta_k$. As a consequence $\rho_i\gamma_k/4 \leq \zeta_k\,(i \in \{1, 2, \ldots, k\})$ and $\max_{j\leq k}\rho_j\gamma_k/4 = \zeta_k$. So the control parameter value is given by $\gamma_k = 4\zeta_k/\max_{j\leq k}\rho_j$, then $\gamma_k/4 = \zeta_k/\max_{j\leq k}\rho_j$. We also know that there are m different values of γ: $\gamma_1 \neq \gamma_2 \neq \ldots \neq \gamma_{m-1} \neq \gamma_m$. Since dividing the γ values by four does not affect the inequalities, then the condition (6) is fulfilled.

$$\frac{\zeta_i}{\max_{l\leq i}\rho_l} \neq \frac{\zeta_j}{\max_{l\leq j}\rho_l}. \tag{13}$$

Due to $\gamma_k \in (0, 4)$, for $k = 1, \ldots, m-1$ and $\gamma_m = 4$, then $\gamma_k < \gamma_m$, for $k = 1, \ldots, m-1$. So $\gamma_k/4 < \gamma_m/4 = 1$, therefore

$$\frac{\zeta_j}{\max_{i\leq j}\rho_i} < 1, \tag{14}$$

which fulfills equation (7) and the proof is completed. □

Example 2. As an example, it is selected $\{\zeta_i\}_{i=0}^{3} = \{0, 0.2, 0.6, 1\}$ and $\{\rho_i\}_{i=1}^{3} = \{1, 0.7, 0.6\}$ which determine a nonuniform partition of I and the local maximum for each subinterval $\{S_i\}_{i=1}^{3}$. With the aforementioned parameters, a trimodal map is defined by (5) as follows:

$$g_{\gamma_k}(x) = \gamma_k \begin{cases} 25\,(0.2 - x)\,(x - 0), & \text{for} \quad 0 \leq x \leq 0.2; \\ 4.375\,(0.6 - x)\,(x - 0.2), & \text{for} \quad 0.2 < x \leq 0.6;, \\ 3.75\,(1 - x)\,(x - 0.6), & \text{for} \quad 0.6 < x \leq 1; \end{cases} \tag{15}$$

because $\max_{j\leq 3}\rho_i = 1$, then the trimodal map g_4 is surjective on I. Besides, it is easy to check that the trimodal map given by (15) fulfills the equation (6) and (7) of Theorem 1; thus there exist control parameters that satisfy Definition 4 by calculating them with (10) results in $\gamma_1 = 0.8$, $\gamma_2 = 2.4$, and $\gamma_3 = 4$. Therefore, g_γ given by (15) is a multimodal family comprises by a unimodal map, bimodal map and trimodal map, i.e., $g_{0.8}$: $[0, 0.2] \longrightarrow [0, 0.2]$ defines a unimodal map, $g_{2.4}$: $[0, 0.6] \longrightarrow [0, 0.6]$ defines a bimodal map, and g_4: $I \longrightarrow I$ defines a trimodal map. It is worth mentioning that the trimodal map does not have a chaotic behavior for all the considered values of γ's. The Lyapunov exponents are 0.6931, -1.0312, and 1.1162 for $\gamma_1 = 0.8$, $\gamma_2 = 2.4$, and $\gamma_3 = 4$, respectively. Note that for $\gamma_2 = 2.4$ the Lyapunov exponent is -1.0312, then the orbits converge at a fixed point. The interest is to consider multimodal families with chaotic behavior for all considered values of γ's. Therefore, we need to consider values of γ such that the multimodal families present unstable fixed points.

Corollary 1 (uniform partition). *Let $g_\gamma(x)$ be a m-modal map given by equation (5) on I with $\gamma = 4$, uniform partition Π, and $\rho_1 = \rho_2 = \cdots = \rho_m$. Then there exist control parameter values $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ such that the family $\{g_{\gamma_k}\}_{k=1}^{m}$ is a multimodal family of m m-modal map.*

Proof. This corollary is a direct consequence of Theorem 1. □

4. Dynamics of the Multimodal Family of m-Modal Maps

Once the necessary and sufficient conditions to choose a multimodal family of m-modal maps are given, the next step is to analyze the dynamics of each m-modal map of a family. As the interest of this work is to develop chaotic behavior, the m-modal maps must display chaotic behavior in all of its members of the family. In this section, the necessary and sufficient conditions are given to avoid only regular motion and obtain a multimodal family of chaotic maps.

4.1. Fixed Points Analysis. It is well known that there exist orbits of maps which dynamics are not useful to create chaotic behaviour, so it is better to avoid them. For example, when a map convergences asymptotically to a stable fixed point, it generates a time series without fluctuations, useless to produce chaotic behavior.

To avoid the existence of multimodal families with stable fixed points, it is made a local stability analysis and the result is given in the following theorem.

Theorem 2 (stable fixed points). *Let $g_\gamma(x)$ be a m-modal map given by equation (5) on I if the following interval exists*

$$\Upsilon_j = \left(\frac{\zeta_{j-1} + \zeta_j + 2\sqrt{\zeta_{j-1}\zeta_j}}{\rho_j}, \frac{\zeta_{j-1} + \zeta_j + 2\sqrt{\sigma_j^2 + \zeta_{j-1}\zeta_j}}{\rho_j} \right), \tag{16}$$

and $\gamma_j \in \Upsilon_j$, then two fixed points of system (5) exist in each subinterval $[\zeta_{j-1}, \zeta_j]$ and at least one is stable.

Proof. We start by calculating the fixed points of a multimodal family with m maps $\{g_{\gamma_j}\}_{j=1}^{m}$, then the following equation is solved

$$x_{n+1} = g_{\gamma_j}(x_n) = x_n, \text{ for } x_n \in I, \tag{17}$$

where g_{γ_k} is a multimodal map given by (5) and defined in m intervals $S_1 = [\zeta_0, \zeta_1]$, $S_2 = (\zeta_1, \zeta_2], \ldots, S_m = (\zeta_{m-1}, \zeta_m]$. The fixed points are calculated by the following equation

$$\gamma_j \frac{\rho_j}{\sigma_j^2}\left(\zeta_j - x_n\right)\left(x_n - \zeta_{j-1}\right) = x_n, \text{ for } x_n \in S_j, \tag{18}$$

where $j \in \{1, 2, \ldots, m\}$. Each part of the function must have a maximum of two fixed points; thus an m-modal map must have a maximum of $2m$ fixed points. The fixed points are given as follows:

$$\overline{x}_{L,R}^{(j)} = \frac{\sigma_j^2 - \gamma_j\rho_j\left(\zeta_j + \zeta_{j-1}\right) \pm \sigma_j\sqrt{\gamma_j^2\rho_j^2 - 2\gamma_j\rho_j\left(\zeta_j + \zeta_{j-1}\right) + \sigma_j^2}}{-2\gamma_j\rho_j}. \tag{19}$$

There are two fixed points if the discriminant is positive. $\gamma_k > 0$ for each mutimodal family, thus the fixed points exist if the following inequality is fulfills:

$$\gamma_k > \frac{\zeta_{j-1} + \zeta_j + 2\sqrt{\zeta_{j-1}\zeta_j}}{\rho_j}. \tag{20}$$

Now, we need to prove that at least one of the fixed points is stable. In our case $\overline{x}_R^{(j)}$ is stable if $|g_{\gamma_j}{}'(\overline{x}_R^{(j)})| < 1$ for g_γ defined in S_j which is satisfied when

$$\gamma_k \in \left(\frac{\zeta_{j-1} + \zeta_j + 2\sqrt{\zeta_{j-1}\zeta_j}}{\rho_j}, \frac{\zeta_{j-1} + \zeta_j + 2\sqrt{\sigma_j^2 + \zeta_{j-1}\zeta_j}}{\rho_j} \right). \tag{21}$$

The proof is completed.

The inequality (20) warranties the existence of two fixed points in each interval S_i, with $i = 1, \ldots, m$. However, when g_{γ_j} domain is restricted to the interval $S_1 = [\zeta_0 = 0, \zeta_1]$, there always exists the fixed point $\overline{x}_L^{(1)} = 0$ which is asymptotically stable when the condition $|g_{\gamma_1}{}'(\overline{x}_L^{(1)})| < 1$ is fulfilled.

$$\left|g_{\gamma_1}{}'(0)\right| = \left|-2\gamma_k\frac{\rho_1}{\sigma_1^2}(0) + \gamma_k\frac{\rho_1}{\sigma_1^2}(\zeta_0 + \zeta_1)\right| = \left|\gamma_k\frac{\rho_1}{\zeta_1}\right| < 1. \tag{22}$$

This results in $\gamma_1 \in (0, \zeta_1/\rho_1)$, then $\overline{x}_L^{(1)} = 0$ is asymptotically stable and is unique. When $\gamma_k > \zeta_1/\rho_1$ there exist two fixed points and the second fixed point is given by

$$\overline{x}_R^{(1)} = \frac{\gamma_1\zeta_1\rho_1 - \zeta_1^2}{\gamma_1\rho_1}. \tag{23}$$

This fixed point $\overline{x}_R^{(1)}$ is asymptotically stable always that $\gamma_1 \in (\zeta_1/\rho_1, 3\zeta_1/\rho_1)$ is fulfilled, according to (21). □

Definition 5. The set Υ is given by the union of the intervals $[0, \zeta_1/\rho_1]$ and Υ_j, if Υ_j exists. Thus, Υ is called the stable set of γ parameter values.

Therefore, if $\gamma \in \Upsilon$, then the m-modal map $g_\gamma(x)$ given by (5) on I presents at least one stable fixed point.

Definition 6. The set $\Upsilon^* = [0,4] - \overline{\Upsilon}$ is called the set of γ parameter values such that the m-modal map $g_\gamma(x)$ given by Equation (5) on I presents unstable fixed points, where $\overline{\Upsilon}$ is the closure of Υ.

If any γ_k of the multimodal family $\{g_{\gamma_1}, g_{\gamma_2}, \dots, g_{\gamma_m}\}$ belongs to the set Υ, then the data series from the multimodal map are useless to generate chaotic behaviour.

4.2. Transitivity. One of the important characteristics of dynamical systems is the transitivity property, which describes that given any open subsets $U_1, U_2 \in X$, there exists an $x_0 \in U_1$, and $n > 0$, such that $f^n(x_0) \in U_2$. It is worth to mention that Theorem 1 only states the parameters useful to get a multimodal family of maps, but these maps could be non-transitive. Now, it is important to establish the necessary conditions to obtain the transitivity property for each map of a multimodal family $\{g_{\gamma_k}\}_{k=1}^{m}$ on the interval $J = [0, \zeta_k]$, after the parameter values ζ_i and ρ_i are selected, so the transitivity is warrantied by the next proposition:

Theorem 3. *Let $\{g_{\gamma_k}\}_{k=1}^{m}$ be a multimodal family of m-modal maps given by equation (5) on I, with $\gamma_k \in \Upsilon^*$. Then the multimodal family of m-modal maps is transitive for $\gamma_1, \dots, \gamma_m$. If any of the following cases occur:*

(a)

$$g_{\gamma_k}\left(x_c^{(1)}\right) = g_{\gamma_k}\left(x_c^{(2)}\right) = \cdots = g_{\gamma_k}\left(x_c^{(m)}\right), \text{ with } k = 1, \dots, m. \tag{24}$$

(b)

$$g_{\gamma_k}\left(x_c^{(1)}\right) > g_{\gamma_k}\left(x_c^{(2)}\right) > \dots > g_{\gamma_k}\left(x_c^{(m)}\right), \text{ with } k = 1, \dots, m. \tag{25}$$

and $\overline{x}_L^{(j)} > g^2_{\gamma_k}(x_c^{(j)})$ for all $j = 2, \dots, m$, if there exist $\overline{x}_L^{(j)}$ and $g_{\gamma_k}(x_c^{(j)})$ in the S_j interval.

(c)

$$g_{\gamma_k}\left(x_c^{(1)}\right) < g_{\gamma_k}\left(x_c^{(2)}\right) < \dots < g_{\gamma_k}\left(x_c^{(m)}\right), \text{ with } k = 1, \dots, m. \tag{26}$$

and the following inequality is always preserved

$$g_{\gamma_k}\left(x_c^{(j)}\right) > x_L^{(j+1)}, \text{ for } j = 1, \dots, m-1, \tag{27}$$

where $x_c^{(j)}$ and $\overline{x}_L^{(j)}$ are the critical point and the left fixed point in the j-th interval, with $j = 1, \dots, m$, and $\overline{x}_L^{(j+1)}$ is the fixed point in the S_{j+1} interval.

Proof. See appendix A.

If the m-modal map presents combined modals, i.e., $g_{\gamma_k}(x_c^{(j)}) > g_{\gamma_k}(x_c^{(j+1)}) < g_{\gamma_k}(x_c^{(j+2)})$, with $j = 1, \dots, m-2$, then Theorem 3 (b) and (c) needs to be checked. □

Example 3. Analysis of an example of transitivity of multimodal maps. Specifically, the aim is to design a trimodal family of transitive maps $\{g_{\gamma_i}\}_{i=1}^{3}$, such that Theorem 3 (c) is fulfilled. Therefore, the parameter $m = 3$ determines that the interval I is partitioned in three subintervals. Arbitrarily, we propose the following parameters to generate the partition: $\{\zeta_i\}_{i=0}^{3} = \{0, 1/4, 1/2, 1\}$. We also know that $g_{\gamma_3}(x_c^{(1)}) < g_{\gamma_3}(x_c^{(2)}) < g_{\gamma_3}(x_c^{(3)}) = 1$, then $\rho_3 = 1$ and $\rho_3/\sigma_3^2 = 4$. Therefore, $g_{\gamma_3}(x_n)$ is defined in the subinterval $(0.5, 1]$ as follows:

$$x_{n+1} = \gamma_k \cdot 4(1 - x_n)(x_n - 0.50), \tag{28}$$

the fixed points $\overline{x}_L^{(3)}$ and $\overline{x}_R^{(3)}$ are given by 0.5899 and 0.8475, respectively. Accordingly Theorem 3 c)$0.5899 < \rho_2 < 1$, so we set $\rho_2 = 0.7$ that generates $\rho_2/\sigma_2^2 = 11.2$. Now $g_{\gamma_3}(x_n)$ is defined in the subinterval $(0.25, 0.5]$ as follows:

$$x_{n+1} = \gamma_k \cdot 11.2(0.5 - x_n)(x_n - 0.25), \tag{29}$$

the fixed points $\overline{x}_L^{(2)}$ and $\overline{x}_R^{(2)}$ are given by 0.2779 and 0.4497, respectively. Then $0.2779 < \rho_1 < 0.7$, so $\rho_1 = 0.5$ and $\rho_1/\sigma_2^2 = 8$. Therefore, $g_{\gamma_3}(x_n)$ is defined in the subinterval $[0, 0.25]$ as follows:

$$x_{n+1} = \gamma_k \cdot 8(0.25 - x_n)(x_n - 0), \tag{30}$$

the fixed points $\overline{x}_L^{(2)}$ and $\overline{x}_R^{(2)}$ are given by 0 and 0.2187, respectively. Therefore, $\{\rho_i\}_{i=1}^{3} = \{0.5, 0.7, 1\}$ forms a trimodal map given as follows:

$$x_{n+1} = g_{\gamma_k}(x_n) = \gamma_k \begin{cases} 8(0.25 - x_n)x_n, & \text{for } 0 \le x_n \le 0.25; \\ 11.2(0.50 - x_n)(x_n - 0.25), & \text{for } 0.25 < x_n \le 0.50; \\ 4(1 - x_n)(x_n - 0.50), & \text{for } 0.50 < x_n \le 1. \end{cases} \tag{31}$$

Due to $\max_{i\le 3} \rho_i = 1$, $\zeta_1/\rho_1 = 0.5 < \zeta_2/\max_{i\le 2} \rho_i = 0.7143 < 1$, conditions of Theorem 1 are fulfilled, then there exist control parameter values that form a multimodal family which are $\gamma_1 = 2$, $\gamma_2 = 2.8571$ and $\gamma_3 = 4$, see Figure 1. Meanwhile fixed points and maximum values of this family of multimodal maps g_2, $g_{2.8571}$, and g_4 are shown in Table 1.The cobweb

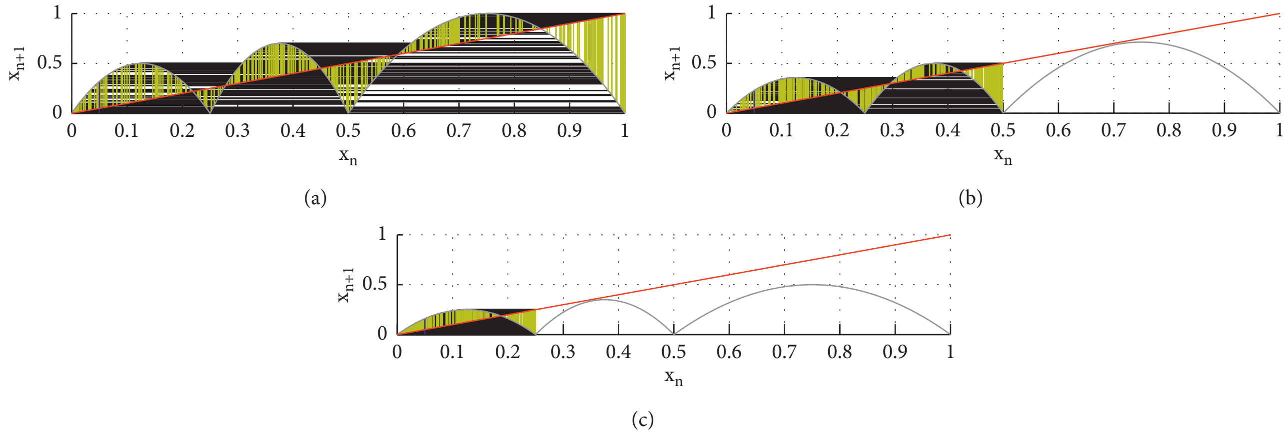

FIGURE 1: Cobweb diagram of the maps: (a)$g_{\gamma_3=4}$: $I \longrightarrow I$, (b)$g_{\gamma_2=2.8571}$: $[0, 0.5] \longrightarrow [0, 0.5]$ and (c)$g_{\gamma_1=2}$: $[0, 0.25] \longrightarrow [0, 0.25]$, using equation (31) with $x_0 = 0.05$ which a long evolution covers completely the intervals I, $[0, \zeta_2]$ and $[0, \zeta_1]$, for γ_3, γ_2, γ_1, respectively.

TABLE 1: Maximum values and fixed points of the multimodal family $\{g_{\gamma_k}\}_{k=1}^{3}$, for $\gamma_1 = 2$, $\gamma_2 = 2.85$, and $\gamma_3 = 4$.

k	$g_{\gamma_k}(x_c^{(1)})$	$g_{\gamma_k}(x_c^{(2)})$	$g_{\gamma_k}(x_c^{(3)})$	$\overline{x}_{L,R}^{(1)}$	$\overline{x}_{L,R}^{(2)}$	$\overline{x}_{L,R}^{(3)}$
1	0.25	0.35	0.5	0, 0.1875	–, –	–, –
2	0.3563	0.5	0.7143	0, 0.2062	0.2949, 0.4238	–, –
3	0.5	0.7	1	0, 0.2187	0.2779, 0.4497	0.5899, 0.8475

diagrams of the maps: $g_{\gamma_3=4}$: $I \longrightarrow I$, see Figure 1(a); $g_{\gamma_2=2.8571}$: $[0, 0.5] \longrightarrow [0, 0.5]$, see Figure 1(b); and $g_{\gamma_1=2}$: $[0, 0.25] \longrightarrow [0, 0.25]$ (see Figure 1(c)), using the (31) with $x_0 = 0.05$. These maps g_{γ_i} are surjective on $J_i = [0, \zeta_i]$, with $i = 1, 2, 3$; besides, these maps fulfill theorem (3). Then maps g_{γ_1}, g_{γ_2} and g_{γ_3} are transitive on $[0, \zeta_1]$, $[0, \zeta_2]$ and I, respectively, as is shown in Figure 1. Figure 2 shows the bifurcation diagram where it is possible to observe that period 2 is a route to chaos because the family is based on the logistic map.

The Lyapunov exponents of these sequences are 0.6932, 1.0127 and 1.3374 for $\gamma_1 = 2$, $\gamma_2 = 2.85$, and $\gamma_3 = 4$, respectively. Then, the family composed of a unimodal map, a bimodal map, and a trimodal map is given by (31) exhibits chaotic behavior.

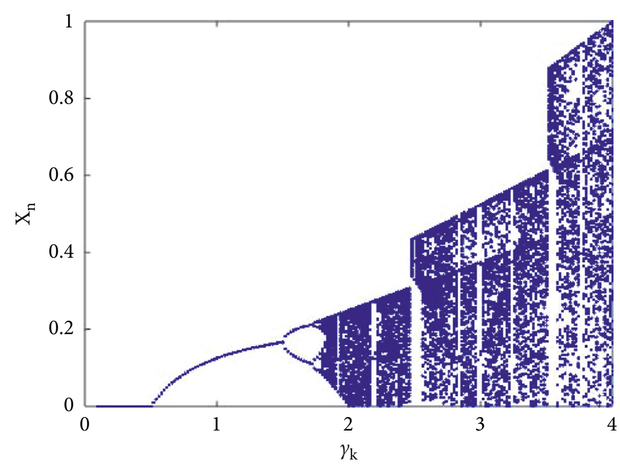

FIGURE 2: Bifurcation diagram of system (31).

Example 4. Analysis of a non-transitivity example, we define a multimodal family with the following parameters: $m = 3$, $\{\zeta_i\}_{i=0}^{3} = \{0, 1/4, 1/2, 1\}$. $\{\rho_i\}_{i=1}^{3} = \{0.26, 0.51, 1\}$, which form a trimodal map given as follows:

$$x_{n+1} = g_{\gamma_k}(x_n) = \gamma_k \begin{cases} 4.16(0.25 - x_n)x_n, & \text{for } 0 \le x_n \le 0.25; \\ 8.16(0.50 - x_n)(x_n - 0.25), & \text{for } 0.25 < x_n \le 0.50; \\ 4(1 - x_n)(x_n - 0.50), & \text{for } 0.50 < x_n \le 1. \end{cases} \quad (32)$$

Notice that the values of $\{\rho_i\}_{i=1}^{3}$ do not fulfill Theorem 3 (c), only the conditions of Theorem 1 are fulfilled, $\max_{i\le 3} \rho_i = 1$, $\zeta_1/\rho_1 = 0.9615 < \zeta_2/\max_{i\le 2} \rho_i = 0.9804 < 1$. Then there exist control parameter values that form a multimodal family which are $\gamma_1 = 3.8462$, $\gamma_2 = 3.9216$ and $\gamma_3 = 4$, see Figure 3(a)–3(c)), respectively. Meanwhile, fixed points and maximum values of this family of multimodal family$\{g_{\gamma_i}\}_{i=1}^{3}$ are shown in Table 2.

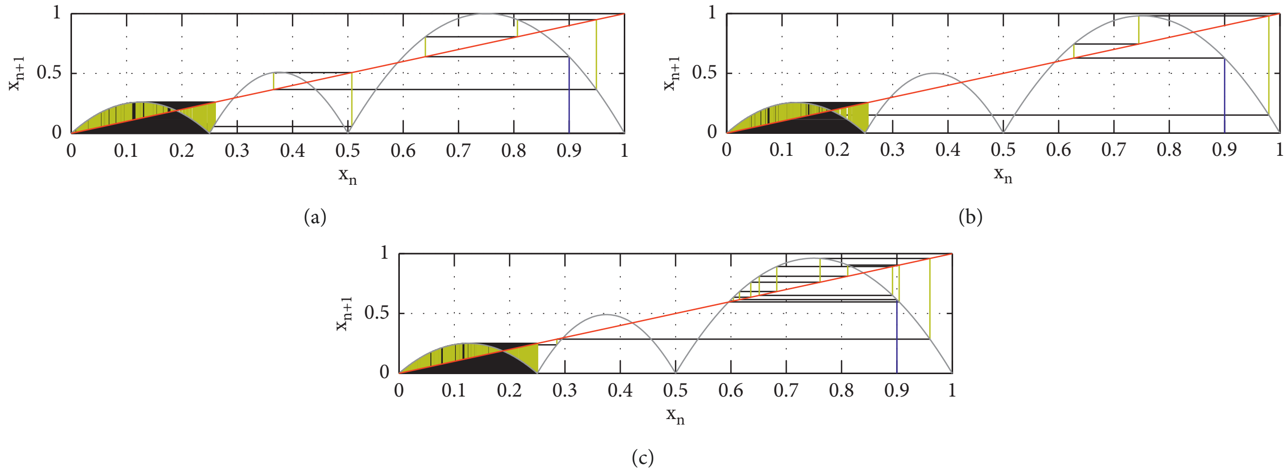

FIGURE 3: Cobweb diagrams of the maps $\{g_{\gamma_i}\}_{i=1}^{3}$ given by equation (32) with $x_0 = 0.9$ for: (a)$\gamma_3 = 4$, (b)$\gamma_2 = 3.9216$ and (c)$\gamma_1 = 3.8462$.

TABLE 2: Maximum values and fixed points of the multimodal family $\{g_{\gamma_k}\}_{k=1}^{3}$, for $\gamma_1 = 3.846$, $\gamma_2 = 3.922$, and $\gamma_3 = 4$.

k	$g_{\gamma_k}(x_c^{(1)})$	$g_{\gamma_k}(x_c^{(2)})$	$g_{\gamma_k}(x_c^{(3)})$	$\overline{x}_{L,R}^{(1)}$	$\overline{x}_{L,R}^{(2)}$	$\overline{x}_{L,R}^{(3)}$
1	0.25	0.49	0.962	0, 0.188	0.296, 0.422	0.596, 0.839
2	0.255	0.5	0.98	0, 0.189	0.295, 0.424	0.593, 0.843
3	0.26	0.51	1	0, 0.19	0.294, 0.426	0.59, 0.848

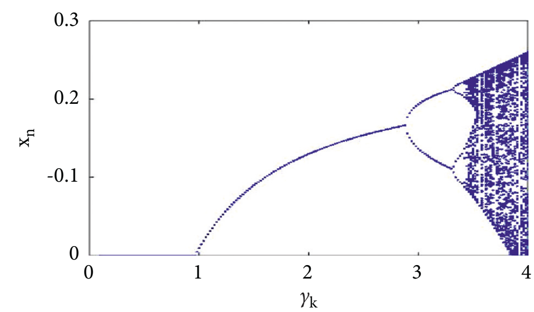

FIGURE 4: Bifurcation diagram of system (32) by using $x_0 = 0.3$.

Each map $\{g_{\gamma_i}\}$ is surjective on $J_i = [0, \zeta_i]$, with $i = 1, 2, 3$; however, all of them are not transitive because Theorem 3 is not fulfilled. Then the maps g_{γ_2}, g_{γ_3} are not transitive on $[0, \zeta_2]$ and I, respectively. Only the map g_{γ_1} is transitive, see Figure 3(c).

Figure 4 shows the bifurcation diagram of system (32), where it is possible to observe that this bifurcation diagram resembles the bifurcation diagram of the logistic map. It is worth mentioning that the initial condition used to compute the bifurcation diagram is $x_0 = 0.3$. Multistability was numerically observed for this system, however, multistability is not addressed in this work because it is off target.

5. Numerical Example of Trimodal Family

Now, in this Section 5 we use the developed theory of this class of discrete maps to create a trimodal family which is comprised by a unimodal map, a bimodal map and a trimodal map. Consequently, a procedure to construct multimodal family of maps is given and a numerical example is also provided as follows: Select m parameters to generate a partition of the interval I, $(\zeta_i <)_{i=0}^{m}$ (where $\zeta_0 = 0 < \zeta_1 \cdots < \zeta_m = 1$). Set the parameter values $\{\rho_i\}_{i=1}^{m}$ taking into account that Theorems 1 and 3 are fulfilled, and select the control parameters γ_k outside the regions where there exists asymptotically stable fixed points.

Example 5. Now, we give an example using the aforementioned steps to design a multimodal family based on a unimodal map, bimodal map, and a trimodal map. So $m = 3$, and we select $\{\zeta\}_{i=0}^{3} = \{0, 1/5, 1/2, 1\}$.

We want $g_{\gamma_3}(x_c^{(1)}) > g_{\gamma_3}(x_c^{(2)}) < g_{\gamma_3}(x_c^{(3)})$, but $g_{\gamma_3}(x_c^{(1)}) > g_{\gamma_3}(x_c^{(3)})$, then $\rho_1 = 1$ and $\rho_1/\sigma_1^2 = 25$. Therefore, we define $g_{\gamma_3}(x_n)$ in the subinterval $(0, 1/5]$ as follows:

$$x_{n+1} = \gamma_k \cdot 25 (1/5 - x_n)(x_n - 0). \tag{33}$$

The fixed points $\overline{x}_L^{(1)}$ and $\overline{x}_R^{(1)}$ are given by 0 and 0.1900, respectively. Because $\rho_1 = 1$ and $\zeta_1 \neq \zeta_2 \neq \zeta_3$, then Theorem1 is fulfilled. We have $g_{\gamma_3}(x_c^{(1)}) > g_{\gamma_3}(x_c^{(2)})$ then $\rho_1 = 1 > \rho_2$, so we set $\rho_2 = 0.91$ that generates $\rho_2/\sigma_2^2 = 10.1111$. Therefore, we have that $g_{\gamma_3}(x_n)$ is defined in the subinterval $(1/5, 1/2]$ as follows:

$$x_{n+1} = \gamma_k \cdot 10.1111 (1/2 - x_n)(x_n - 1/5). \tag{34}$$

The fixed points $\overline{x}_L^{(2)}$ and $\overline{x}_R^{(2)}$ are given by 0.2193, and 0.4560, respectively. Due to $g_4^2(x_c^{(2)}) \notin S_2$ then Theorem 3 b) is fulfilled. Now we want that $\rho_2 < \rho_3 < 1$, so we set $\rho_3 = 0.95$ and $\rho_3/\sigma_3^2 = 3.8$. Therefore, $g_{\gamma_3}(x_n)$ is defined in the subinterval $[1/2, 1]$ as follows:

$$x_{n+1} = \gamma_k \cdot 3.8 (1 - x_n)(x_n - 1/2), \tag{35}$$

the fixed points $\overline{x}_L^{(3)}$ and $\overline{x}_R^{(3)}$ are given by 0.5978 and 0.8364, respectively. Because $g_4^2(x_c^{(3)}) = 0.3420 < x_L^{(3)} = 0.5978$ and $\overline{x}_L^{(3)} < g_4(x_c^{(2)})$ then Theorem 3 b) and c) is fulfilled. So $\{\rho_i\}_{i=1}^{3} = \{1, 0.91, 0.95\}$ is given and determine different

Table 3: The maximum values and fixed points of the multimodal family $\{g_{\gamma_k}\}_{k=1}^{3}$, with $\gamma_1 = 0.8$, $\gamma_2 = 2$, and $\gamma_3 = 4$.

k	$g_{\gamma_k}(x_c^{(1)})$	$g_{\gamma_k}(x_c^{(2)})$	$g_{\gamma_k}(x_c^{(3)})$	$\overline{x}_{L,R}^{(1)}$	$\overline{x}_{L,R}^{(2)}$	$\overline{x}_{L,R}^{(3)}$
1	0.2	0.18	0.19	0, 0.15	–	–
2	0.5	0.45	0.475	0, 0.18	0.2491, 0.4015	–
3	1	0.91	0.95	0, 0.19	0.2193, 0.4560	0.5978, 0.8364

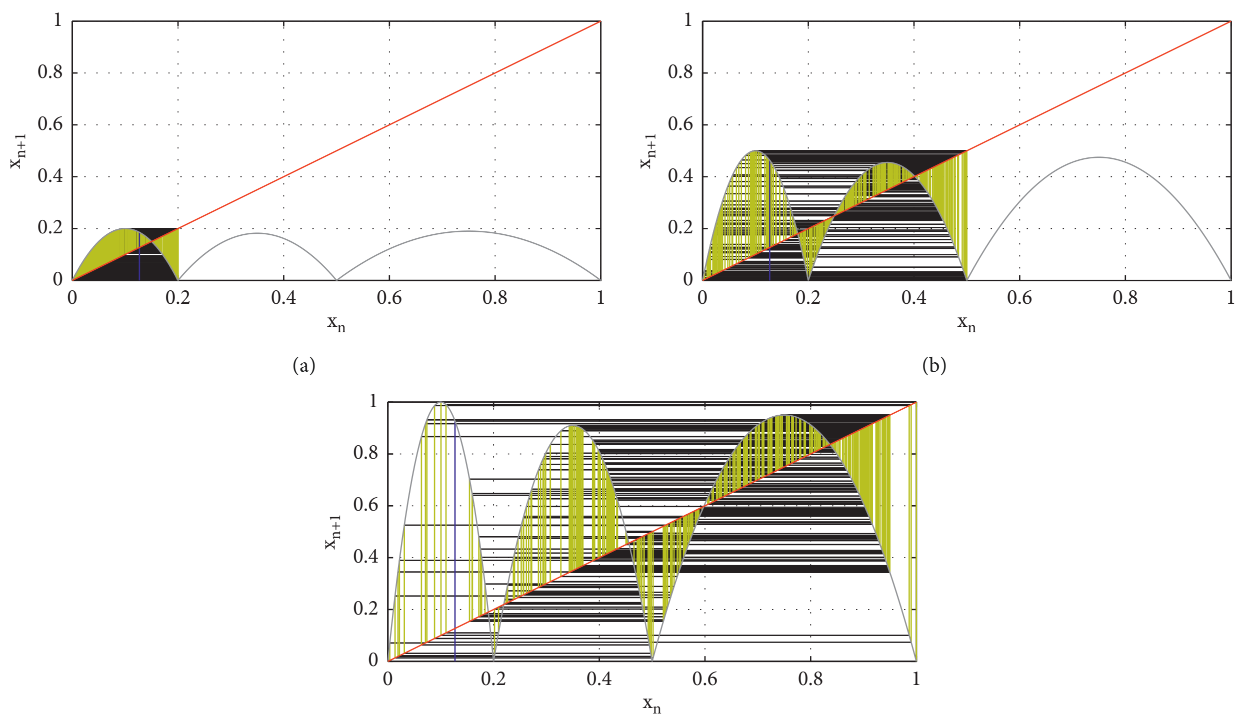

Figure 5: Cobweb Diagrams obtained from trimodal family equation (36) for initial condition $x_0 = 0.127$: (a)$g_{\gamma_1=0.8}(x_n)$; (b)$g_{\gamma_2=2}(x_n)$; and (c)$g_{\gamma_3=4}(x_n)$.

modal values for a not uniform partition on I. Thus $\{\zeta\}_{i=0}^{3}$ and $\{\rho_i\}_{i=1}^{3}$ define a trimodal map as follows:

$$x_{n+1} = g_{\gamma_k}(x_n) = \gamma_k \begin{cases} 25(0.2 - x_n)(x_n - 0), & \text{for } 0 \le x_n \le 0.2; \\ 10.1111(0.5 - x_n)(x_n - 0.2), & \text{for } 0.2 < x_n \le 0.5; \\ 3.8(1 - x_n)(x_n - 0.5), & \text{for } 0.5 < x_n \le 1. \end{cases} \quad (36)$$

The control parameter values for a multimodal family are $\gamma_1 = 4\zeta_1 = 0.8$, $\gamma_2 = 4\zeta_2 = 2$, $\gamma_3 = 4$. In Table 3, the maximum values and fixed points of the multimodal family $\{g_{\gamma_k}\}_{k=1}^{3}$ are given. The control parameter values where the fixed points are asymptotically stable are given as follows

$$\gamma_k \in \Upsilon = (0, 0.2) \cup (0.2, 0.6) \cup (1.4642, 1.7272) \cup (3.0676, 3.4022). \quad (37)$$

Since none of the control parameter values of the multimodal family $\{g_{\gamma_k}\}_{k=1}^{3}$ belongs to the above interval Υ, then there are not asymptotic stable fixed points. Now, we only need to verify transitivity for g_2, because for the unimodal case $g_{0.8}$ the map is transitive due to it is unimodal. For $\gamma = 2$, we have $g_2^2(x_c^{(2)}) = 0.2320 < \overline{x}_L^{(2)} = 0.2500$, therefore Theorem 3 is fulfilled. Figure 5 shows a cobweb diagrams under the map: (a) $x_{n+1} = g_{\gamma_1=0.8}(x_n)$, (b) $x_{n+1} = g_{\gamma_2=2}(x_n)$, and (c) $x_{n+1} = g_{\gamma_3=4}(x_n)$.

Figure 6 shows three time series obtained from trimodal map calculated with (36): (a) $g_{0.8}$, (b) g_2, and (c) g_4, with $x_0 = 0.127$.

The maps of the multimodal family are iterated up to one million data are generated, where each datum (with 16 significant digits) is distributed in the interval $(0, 1)$. Afterwards, the Lyapunov exponents of these sequences are computed. For these numerical computation, more than 2000 initial values are employed for each map and, we do not

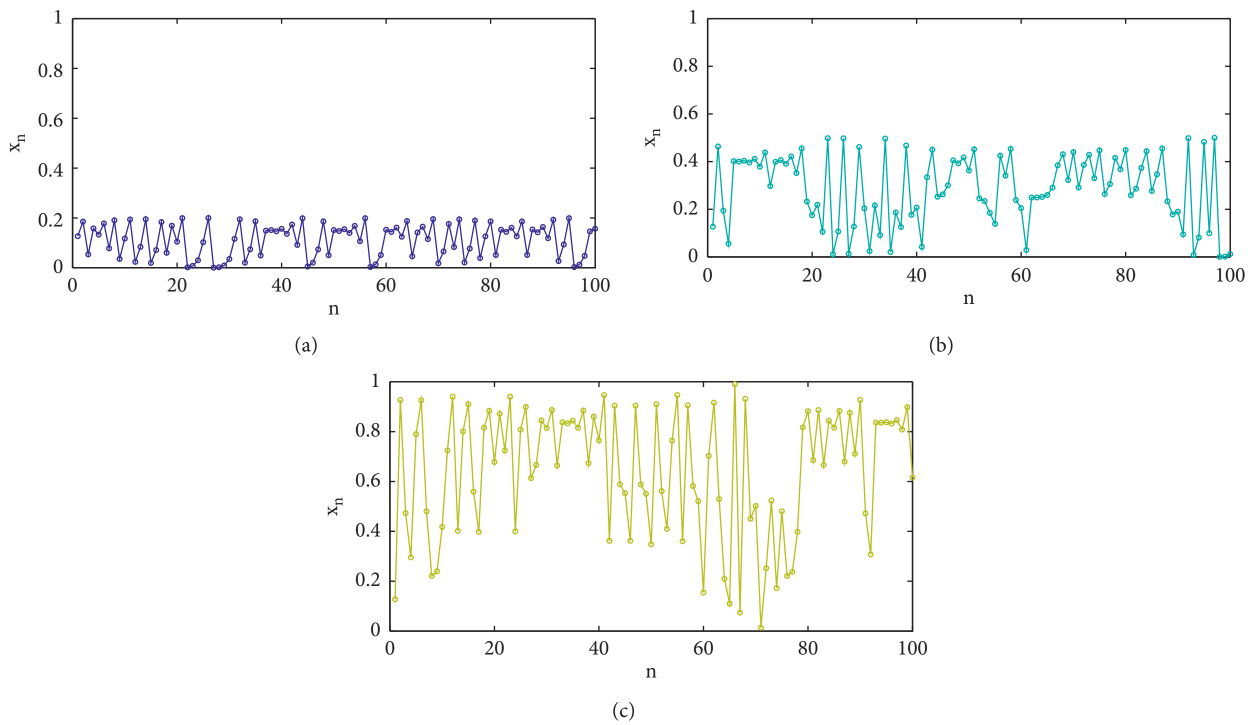

Figure 6: Time series obtained from trimodal map calculated with equation (36): (a)$g_{0.8}$, (b)g_2, and (c)g_4, with $x_0 = 0.127$.

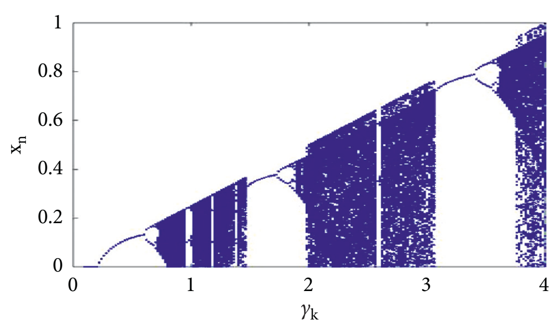

Figure 7: Bifurcation diagram of system (36) by using $x_0 = 0.127$.

find an orbit that has no positive Lyapunov exponent. Then the family comprise by a unimodal map, a bimodal map, and a trimodal map is given by (36) and exhibit chaotic behavior.

Figure 7 shows the bifurcation diagram of the system given by (36), where it is possible to observe that this bifurcation diagram resembles the bifurcation diagram of the logistic map. It is worth mentioning that the initial condition used to compute the bifurcation diagram is $x_0 = 0.127$.

In the literature, others piecewise multimodal maps have been reported that show regular partition intervals and use the same height for all modals. For example, in reference [25] a multimodal map and its basin of attraction were presented, where all modals have the same height and only numerical results were presented. Also in [27] the authors derived analytic expressions for the autocorrelation function and the auto-spectral density function of chaotic signals generated by a multimodal skew tent map and all modals with the same height.

6. Conclusions

In this work, the concept of multimodal family of m-modal maps based on the logistic map was introduced, we also gave the necessary and sufficient conditions to build a multimodal family in regular and irregular intervals. These families can display a unimodal map, bimodal map, up to m-modal map. The generation of a multimodal family of m-modal maps is warranty by means of the transitivity property. Also the stability of the fixed points was analyzed.

This work could be continued in applications about pseudo-random bit generators to use them in the development of cryptographic systems.

Appendix

A. Transitivity

Proof (a). $\Longrightarrow$We need to prove that any open subsets $U_1, U_2 \in J_k$, $k = 1, \ldots, m$, there exists an $x_0 \in U_1$, and $n > 0$, such that $g^n_{\gamma_k}(x_0) \in U_2$.

For the case $k = 1$, we have $g_{\gamma_1}: J_1 \longrightarrow J_1$, with $J_1 = [0, \zeta_1]$ and the critical point $x_c^{(1)} \in J_1$ divides this interval in two intervals $J_{21} = [0, x_c^{(1)}]$ and $J_{22} = [x_c^{(1)}, \zeta_1]$. Therefore, we have $g_{\gamma_1}: J_{21} \longrightarrow J_1$ and $g_{\gamma_1}: J_{22} \longrightarrow J_1$. Then there are two points τ_{21} and τ_{22}, $g^2_{\gamma_1}(\tau_{21}) = g^2_{\gamma_1}(\tau_{22}) = \zeta_1$, that divide the intervals J_{21} and J_{22}, respectively, such that $g^2_{\gamma_1}: J_{31} \longrightarrow J_1$, $g^2_{\gamma_1}: J_{32} \longrightarrow J_1$, $g^2_{\gamma_1}: J_{33} \longrightarrow J_1$, and $g^2_{\gamma_1}: J_{34} \longrightarrow J_1$, with $J_{31} = [0, \tau_{21}]$, $J_{32} = [\tau_{21}, x_c^{(1)}]$, $J_{34} = [x_c^{(1)}, \tau_{22}]$, and $J_{34} = [\tau_{22}, \zeta_1]$.

The intervals J_{31}, J_{32}, J_{33} and J_{34} contain points $\tau_{31}, \tau_{32}, \tau_{33}$, and τ_{34}, respectively, such that $g^3_{\gamma_1}(\tau_{31}) = g^3_{\gamma_1}(\tau_{32}) = g^3_{\gamma_1}(\tau_{33}) = g^3_{\gamma_1}(\tau_{34}) = \zeta_1$, and each interval J_{3i}, with $i = 1, \ldots, 4$, is divided into two, generating eight subintervals $J_{41}, J_{42}, \ldots, J_{47}, J_{48}$. Now we have that each of these intervals are mapped by $g^3_{\gamma_1}: J_{4i} \longrightarrow J_1$, with $i = 1, \ldots, 2^3$.

Notice that the points τ's always exit in the intervals J's. We can continue up to any of the intervals $J_{(n+1)1}, J_{(n+1)2}, \ldots, J_{(n+1)(2^n-1)}, J_{(n+1)(2^n)}$ is contained in U_1. Suppose that the interval $J_{(n+1)i}$ is contained in U_1, then this interval $J_{(n+1)i}$ is mapped to the whole interval J_1, $g^n_{\gamma_1}: J_{(n+1)i} \longrightarrow J_1$, with $i = 1, \ldots, 2^n$. This implies that there exists an $x_0 \in U_1$, and $n > 0$, such that $g^n_{\gamma_1}(x_0) \in U_2$. The dynamical system g_{γ_1} is transitive.

For the case $k = 2$, there are two critical points in $J_2 = [0, \zeta_2] = S_i \cup S_2$, i.e., $x_c^{(1)} \in S_1 = [0, \zeta_1]$ and $x_c^{(2)} \in S_2 = (\zeta_1, \zeta_2]$, and $g_{\gamma_2}: J_2 \longrightarrow J_2$. The critical points $x_c^{(1)}$ and $x_c^{(2)}$ divide these intervals into intervals $J_{21} = [0, x_c^{(1)}]$, $J_{22} = [x_c^{(1)}, \zeta_1]$, $J_{23} = [\zeta_1, x_c^{(2)}]$, and $J_{24} = [x_c^{(2)}, \zeta_2]$, such that $g_{\gamma_2}: J_{2i} \longrightarrow J_2$, with $i = 1, \ldots, 4$, then there are four points τ_{2i}, with $i = 1, \ldots, 4$ that divide the intervals J_{2i}, respectively, such that $g^2_{\gamma_2}: J_{3i} \longrightarrow J_1$, with $i = 1, \ldots, 8$, with $J_{31} = [0, \tau_{21}]$, $J_{32} = [\tau_{21}, x_c^{(1)}]$, $J_{34} = [x_c^{(1)}, \tau_{22}]$, $J_{34} = [\tau_{22}, \zeta_1]$. $J_{35} = [\zeta_1, \tau_{23}]$, $J_{36} = [\tau_{23}, x_c^{(2)}]$, $J_{37} = [x_c^{(2)}, \tau_{24}]$, $J_{38} = [\tau_{24}, \zeta_2]$.

Now, the intervals J_{3i}, with $i = 1, \ldots, 8$, contain points τ_{3i}, with $i = 1, \ldots, 8$, respectively, such that each interval is divided into two, generating sixteen subintervals J_{4i}, with $i = 1, \ldots, 16$. Now we have that each of these intervals maps $g^3_{\gamma_1}: J_{4i} \longrightarrow J_1$, with $i = 1, \ldots, 2^4$.

Notice that the points τ's always exist in the intervals J's. We can continue up to any of the intervals $J_{(n+1)1}, J_{(n+1)2}, \ldots, J_{(n+1)(2^{n+1}-1)}, J_{(n+1)(2^{n+1})}$ is contained in U_1. Suppose that the interval $J_{(n+1)i}$ is contained in U_1, then this interval $J_{(n+1)i}$ is mapped to the whole interval J_1, $g^n_{\gamma_2}: J_{(n+1)i} \longrightarrow J_1$, with $i = 1, \ldots, 2^{n+1}$. This implies that there exists an $x_0 \in U_1$, and $n > 0$, such that $g^n_{\gamma_2}(x_0) \in U_2$. The dynamical system g_{γ_2} is transitive.

In general, for $k = m$, we have $g_{\gamma_m}: J_m = [0, \zeta_m] \longrightarrow [0, \zeta_m] = I$ and there are m critical points $x_c^{(i)}$, with $i = 1, \ldots, m$ that divide the intervals S_i, with $i = 1, \ldots, m$, into $2 * m$ intervals J_{2i} such that $g_{\gamma_m}: J_{2i} \longrightarrow I$, with $i = 1, \ldots, 2 * m$. Therefore, there exist $2 * m$ τ's that divide the intervals J_{2i}, with $i = 1, \ldots, 2 * m$, into $2^2 * m$ intervals J_{3i} such that $g^2_{\gamma_m}: J_{3i} \longrightarrow I$, with $i = 1, \ldots, 2^2 * m$. The points τ's always exist into the intervals J's. We can continue up to any of the intervals $J_{(n+1)i}$, with $i = 1, \ldots, 2^n * m$, is contained in U_1. Suppose that the interval $J_{(n+1)i}$ is contained in U_1, then this interval $J_{(n+1)i}$ is mapped to the whole interval I, $g^n_{\gamma_m}: J_{(n+1)i} \longrightarrow I$, with $i = 1, \ldots, 2^n * m$. This implies that there exists an $x_0 \in U_1$, and $n > 0$, such that $g^n_{\gamma_m}(x_0) \in U_2$. The dynamical system g_{γ_m} is transitive for the case (a).

(b) $\Rightarrow$

We need to prove that any open subsets $U_1, U_2 \in J_k$, with $k = 1, \ldots, m$, there exists an $x_0 \in U_1$, and $n > 0$, such that $g^n_{\gamma_k}(x_0) \in U_2$. For the case $k = 1$, we have $g_{\gamma_1}: J_1 \longrightarrow J_1$, with $J_1 = [0, \zeta_1]$. This case is proved in the same way that the previous case (a) for $k = 1$.

For the case $k = 2$, $g_{\gamma_2}: J_2 \longrightarrow J_2$, with $J_2 = [0, \zeta_2]$, and there are two critical points $x_c^{(1)} \in S_1 = [0, \zeta_1]$ and $x_c^{(2)} \in S_2 = (\zeta_1, \zeta_2]$, with $g_{\gamma_2}(x_c^{(1)}) = \zeta_2$ and $g_{\gamma_2}(x_c^{(2)}) < \zeta_2$. The critical points $x_c^{(1)}$ and $x_c^{(2)}$ divide the intervals S_1 and S_2, respectively, into four intervals $J_{21} = [0, x_c^{(1)}]$, $J_{22} = [x_c^{(1)}, \zeta_1]$, $J_{23} = [\zeta_1, x_c^{(2)}]$, $J_{24} = [x_c^{(2)}, \zeta_2]$. Therefore, we have $g_{\gamma_2}: J_{2i} \longrightarrow J_2$, with $i = 1, 2$, and $g_{\gamma_2}: J_{2i} \longrightarrow [0, g_{\gamma_2}(x_c^{(2)})] \subset J_2$, with $i = 3, 4$. Firstly, we analyze the cases $i = 1, 2$, and later cases $i = 3, 4$.

For $i = 1, 2$, there are $\tau_{21} \in J_{21}$ and $\tau_{22} \in J_{22}$ such that $g^2_{\gamma_2}(\tau_{21}) = g^2_{\gamma_2}(\tau_{22}) = \zeta_2$. Then it is possible to generate four intervals $J_{31} = [0, \tau_{21}]$, $J_{32} = [\tau_{21}, x_c^{(1)}]$, $J_{33} = [x_c^{(1)}, \tau_{22}]$, and $J_{34} = [\tau_{22}, \zeta_1]$. Each of these intervals fulfills $g^2_{\gamma_2}: J_{3i} \longrightarrow J_2$, with $i = 1, \ldots, 4$. This implies that there are $\tau_{3i} \in J_{3i}$ such that $g^3_{\gamma_2}(\tau_{3i}) = \zeta_2$, with $i = 1, \ldots, 2^3$. Then it is possible to generate eight (2^3) intervals J_{4i} such that $g^3_{\gamma_2}: J_{4i} \longrightarrow J_2$, with $i = 1, \ldots, 2^3$. Once again, the τ's always exist because we have that each of the (2^n) intervals $J_{(n+1)i}$ fulfills $g^n_{\gamma_2}: J_{(n+1)i} \longrightarrow J_2$, with $i = 1, \ldots, 2^n$. The refinement of the intervals continues up to any of the intervals $J_{(n+1)i}$ is contained in $U_1 \subset S_1$. Suppose that the interval $J_{(n+1)i}$ is contained in U_1, then this interval $J_{(n+1)i}$ is mapped to the whole interval J_2, $g^n_{\gamma_2}: J_{(n+1)i} \longrightarrow J_2$, with $i = 1, \ldots, 2^n$. This implies that there exists an $x_0 \in U_1 \subset S_1$, and $n > 0$, such that $g^n_{\gamma_2}(x_0) \in U_2 \subset J_2$.

For the cases $i = 3, 4$, $g_{\gamma_2}: J_{2i} \longrightarrow [0, g_{\gamma_2}(x_c^{(2)})]$. There are three cases: $g_{\gamma_2}(x_c^{(2)}) \le \zeta_1 < x_c^{(2)}$; $\zeta_1 < g_{\gamma_2}(x_c^{(2)}) < x_c^{(2)}$; and $x_c^{(2)} < g_{\gamma_2}(x_c^{(2)}) < \zeta_2$.

For the first case, if $g_{\gamma_2}(x_c^{(2)}) \le \zeta_1$, then $g_{\gamma_2}: S_2 \longrightarrow [0, g_{\gamma_2}(x_c^{(2)})] \subseteq S_1$. We know that there exist 2^{n-1} intervals $J_{(n)i} \in S_1$, with $i = 1, \ldots, 2^{n-1}$, such that each interval fulfills $g^{n-1}_{\gamma_2}: J_{(n)i} \longrightarrow J_2$. If $n \longrightarrow \infty$, then the diameter of each interval $J_{(n)i} \longrightarrow 0$ and each of these intervals has four preimages $g^{-1}_{\gamma_2}(J_{(n)i}) \subset J_2$. Two in S_1 and two in S_2. So, if $J_{(n+1)i} = g^{-1}_{\gamma_2}(J_{(n)i}) \in U_1 \subset S_2$ then $g^n_{\gamma_2}: J_{(n+1)i} \longrightarrow J_2$. This implies that there exists an $x_0 \in U_1 \subset S_2$, and $n > 0$, such that $g^n_{\gamma_2}(x_0) \in U_2 \subset J_2$. Therefore, $g_{\gamma_2}(x_0)$ is transitive in J_2 for $x_0 \in S_2$. • The second case $\zeta_1 < g_{\gamma_2}(x_c^{(2)}) < x_c^{(2)}$. There are points $\kappa_1, \kappa_2 \in S_2$ such that $g_{\gamma_2}(\kappa_1) = g_{\gamma_2}(\kappa_2) = \zeta_1$, and $\zeta_1 < \kappa_1 < \kappa_2 < \zeta_2$, then the intervals $(\zeta_1, \kappa_1]$ and $[\kappa_2, \zeta_2]$ are mapped to the whole interval S_1. Therefore, these two intervals can be considered as the first case, however, the other interval g_{γ_2}: $(\kappa_1,\kappa_2) \longrightarrow (\zeta_1, g_{\gamma_2}(x_c^{(2)})] \subset (\zeta_1, x_c^{(2)}] \subset S_2$. $(\kappa_1,\kappa_2) \ni x_c^{(2)}$, such that $g_{\gamma_2}^k(x_c^{(2)}) < \cdots < g_{\gamma_2}^2(x_c^{(2)}) < g_{\gamma_2}(x_c^{(2)}) < x_c^{(2)}$, with $k \in N$ because $g_{\gamma_2}(x) < x$ for all $x \in S_2$. There exists a $k \in N$ such that $g_{\gamma_2}^k(x_c^{(2)}) \in S_1$ and $g_{\gamma_2}^{k-1}(x_c^{(2)}) \in S_2$, then $g_{\gamma_2}^k$: $S_2 \longrightarrow S_1$ at least once any point $x \in S_2$ has been mapped to the interval S_1. This implies that there exists an $x_0 \in U_1 \subset S_2$, and $0 < n \in N$, such that $g_{\gamma_2}^n(x_0) \in U_2 \subset J_2$. Therefore, $g_{\gamma_2}(x_0)$ is transitive in J_2 for $x_0 \in S_2$. • For the third case $x_c^{(2)} < g_{\gamma_2}(x_c^{(2)}) < \zeta_2$, there are two unstable fixed points $\overline{x}_L^{(2)}, \overline{x}_R^{(2)} \in S_2$, with $\overline{x}_L^{(2)} > g_{\gamma_2}^2(x_c^{(2)})$. There are two intervals $(\zeta_1, \overline{x}_L^{(2)})$ and $(\overline{x}_{L'}^{(2)}, \zeta_2)$, with $g_{\gamma_2}(\overline{x}_L^{(2)}) = g_{\gamma_2}(\overline{x}_{L'}^{(2)})$ that fall in the previous case because $g_{\gamma_2}(x) < x$, for all $x \in (\zeta_1, \overline{x}_L^{(2)}) \cup (\overline{x}_{L'}^{(2)}, \zeta_2]$. Therefore, $g_{\gamma_2}(x_0)$ is transitive in J_2 for $(\zeta_1, \overline{x}_L^{(2)})t \cup n(q\overline{x}_{L'}^{(2)}, \zeta_2]$. The middle interval $(\overline{x}_L^{(2)}, \overline{x}_{L'}^{(2)})$ contains an interval $(x_c^{(2)} - \delta, x_c^{(2)} + \delta)$, with $g_{\gamma_2}(x_c^{(2)} - \delta) = \overline{x}_L^{(2)}$, such that g_{γ_2}: $(x_c^{(2)} - \delta) \longrightarrow (\overline{x}_{L'}^{(2)}, \zeta_2]$, so this case falls in the previous case. For the intervals $(\overline{x}_L^{(2)}, x_c^{(2)} - \delta)$ and $(x_c^{(2)} + \delta, \overline{x}_{L'}^{(2)})$, we have $g_{\gamma_2}^k$: $(\overline{x}_L^{(2)}, x_c^{(2)} - \delta) \cup (x_c^{(2)} + \delta, \overline{x}_{L'}^{(2)}) \longrightarrow (x_c^{(2)} - \delta, x_c^{(2)} + \delta)$, for some $k \in N$. This implies that there exists an $x_0 \in U_1 \subset S_2$, and $0 < n \in N$, such that $g_{\gamma_2}^n(x_0) \in U_2 \subset J_2$. Therefore, $g_{\gamma_2}(x_0)$ is transitive in J_2 for $x_0 \in S_2$.

The general case $k = m$, because $g_{\gamma_k}(x_c^{(1)}) > g_{\gamma_k}(x_c^{(2)}) > \cdots > g_{\gamma_k}(x_c^{(m)})$, with $k = 1, \ldots, m$ and $\overline{x}_L^{(j)} > g_{\gamma_k}^2(x_c^{(j)})$ for all $j = 2, \ldots, m$, then there are $k_i \in N$ such that $g_{\gamma_m}^{k_i}$: $S_i \longrightarrow S_1$, with $i = 2, 3 \ldots, m$. And g_{γ_m} is always transitive in I because g_{γ_m}: $S_1 \in I$. This implies that there exists an $x_0 \in U_1 \subset I$, and $n > 0$, such that $g_{\gamma_m}^n(x_0) \in U_2 \subset I$. The dynamical system g_{γ_m} is transitive for the case (b).

(c) ⇒For the case $k = 1$, we have g_{γ_1}: $J_1 \longrightarrow J_1$, with $J_1 = [0, \zeta_1]$. This case is proved in the same way that the previous case (a) for $k = 1$.

For the case $k = 2$, g_{γ_2}: $J_2 \longrightarrow J_2$, with $J_2 = [0, \zeta_2]$, and there are two critical points $x_c^{(1)} \in S_1 = [0, \zeta_1]$ and $x_c^{(2)} \in S_2 = (\zeta_1, \zeta_2]$, with $x_L^{(2)} < g_{\gamma_2}(x_c^{(1)}) < \zeta_2$ and $g_{\gamma_2}(x_c^{(2)}) = \zeta_2$. There are two points $\kappa_1, \kappa_2 \in S_1$ such that $g_{\gamma_2}(\kappa_1) = g_{\gamma_2}(\kappa_2) = x_L^{(2)}$, then g_{γ_2}: $(\kappa_1, \kappa_2) \longrightarrow S_2$. For all $x_0 \in (0, \kappa_1)$ there is a $k - 1 \in N$ such that $g_{\gamma_2}^{k-1}(x_0) \in (\kappa_1, \kappa_2)$, so $g_{\gamma_2}^k(x_0) \in S_2$, and the same for all $x_0 \in (\kappa_2, \zeta_1)$. Therefore, for all $x_0 \in S_1$ there is a $k \in N$ such that $g_{\gamma_2}^k(x_0) \in S_2$. Now, to prove transitivity, we only need to show that there exist intervals of whatever tiny diameter in S_2 that are mapped by g_{γ_2} to the whole interval J_2.

The critical point $x_c^{(2)}$ divides the interval S_2, into two intervals $J_{21} = [\zeta_1, x_c^{(2)}]$, and $J_{22} = [x_c^{(2)}, \zeta_2]$. So, we have g_{γ_2}: $J_{2i} \longrightarrow J_2$, with $i = 1, 2$, because $g_{\gamma_2}(x_c^{(2)}) = \zeta_2$. There are $\tau_{21} \in J_{21}$ and $\tau_{22} \in J_{22}$ such that $g_{\gamma_2}^2(\tau_{21}) = g_{\gamma_2}^2(\tau_{22}) = \zeta_2$. Then it is possible to generate four intervals $J_{31} = [0, \tau_{21}]$, $J_{32} = [\tau_{21}, x_c^{(1)}]$, $J_{33} = [x_c^{(1)}, \tau_{22}]$, and $J_{34} = [\tau_{22}, \zeta_1]$. Each of these intervals fulfills $g_{\gamma_2}^2$: $J_{3i} \longrightarrow J_2$, with $i = 1, \ldots, 4$. This implies that there are $\tau_{3i} \in J_{3i}$ such that $g_{\gamma_2}^3(\tau_{3i}) = \zeta_2$, with $i = 1, \ldots, 2^3$. Then it is possible to generate eight (2^3) intervals J_{4i} such that $g_{\gamma_2}^3$: $J_{4i} \longrightarrow J_2$, with $i = 1, \ldots, 2^3$. Once again, the τ's always exist because we have that each of the (2^n) intervals $J_{(n+1)i}$ fulfills $g_{\gamma_2}^n$: $J_{(n+1)i} \longrightarrow J_2$, with $i = 1, \ldots, 2^n$. The refinement of the intervals continues up to any of the intervals $J_{(n+1)i}$ is contained in $U_1 \subset S_2$. Suppose that the interval $J_{(n+1)i}$ is contained in U_1, then this interval $J_{(n+1)i}$ is mapped to the whole interval J_2, $g_{\gamma_2}^n$: $J_{(n+1)i} \longrightarrow J_2$, with $i = 1, \ldots, 2^n$. For all $x \in S_1$ is mapped to S_2 then there is a $n \in N$ such that $g_{\gamma_2}^n$: $U_1 \longrightarrow J_2$, so there are preimages of the $J_{(n+1)i}$ such that $U_1 \subset g_{\gamma_2}^{-n_2}(J_{(n_1+1)i})$, with $n = n_1 + n_2$. This implies that there exists an $x_0 \in U_1 \subset J_2$, and $n > 0$, such that $g_{\gamma_2}^n(x_0) \in U_2 \subset J_2$.

We prove an arbitrary case $g_{\gamma_2}(x_c^{(1)}) < g_{\gamma_2}(x_c^{(2)}) = \zeta_2$. Then the general case $k = m$ is a direct consequence of this case because $g_{\gamma_k}(x_c^{(1)}) < g_{\gamma_k}(x_c^{(2)}) < \cdots < g_{\gamma_k}(x_c^{(m)}) = \zeta_m$, with $k = 1, \ldots, m$, and the following inequality is always preserved $g_{\gamma_k}(x_c^{(j)}) > x_L^{(j+1)}$, for $j = 1, \ldots, m - 1$.

Therefore, for all $x_0 \in U_1 \subset I$ there is a $k \in N$ such that $g_{\gamma_2}^k(x_0) \in U_2 \subset I$. Then g_{γ_k} is transitive, for $k = 1, \ldots, m$. This completes the proof. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this paper.

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